Approximate sequences and characterization of their limit points on Hadamard spaces

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Contents

Chapter 1

Introduction

Approximation theory of fixed points is one of the most important problems in nonlinear analysis and many researchers have been studying various techniques. Among them, Mann and Halpern type methods are famous techniques.

Mann type methods were introduced by Mann [15] in 1953. An iteration using this method is said be a Mann type method, which is iteratively generated by a convex combination of itself and its mapped image as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n.$$

Mann showed the iteration with a nonexpansive mapping is convergent to a fixed point in a compact convex set in a Banach space.

On the other hand, An iteration using the Halpern type method were considered by Halpern [7] in 1967. The iteration is said to be a Halpern type method, which is iteratively generated by a convex combination of a definite point, which is called an anchor point, and the image mapped from the current point as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n.$$

Later, Wittmann [24] showed the iteration with a nonexpansive mapping is strong convergent to a fixed point in a Hilbert space. He laid the foundation for Halpern type methods.

These methods have been proved in various setting by many mathematicians. They have already been showed in a Banach space with some assumptions [18, 22], in a Hadamard space [5, 19], and in a complete CAT(1) space

[10, 11, 17]. Moreover, these methods are applied to an approximation to a common fixed point. For example, Takahashi and Tamura [23], Kimura, Saejung, and Yotkaew [10], Kimura and Nakagawa [9] introduced Mann type iterations with some mappings converging to a common fixed point, respectively. Shimizu and Takahashi [21] proved that a Halpern type iteration with two nonexpansive mappings converges strongly to a common fixed point in a Hilbert space, and Saejung [19] introduced an approximation theorem to a common fixed point used a finite family of nonexpansive mappings in a Hadamard space.

In this thesis, we introduce a new iteration with a Halpern type construction which is used a finite family of mappings in a Hadamard space. The iteration is strongly convergent to a common fixed point under similar conditions to the following theorem.

Theorem 1.1 (Kimura-Takahashi-Toyoda [12]). *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and C a closed convex subset of E. Let $T_1, T_2, \ldots, T_r : C \to C$ be nonexpansive mappings such that the common fixed point set F is nonempty. Let u, x_1 be arbitrary points in C and let $\{x_n\}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \sum_{k=1}^{r} \beta_n^k T_k x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $]0, 1[$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

and, for each $k = 1, 2, \ldots, r$, $\{\beta_n^k\}$ are sequences in $[a, b] \subset\,]0, 1[$ such that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{r} |\beta_{n+1}^k - \beta_n^k| < \infty, \quad \lim_{n\to\infty} \beta_n^k = \beta^k \in\,]0, 1[\,, \quad \sum_{k=1}^{r} \beta_n^k = 1. \quad (n \in \mathbb{N})$$

Then, $\{x_n\}$ converges to the point Pu, where P is a sunny nonexpansive retraction of C onto F.

The sequence $\{\beta_n\}$ of Theorem 1.1 must belong to $[a, b]$ for some $a, b \in\,]0, 1[$. We also consider a convergence result under the condition which $\{\beta_n\}$ does not belong to $[a, b]$ for any $a, b \in\,]0, 1[$.

Furthermore, we introduce an iteration generated by the combination of Mann and Halpern types in the following theorem.

Theorem 1.2 (Aoyama-Kimura [1]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $S, T : C \to C$ nonexpansive mappings with a common fixed point. Suppose S or T is a strongly nonexpansive mapping. Let $u, x_1 \in X$ and $\{x_n\}$ is iteratively generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)S(\alpha_n u + (1-\alpha_n)Tx_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset]0,1[$ satisfies

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

and $\{\beta_n\} \subset [a,b] \subset]0,1[$. Then, $\{x_n\}$ converges strongly to the nearest point of $F(S) \cap F(T)$ to u.

We consider an iterative scheme in Theorem 1.2 converges strongly to a common fixed point under similar conditions in a Hadamard space. Then, since we can not define a strongly nonexpanisve mapping in a Hadamard space in a natural way, we use strongly quasinonexpansive mappings, originally proposed by Bruck [4], instead of strongly nonexpansive mappings.

In this thesis, we prepare definitions and notions for main results in Chapter 2, and consider main results in Chapter 3 and Chapter 4. We introduce an iteration extending Halpern type in Chapter 3 and an iteration generated by combination of Mann and Halpern types in Chapter 4. Finally, we make a summary of our results in Chapter 5.

Chapter 2

Preliminaries

In this chapter, we introduce some definitions and their properties for main theorems.

2.1 Hadamard spaces

Let (X, d) be a metric space. For $x, y \in X$, a mapping $c : [0, l] \to X$ is called a geodesic with endpoints x, y if c satisfies $c(0) = x, c(l) = y$ and $d(c(u), c(v)) = |u - v|$ for $u, v \in [0, l]$. Then, the image of a geodesic c with endpoints x, y is called a geodesic segment joining x and y, and is denoted by $[x, y]$. If a geodesic segment exists for any $x, y \in X$, then we call X a geodesic metric space. Moreover, if a geodesic segment exists uniquely for each $x, y \in X$, then we call X a uniquely geodesic space. Then, for $t \in [0, 1]$ and $x, y \in X$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y)$. We denote z by $tx \oplus (1 - t)y$.

Let X be a uniquely geodesic space. A geodesic triangle $\triangle(x_1, x_2, x_3)$ with vertices x_1, x_2, x_3 in X is the union of geodesic segments joining each pair of vertices. A comparison triangle $\overline{\triangle}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ for $\triangle(x_1, x_2, x_3)$ is a triangle such that $d(x_i, x_j) = \|\bar{x}_i - \bar{x}_j\|$ for all $i, j = 1, 2, 3$. A point $\bar{p} \in [\bar{x}_1, \bar{x}_2]$ is a comparison point of $p \in [x_1, x_2]$ if $d(x_1, p) = \|\bar{x}_1 - \bar{p}\|$. For any $p, q \in \triangle(x_1, x_2, x_3)$ and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, if the inequality

$$d(p, q) \leq \|\bar{p} - \bar{q}\|$$

is satisfied for all triangle in X, then X is called a CAT(0) space, and this inequality is called the CAT(0) inequality. A Hadamard space is defined as a complete CAT(0) space.

The following lemma plays an important role in this thesis.

Theorem 2.1. *Let X be a Hadamard space. Then, for any $x, y, z \in X$ and $t \in]0,1[$, it follows that*

$$d(x, ty \oplus (1-t)z)^2 \leq td(x,y)^2 + (1-t)d(x,z)^2 - t(1-t)d(y,z)^2.$$

Proof. Consider the geodesic triangle $\triangle(x,y,z)$. From the CAT(0) inequality, we have

$$\begin{aligned} d(x, ty \oplus (1-t)z)^2 &\leq \|\bar{x} - (t\bar{y} - (1-t)\bar{z})\|^2 \\ &= \|t(\bar{x} - \bar{y}) + (1-t)(\bar{x} - \bar{z})\|^2 \\ &= t\|\bar{x} - \bar{y}\|^2 + (1-t)\|\bar{x} - \bar{z}\|^2 - t(1-t)\|\bar{y} - \bar{z}\|^2 \\ &= td(x,y)^2 + (1-t)d(x,z)^2 - t(1-t)d(y,z)^2. \end{aligned}$$

Thus, we get the desired result. □

Further, using Theorem 2.1, we get the following result.

Corollary 2.2. *Let $\{x_n\}, \{y_n\}$ be bounded sequences of a Hadamard space. For $\{\alpha_n\} \subset]0,1[$, define a sequence $\{z_n\}$ by $z_n = \alpha_n x_n \oplus (1-\alpha_n)y_n$. Then $\{z_n\}$ is bounded.*

A subset C in X is said to be convex if, for any $x, y \in C$, $[x, y]$ is included in C. Then, there exists a metric projection in a Hadamard space.

Theorem 2.3. *Let X be a Hadamard space, C a closed convex subset of X. Then, for any point $u \in X$, there exists a unique nearest point x_0 in C to u. That is, the following holds.*

$$d(u, x_0) = \inf_{x \in C} d(u, x).$$

Proof. Put $k = \inf_{x \in C} d(u,x)$. Assume $\{x_n\}$ is a sequence on C such that $d(u,x_n)^2 \leq k^2 + \frac{1}{n}$. Then we have $\lim_{n \to \infty} d(u, x_n) = k$. For $n \leq m$, it follows from Theorem 2.1 that

$$d(u, \frac{1}{2}x_n \oplus \frac{1}{2}x_m)^2 \leq \frac{1}{2}d(u,x_n)^2 + \frac{1}{2}d(u,x_m)^2 - \frac{1}{4}d(x_n,x_m)^2.$$

Since C is a convex, we have

$$k^2 \leq \frac{1}{2}d(u,x_n)^2 + \frac{1}{2}d(u,x_m)^2 - \frac{1}{4}d(x_n,x_m)^2.$$

Thus,

$$\begin{aligned}
\frac{1}{4}d(x_n,x_m)^2 &\le \frac{1}{2}d(u,x_n)^2+\frac{1}{2}d(u,x_m)^2-k^2\\
&=\frac{1}{2}(d(u,x_n)^2-k^2)+\frac{1}{2}(d(u,x_m)^2-k^2)\\
&\le \frac{1}{2}(k^2+\frac{1}{n}-k^2)+\frac{1}{2}(k^2+\frac{1}{n}-k^2)\\
&=\frac{1}{n}.
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence, and since X is complete and C is closed, there exists $x_0 \in C$ such that $\lim_{n\to\infty} x_n = x_0$. Thus,

$$d(u,x_0)=\lim_{n\to\infty} d(u,x_n)=k.$$

Next, we show that x_0 is unique. Let $y_0 \in C$ satisfy that $d(u,y_0)=k$. Then, we have

$$\begin{aligned}
k^2 &\le d(u,\frac{1}{2}x_0\oplus\frac{1}{2}y_0)^2\\
&\le \frac{1}{2}d(u,x_0)^2+\frac{1}{2}d(u,y_0)^2-\frac{1}{4}d(x_0,y_0)^2\\
&=k^2-\frac{1}{4}d(x_0,y_0)^2.
\end{aligned}$$

Then, we get

$$\frac{1}{4}d(x_0,y_0)^2 \le k^2-k^2=0.$$

Thus, we have $x_0=y_0$. □

For more details on Hadamard space, see [3].

Let $\{x_n\}$ be a bounded sequence in a metric space X. For any $x \in X$, we put

$$r(x,\{x_n\})=\limsup_{n\to\infty} d(x,x_n), \quad r(\{x_n\})=\inf_{x\in X} r(x,\{x_n\}).$$

Then, if there exists $x \in X$ such that $r(x,\{x_n\})=r(\{x_n\})$, we call x an asymptotic center of $\{x_n\}$. Moreover if, for any subsequence of $\{x_n\}$, its asymptotic center is a unique point x, we say that $\{x_n\}$ is Δ-convergent to x.

Theorem 2.4 (He-Fang-López-Li [8])**.** *Let X be a Hadamard space and $\{x_n\}$ be a bounded sequence of X. If $\{x_n\}$ is Δ-convergent to $x \in X$, then*

$$d(u, x)^2 \le \liminf_{n\to\infty} d(u, x_n)^2$$

for all $u \in X$.

Proof. Let $u \in X$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} d(u, x_{n_i})^2 = \liminf_{n\to\infty} d(u, x_n)^2.$$

For $t \in]0, 1[$, we have

$$\begin{aligned}\limsup_{i\to\infty} d(x_{n_i}, x)^2 &\le \limsup_{i\to\infty} d(x_{n_i}, tu \oplus (1-t)x)^2 \\ &\le \limsup_{i\to\infty}(td(x_{n_i}, u)^2 + (1-t)d(x_{n_i}, x)^2 - t(1-t)d(u, x)^2) \\ &\le t \limsup_{i\to\infty} d(x_{n_i}, u)^2 \\ &\quad + (1-t)\limsup_{i\to\infty} d(x_{n_i}, x)^2 - t(1-t)d(u, x)^2.\end{aligned}$$

Thus,

$$0 \le \limsup_{i\to\infty} d(x_{n_i}, x)^2 \le \limsup_{i\to\infty} d(x_{n_i}, u)^2 - (1-t)d(u, x)^2.$$

Tending $t \to 0$, we obtain

$$0 \le \limsup_{i\to\infty} d(x_{n_i}, u)^2 - d(u, x)^2,$$

and hence

$$d(u, x)^2 \le \limsup_{i\to\infty} d(x_{n_i}, u)^2 = \lim_{i\to\infty} d(x_{n_i}, u)^2 = \liminf_{n\to\infty} d(x_n, u)^2,$$

so we get the desired result. □

Now, we introduce an important property of Δ-convergence.

Theorem 2.5 (Goebel-Kirk [6], Kirk-Panyanak [14])**.** *In a Hadamard space, any bounded sequence has a Δ-converging subsequence.*

2.2 Mappings and fixed points

Let T be a mapping defined on a metric space (X, d). T is called a nonexpansive mapping if the inequality $d(Tx, Ty) \leq d(x, y)$ is satisfied for any $x, y \in X$. A point $z \in X$ is called a fixed point of T if $Tz = z$ holds. We denote the set of all fixed points of T by $F(T)$.

A mapping T is quasinonexpansive if T has a fixed point and satisfies $d(Tx, z) \leq d(x, z)$ for any $x \in X$ and $z \in F(T)$. We know that a nonexpansive mapping with a fixed point is quasinonexpansive.

Theorem 2.6. *Let T be a quasinonexpansive mapping in a Hadamard space X. Then, $F(T)$ is closed and convex.*

Proof. Let $\{x_n\} \subset F(T)$ be convergent to $x \in X$. Then,

$$d(x, Tx) \leq d(x, x_n) + d(x_n, Tx) \leq 2d(x, x_n).$$

Tending $n \to 0$, we have $d(x, Tx) \leq 0$. Thus x is a fixed point of T, so $F(T)$ is closed. Next, we show $F(T)$ is convex. For any $x, y \in F(T)$, suppose $u \in [x, y]$, that is $u = tx \oplus (1-t)y$ for $t \in]0, 1[$. We have

$$\begin{aligned} d(u, Tu)^2 &= d(tx \oplus (1-t)y, Tu)^2 \\ &\leq td(x, Tu)^2 + (1-t)d(y, Tu)^2 - t(1-t)d(x, y)^2 \\ &\leq td(x, u)^2 + (1-t)d(y, u)^2 - t(1-t)d(x, y)^2 \\ &= t(1-t)^2 d(x, y)^2 + t^2(1-t)d(y, x)^2 - t(1-t)d(x, y)^2 \\ &= 0. \end{aligned}$$

We get $u = Tu$, so $F(T)$ is convex. □

The following theorems show properties of a convex combination of mappings.

Theorem 2.7. *Let X be a Hadamard space and $S, T : X \to X$ quasinonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. For any $\beta \in]0, 1[$, define a mapping U by $Ux = \beta Sx \oplus (1-\beta)Tx$ for all $x \in X$. Then U is a quasinonexpansive mapping such that $F(U) = F(S) \cap F(T)$. In particular, if S, T are nonexpansive, then so is U.*

Proof. First, we show $F(U) = F(S) \cap F(T)$. It is obvious that $F(U) \supset F(S) \cap F(T)$, so we show $F(U) \subset F(S) \cap F(T)$. Let x be a point of $F(U)$. Then, for any point $p \in F(S) \cap F(T)$, we have

$$\begin{aligned}
d(x,p)^2 &= d(Ux,p)^2 \\
&= d(\beta Tx \oplus (1-\beta)Sx, p)^2 \\
&\le \beta d(Tx,p)^2 + (1-\beta)d(Sx,p)^2 - \beta(1-\beta)d(Tx,Sx)^2 \\
&\le d(x,p)^2 - \beta(1-\beta)d(Tx,Sx)^2.
\end{aligned}$$

Since $\beta \in]0,1[$, we have $d(Tx,Sx) = 0$ and hence $Tx = Sx$. Therefore, we have

$$x = Ux = Tx = Sx,$$

and hence $x \in F(S) \cap F(T)$. Thus, we get $F(U) = F(S) \cap F(T)$.

By this result, $F(U)$ is not empty since $F(S) \cap F(T) \neq \emptyset$. Moreover, for any $x \in X$ and $z \in F(U)$, we get

$$d(Ux,z)^2 \le \beta d(Sx,z)^2 + (1-\beta)d(Tx,z)^2 \le d(x,z)^2,$$

thus $d(Ux,z) \le d(x,z)$. Therefore, U is a quasinonexpansive.

Moreover, let S,T be nonexpansive. For any $x,y \in X$, considering $\overline{\triangle}(\overline{Sx},\overline{Tx},\overline{Ty})$, $\overline{\triangle}(\overline{Ty},\overline{Sx},\overline{Sy})$, we have

$$\begin{aligned}
d(Ux,Uy) &= d(\beta Sx \oplus (1-\beta)Tx, \beta Sy \oplus (1-\beta)Ty) \\
&\le d(\beta Sx \oplus (1-\beta)Tx, \beta Sx \oplus (1-\beta)Ty) \\
&\quad + d(\beta Sx \oplus (1-\beta)Ty, \beta Sy \oplus (1-\beta)Ty) \\
&\le \left\|(\beta\overline{Sx} + (1-\beta)\overline{Tx}) - (\beta\overline{Sx} + (1-\beta)\overline{Ty})\right\| \\
&\quad + \left\|(\beta\overline{Sx} + (1-\beta)\overline{Ty}) - (\beta\overline{Sy} + (1-\beta)\overline{Ty})\right\| \\
&= (1-\beta)\left\|\overline{Tx} - \overline{Ty}\right\| + \beta\left\|\overline{Sx} - \overline{Sy}\right\| \\
&= (1-\beta)d(Tx,Ty) + \beta d(Sx,Sy) \\
&\le (1-\beta)d(x,y) + \beta d(x,y) \\
&= d(x,y).
\end{aligned}$$

Thus, U is a nonexpansive mapping. □

A sequence $\{x_n\}$ is an asymptotic fixed point sequence of T if $d(Tx_n,x_n) \to 0$. We denote the set of any bounded asymptotic fixed point sequences of T by $\tilde{F}(T)$.

Theorem 2.8. *Let R, S, T be quasinonexpansive mappings with a common fixed point in a Hadamard space. For any $\beta, \gamma \in]0,1[$, we define $U = \beta R \oplus (1-\beta)(\gamma S \oplus (1-\gamma)T)$. Then, $\tilde{F}(U) = \tilde{F}(R) \cap \tilde{F}(S) \cap \tilde{F}(T)$.*

Proof. It is obvious that $\tilde{F}(U) \supset \tilde{F}(R) \cap \tilde{F}(S) \cap \tilde{F}(T)$, so we prove that $\tilde{F}(U) \subset \tilde{F}(R) \cap \tilde{F}(S) \cap \tilde{F}(T)$. Let $\{x_n\} \in \tilde{F}(U)$. Suppose Q is a mapping such that $Q = \gamma S \oplus (1-\gamma)T$ and p is an element of $F(R) \cap F(S) \cap F(T)$. By Theorems 2.1 and 2.7, we get

$$\begin{aligned} d(p, Ux_n)^2 &= d(p, \beta Rx_n \oplus (1-\beta)Qx_n)^2 \\ &\leq \beta d(p, Rx_n)^2 + (1-\beta)d(p, Qx_n)^2 - \beta(1-\beta)d(Rx_n, Qx_n)^2 \\ &\leq d(p, x_n)^2 - \beta(1-\beta)d(Rx_n, Qx_n)^2. \end{aligned}$$

Thus, we have

$$\begin{aligned} \beta(1-\beta)d(Rx_n, Qx_n)^2 &\leq d(p, x_n)^2 - d(p, Ux_n)^2 \\ &\leq (d(p, x_n) + d(p, Ux_n))d(x_n, Ux_n) \\ &\to 0, \end{aligned}$$

and hence we get

$$d(Rx_n, Qx_n) \to 0.$$

Furthermore, we obtain that

$$\begin{aligned} d(x_n, Qx_n) &\leq d(x_n, Ux_n) + d(Ux_n, Qx_n) \\ &= d(x_n, Ux_n) + \beta d(Rx_n, Qx_n) \\ &\to 0. \end{aligned}$$

By the same procedure, it follows that

$$d(p, Qx_n)^2 \leq d(p, x_n)^2 - \gamma(1-\gamma)d(Sx_n, Tx_n)^2,$$

and hence we get

$$d(Sx_n, Tx_n) \to 0.$$

Therefore, we have that

$$\begin{aligned}
d(Rx_n, x_n) &\le d(Rx_n, Ux_n) + d(Ux_n, x_n) \\
&= (1-\beta)d(Rx_n, Qx_n) + d(Ux_n, x_n) \\
&\to 0, \\
d(Sx_n, x_n) &\le d(Sx_n, Qx_n) + d(Qx_n, x_n) \\
&= (1-\gamma)d(Sx_n, Tx_n) + d(Qx_n, x_n) \\
&\to 0, \\
d(Tx_n, x_n) &\le d(Tx_n, Qx_n) + d(Qx_n, x_n) \\
&= \gamma d(Sx_n, Tx_n) + d(Qx_n, x_n) \\
&\to 0,
\end{aligned}$$

and so $\{x_n\} \in \tilde{F}(R) \cap \tilde{F}(S) \cap \tilde{F}(T)$. □

We get the following results by Theorems 2.6 and 2.8.

Corollary 2.9. *Let X be a Hadamard space, and T a nonexpansive mapping in X with $F(T) \neq \emptyset$. Then, $F(T)$ is closed and convex.*

Corollary 2.10. *Let R, S, T be nonexpansive mappings with a common fixed point in a Hadamard space. For any $\beta, \gamma \in]0, 1[$, we define $U = \beta R \oplus (1-\beta)(\gamma S \oplus (1-\gamma)T)$. Then, $\tilde{F}(U) = \tilde{F}(R) \cap \tilde{F}(S) \cap \tilde{F}(T)$.*

We say that T is Δ-demiclosed if $x \in F(T)$ whenever $\{x_n\} \in \tilde{F}(T)$ is Δ-convergent to x.

Theorem 2.11 (Kirk-Panyanak [14])**.** *A nonexpansive mapping defined on a Hadamard space is Δ-demiclosed.*

Proof. Let T be a nonexpansive mapping in a Hadamard space and $\{x_n\} \in \tilde{F}(T)$ Δ-convergent to x. Assume $Tx \neq x$. Then, we have that

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, x) &< \limsup_{n\to\infty} d(x_n, Tx) \\
&\le \limsup_{n\to\infty}(d(x_n, Tx_n) + d(Tx_n, Tx)) \\
&\le \limsup_{n\to\infty} d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, x) \\
&= \limsup_{n\to\infty} d(x_n, x),
\end{aligned}$$

and this is a contradiction. Thus $x \in F(T)$, that is, T is Δ-demiclosed. □

Chapter 3

Halpern type convergence theorems

In this chapter, we consider Halpern type iterations with multiple anchor points in a Hadamard space. First, we show that the iteration with finite family of nonexpansive mappings is strongly convergent to a common fixed point. This result is already published in [13]. For the sake of simplicity, we will only prove the case for triple mappings. On the other hand, the range of the coefficients needs to satisfy some conditions in our result. We attempt to weaken them and obtain a convergence theorems under weaker conditions.

3.1 Lemmas for main results

Lemma 3.1 (Aoyama-Kimura-Takahashi-Toyoda [2], Xu [25]). *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers,* $\{\alpha_n\}$ *a sequence in* $[0,1]$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ *a sequence of nonnegative real numbers with* $\sum_{n=1}^{\infty} u_n < \infty$, *and* $\{t_n\}$ *a sequence of real numbers with* $\limsup_{n\to\infty} t_n \leq 0$. *Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n t_n + u_n \quad \textit{for all } n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} s_n = 0$.

Proof. Let $T_m = \sup_{m \leq n} t_n$ for $m \in \mathbb{N}$. For any $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
& s_{n+m+1} \\
& \leq (1-\alpha_{n+m}) s_{n+m} + \alpha_{n+m} t_{n+m} + u_{n+m} \\
& \leq (1-\alpha_{n+m}) s_{n+m} + \alpha_{n+m} T_m + u_{n+m} \\
& \leq (1-\alpha_{n+m})((1-\alpha_{n+m-1}) s_{n+m-1} + \alpha_{n+m-1} T_m + u_{n+m-1}) \\
& \quad + \alpha_{n+m} T_m + u_{n+m} \\
& \leq (1-\alpha_{n+m})(1-\alpha_{n+m-1}) s_{n+m-1} \\
& \quad + (1-(1-\alpha_{n+m})(1-\alpha_{n+m-1})) T_m + u_{n+m} + u_{n+m-1} \\
& \leq \cdots \\
& \leq \prod_{i=m}^{n+m} (1-\alpha_i) s_m + (1 - \prod_{i=m}^{n+m} (1-\alpha_i)) T_m + \sum_{i=m}^{n+m} u_i,
\end{aligned}$$

and so

$$\begin{aligned}
\lim_{n\to\infty} s_n = & \lim_{n\to\infty} s_{n+m+1} \\
& \leq \prod_{i=m}^{\infty} (1-\alpha_i) s_m + (1 - \prod_{i=m}^{\infty} (1-\alpha_i)) T_m + \sum_{i=m}^{\infty} u_i.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, we get $\prod_{i=m}^{\infty} (1-\alpha_i) = 0$, and so we have

$$\begin{aligned}
\lim_{n\to\infty} s_n & \leq T_m + \sum_{i=m}^{\infty} u_i \\
& = T_m + (\sum_{i=1}^{\infty} u_i - \sum_{i=1}^{m-1} u_i).
\end{aligned}$$

Tending $m \to \infty$, since $\limsup_{n\to\infty} t_n = \lim_{m\to\infty} T_m$, we obtain

$$\begin{aligned}
\lim_{n\to\infty} s_n & \leq \lim_{m\to\infty} T_m + (\sum_{i=1}^{\infty} u_i - \sum_{i=1}^{\infty} u_i) \\
& = \limsup_{n\to\infty} t_n \leq 0.
\end{aligned}$$

Since $\{s_n\}$ is nonnegative, we have $\lim_{n\to\infty} s_n = 0$. □

Lemma 3.2. *Let $\{a_n\}$ be a sequence of real numbers with $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$. Then $\{a_n\}$ is convergent.*

Proof. Let n, m be natural numbers with $n \leq m$. Then, we obtain

$$\begin{aligned}
|a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} - a_{m-2} + \cdots + a_{n+2} - a_{n+1} + a_{n+1} - a_n| \\
&\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n| \\
&= \sum_{k=n}^{m-1} |a_{k+1} - a_k| \\
&\leq \sum_{k=n}^{\infty} |a_{k+1} - a_k| \\
&\to 0. \ (n \to \infty)
\end{aligned}$$

Thus, we have $\{a_n\}$ is a Cauchy sequence, and hence $\{a_n\}$ is convergent. □

In order to obtain an important result, we introduce a following lemma proved by Mayer [16].

Lemma 3.3 (Mayer [16])**.** *Let X be a Hadamard space and $g : X \to \mathbb{R} \cup \{+\infty\}$. If g is convex and lower semicontinuous, then g is bounded from below on bounded subsets of X. Furthermore, g attains its infimum on nonempty bounded convex closed subsets of X. The resulting minimizer is unique if g is strictly convex.*

We obtain the following result by this lemma.

Corollary 3.4. *Let X be a Hadamard space. For any $u_1, u_2, \ldots, u_n \in X$ and $\beta^1, \beta^2, \ldots, \beta^n \in]0, 1[$ with $\sum_{i=1}^{n} \beta^i = 1$, define a function $g : X \to \mathbb{R}$ by*

$$g(x) = \sum_{i=1}^{n} \beta^i d(u_i, x)^2$$

for all $x \in X$. Then g attains its infimum on a nonempty closed convex subset C of X, and its minimizer is unique.

Proof. Let p be an element of X. Since $g(x) \to \infty$ as $d(x, p) \to \infty$, there exists a nonempty bounded closed convex set D such that the minimizers of g on C and D are identical.

For $x, y \in X$ with $x \neq y$ and $t \in]0,1[$, we have

$$\begin{aligned} g(tx \oplus (1-t)y) &= \sum_{i=1}^{n} \beta^i d(u_i, tx \oplus (1-t)y)^2 \\ &\leq \sum_{i=1}^{n} \beta^i (td(u_i, x)^2 + (1-t)d(u_i, y)^2 - t(1-t)d(x,y)^2) \\ &= tg(x) + (1-t)g(y) - t(1-t)d(x,y)^2 \\ &< tg(x) + (1-t)g(y). \end{aligned}$$

Thus g is a strictly convex, and by Lemma 3.3 we get the desired result. □

3.2 Iterations with a finite family of mappings

Let u, v, w be arbitrary points in a Hadamard space X and $R, S, T : X \to X$ nonexpansive mappings with $F = F(R) \cap F(S) \cap F(T) \neq \emptyset$. Then, we consider the following iteration $\{x_n\}$:

$$(*) \begin{cases} x_1 \in X, \\ r_n = \alpha_n u \oplus (1-\alpha_n) R x_n, \\ s_n = \alpha_n v \oplus (1-\alpha_n) S x_n, \\ t_n = \alpha_n w \oplus (1-\alpha_n) T x_n, \\ x_{n+1} = \beta_n r_n \oplus (1-\beta_n)(\gamma_n s_n \oplus (1-\gamma_n) t_n) \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $]0,1[$.

To prove our main results, we show some lemmas. In what follows, we let $y_n = \gamma_n s_n \oplus (1-\gamma_n) t_n$ for $n \in \mathbb{N}$.

Lemma 3.5. *Let $\{x_n\}$ be an iterative scheme on X generated by $(*)$. Then, $\{x_n\}, \{Rx_n\}, \{Sx_n\}$, and $\{Tx_n\}$ are bounded.*

Proof. Let $p \in F$. By Theorem 2.1, we get

$$\begin{aligned}
&d(x_{n+1}, p)^2 \\
&= d(\beta_n r_n \oplus (1-\beta_n) y_n, p)^2 \\
&\leq \beta_n d(r_n, p)^2 + (1-\beta_n) d(y_n, p)^2 \\
&\leq \beta_n (\alpha_n d(u, p)^2 + (1-\alpha_n) d(Rx_n, p)^2) \\
&\quad + (1-\beta_n)(\gamma_n d(s_n, p)^2 + (1-\gamma_n) d(t_n, p)^2) \\
&\leq \beta_n (\alpha_n d(u, p)^2 + (1-\alpha_n) d(x_n, p)^2) \\
&\quad + (1-\beta_n)\gamma_n (\alpha_n d(v, p)^2 + (1-\alpha_n) d(x_n, p)^2) \\
&\quad + (1-\beta_n)(1-\gamma_n)(\alpha_n d(w, p)^2 + (1-\alpha_n) d(x_n, p)^2) \\
&= \alpha_n (\beta_n d(u, p)^2 + (1-\beta_n)(\gamma_n d(v, p)^2 + (1-\gamma_n) d(w, p)^2)) \\
&\quad + (1-\alpha_n) d(x_n, p)^2.
\end{aligned}$$

Putting $M = \max\{d(u, p)^2, d(v, p)^2, d(w, p)^2\}$, we have

$$d(x_{n+1}, p)^2 \leq \max\{M, d(x_n, p)^2\}.$$

By induction, we get

$$d(x_{n+1}, p)^2 \leq \max\{M, d(x_1, p)^2\},$$

and hence we have $\{x_n\}$ is bounded. Since R, S, and T are nonexpansive, we get $\{Rx_n\}, \{Sx_n\}, \{Tx_n\}$ are bounded. □

Lemma 3.6. *Let $\{x_n\}$ be an iterative scheme on X generated by $(*)$ with the following conditions.*

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(iv) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(v) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then, $\{d(x_{n+1}, x_n)\}$ converges to 0.

Proof. From Lemma 3.5 and Corollary 2.2, we have that $\{r_n\}, \{s_n\}, \{t_n\}$, and $\{y_n\}$ are bounded sequences. Considering a geodesic triangle $\triangle(u, Rx_n, Rx_{n-1})$, we obtain

$$\begin{aligned} d(r_n, r_{n-1}) &= d(\alpha_n u \oplus (1-\alpha_n)Rx_n, \alpha_{n-1}u \oplus (1-\alpha_{n-1})Rx_{n-1}) \\ &\leq d(\alpha_n u \oplus (1-\alpha_n)Rx_n, \alpha_n u \oplus (1-\alpha_n)Rx_{n-1}) \\ &\quad + d(\alpha_n u \oplus (1-\alpha_n)Rx_{n-1}, \alpha_{n-1}u \oplus (1-\alpha_{n-1})Rx_{n-1}) \\ &\leq (1-\alpha_n)d(Rx_n, Rx_{n-1}) + |\alpha_n - \alpha_{n-1}|\, d(u, Rx_{n-1}) \\ &\leq (1-\alpha_n)d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|\, d(u, Rx_{n-1}). \end{aligned}$$

From this result, we also get

$$\begin{aligned} &d(y_n, y_{n-1}) \\ &= d(\gamma_n s_n \oplus (1-\gamma_n)t_n, \gamma_{n-1}s_{n-1} \oplus (1-\gamma_{n-1})t_{n-1}) \\ &\leq d(\gamma_n s_n \oplus (1-\gamma_n)t_n, \gamma_n s_{n-1} \oplus (1-\gamma_n)t_n) \\ &\quad + d(\gamma_n s_{n-1} \oplus (1-\gamma_n)t_n, \gamma_n s_{n-1} \oplus (1-\gamma_n)t_{n-1}) \\ &\quad + d(\gamma_n s_{n-1} \oplus (1-\gamma_n)t_{n-1}, \gamma_{n-1}s_{n-1} \oplus (1-\gamma_{n-1})t_{n-1}) \\ &\leq \gamma_n d(s_n, s_{n-1}) + (1-\gamma_n)d(t_n, t_{n-1}) + |\gamma_n - \gamma_{n-1}|\, d(s_{n-1}, t_{n-1}) \\ &\leq \gamma_n((1-\alpha_n)d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|\, d(v, Sx_{n-1})) \\ &\quad + (1-\gamma_n)((1-\alpha_n)d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|\, d(w, Tx_{n-1})) \\ &\quad + |\gamma_n - \gamma_{n-1}|\, d(s_{n-1}, t_{n-1}) \\ &\leq (1-\alpha_n)d(x_n, x_{n-1}) \\ &\quad + |\alpha_n - \alpha_{n-1}|\,(d(v, Sx_{n-1}) + d(w, Tx_{n-1})) + |\gamma_n - \gamma_{n-1}|\, d(s_{n-1}, t_{n-1}). \end{aligned}$$

Therefore, we get

$$\begin{aligned} &d(x_{n+1}, x_n) \\ &= d(\beta_n r_n \oplus (1-\beta_n)y_n, \beta_{n-1}r_{n-1} \oplus (1-\beta_{n-1})y_{n-1}) \\ &\leq d(\beta_n r_n \oplus (1-\beta_n)y_n, \beta_n r_{n-1} \oplus (1-\beta_n)y_n) \\ &\quad + d(\beta_n r_{n-1} \oplus (1-\beta_n)y_n, \beta_n r_{n-1} \oplus (1-\beta_n)y_{n-1}) \\ &\quad + d(\beta_n r_{n-1} \oplus (1-\beta_n)y_{n-1}, \beta_{n-1}r_{n-1} \oplus (1-\beta_{n-1})y_{n-1}) \\ &\leq \beta_n d(r_n, r_{n-1}) + (1-\beta_n)d(y_n, y_{n-1}) + |\beta_n - \beta_{n-1}|\, d(r_{n-1}, y_{n-1}) \\ &\leq (1-\alpha_n)d(x_n, x_{n-1}) \\ &\quad + |\alpha_n - \alpha_{n-1}|\,(d(u, Rx_{n-1}) + d(v, Sx_{n-1}) + d(w, Tx_{n-1})) \\ &\quad + |\gamma_n - \gamma_{n-1}|\, d(s_{n-1}, t_{n-1}) + |\beta_n - \beta_{n-1}|\, d(r_{n-1}, y_{n-1}). \end{aligned}$$

Using conditions (ii), (iii), (iv), (v), and Lemma 3.1, we have $d(x_{n+1}, x_n) \to 0$. □

Lemma 3.7. *Let $\{x_n\}$ be an iterative scheme on X generated by $(*)$ with the following conditions.*

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(iv) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(v) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Suppose $d(x_{n+1}, x_n) \to 0$. If $\{\beta_n\}$ and $\{\gamma_n\}$ belong to $[a, b] \subset]0, 1[$, then $\{x_n\} \in \tilde{F}(R) \cap \tilde{F}(S) \cap \tilde{F}(T)$.

Proof. From the conditions (iv), (v) and Lemma 3.2, there exist β, γ such that $\beta_n \to \beta$ and $\gamma_n \to \gamma$, and by assumption, $\beta, \gamma \in]0, 1[$. We put $Ux = \beta Rx \oplus (1 - \beta)Qx$ for all $x \in X$, where $Qx = \gamma Sx \oplus (1 - \gamma)Tx$.

We show that $d(Ux_n, x_n) \to 0$. Let $q_n = \alpha_n u \oplus (1 - \alpha_n)Qx_n$. Then, considering a geodesic triangle $\triangle(u, Rx_n, Qx_n)$, we have

$$\begin{aligned} &d(Ux_n, \beta_n r_n \oplus (1 - \beta_n)q_n) \\ &\le \alpha_n\beta_n d(Rx_n, u) + \alpha_n(1 - \beta_n)d(Qx_n, u) + |\beta - \beta_n|\, d(Rx_n, Qx_n). \end{aligned}$$

Since $\{\beta_n\}$ converges to β, by condition (i), we get

$$d(Ux_n, \beta_n r_n \oplus (1 - \beta_n)q_n) \to 0.$$

Put $t'_n = \alpha_n v \oplus (1 - \alpha_n)Tx_n$. Then, using this result, we have

$$\begin{aligned} d(Qx_n, y_n) &= d(\gamma Sx_n \oplus (1 - \gamma)Tx_n, \gamma_n s_n \oplus (1 - \gamma_n)t_n) \\ &\le d(\gamma Sx_n \oplus (1 - \gamma)Tx_n, \gamma_n s_n \oplus (1 - \gamma_n)t'_n) \\ &\quad + d(\gamma_n s_n \oplus (1 - \gamma_n)t'_n, \gamma_n s_n \oplus (1 - \gamma_n)t_n) \\ &\le \alpha_n\gamma_n d(Sx_n, v) + \alpha_n(1 - \gamma_n)d(Tx_n, v) \\ &\quad + |\gamma - \gamma_n|\, d(Sx_n, Tx_n) + (1 - \gamma_n)d(t'_n, t_n). \end{aligned}$$

By the CAT(0) inequality, we get

$$\begin{aligned} d(t'_n, t_n) &= d(\alpha_n v \oplus (1 - \alpha_n)Tx_n, \alpha_n w \oplus (1 - \alpha_n)Tx_n) \\ &\le \alpha_n d(v, w) \to 0. \end{aligned}$$

Since $\gamma_n \to \gamma$, we have

$$d(Qx_n, y_n) \to 0.$$

Therefore, by condition (i), we have

$$\begin{aligned}
d(\beta_n r_n \oplus (1-\beta_n)q_n, x_{n+1}) &= d(\beta_n r_n \oplus (1-\beta_n)q_n, \beta_n r_n \oplus (1-\beta_n)y_n) \\
&\leq (1-\beta_n)d(q_n, y_n) \\
&\leq d(\alpha_n u \oplus (1-\alpha_n)Qx_n, Qx_n) + d(Qx_n, y_n) \\
&= \alpha_n d(u, Qx_n) + d(Qx_n, y_n) \\
&\to 0.
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
&d(Ux_n, x_n) \\
&\leq d(Ux_n, \beta_n r_n \oplus (1-\beta_n)q_n) + d(\beta_n r_n \oplus (1-\beta_n)q_n, x_{n+1}) + d(x_{n+1}, x_n) \\
&\to 0.
\end{aligned}$$

This means that $\{x_n\} \in \tilde{F}(U)$, and by Corollary 2.10, we have $\{x_n\} \in \tilde{F}(R) \cap \tilde{F}(S) \cap \tilde{F}(T)$. □

Using the lemmas above, we show the following theorem.

Theorem 3.8. *Let X be a Hadamard space and $R, S, T : X \to X$ nonexpansive mappings with $F = F(R) \cap F(S) \cap F(T) \neq \emptyset$. Let u, v, w be arbitrary points in X and let $\{x_n\}$ be iteratively generated by $(*)$ for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $]0,1[$ such that*

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

and $\{\beta_n\}, \{\gamma_n\}$ are sequences in $[a, b] \subset]0,1[$ such that

(iv) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(v) $\displaystyle\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty$.

Then $\{x_n\}$ *converges to* $x_0 \in F$ *which is a minimizer of* $g(x)=\beta d(u,x)^2+(1-\beta)(\gamma d(v,x)^2+(1-\gamma)d(w,x)^2)$ *on* F*, where* $\beta=\lim_{n\to\infty}\beta_n$ *and* $\gamma=\lim_{n\to\infty}\gamma_n$.

Proof. From Lemma 3.2, there exist $\beta,\gamma\in\,]0,1[$ such that $\beta_n\to\beta$ and $\gamma_n\to\gamma$. Define a function g on X by $g(x)=\beta d(u,x)^2+(1-\beta)h(x)$ for all $x\in X$, where $h(x)=\gamma d(v,x)^2+(1-\gamma)d(w,x)^2$. From Corollary 2.9, F is closed convex, and so there exists $x_0\in F$ which is the unique minimizer of g on F by Corollary 3.4.

Putting

$$\begin{aligned}
f_n &= \beta_n d(u,x_0)^2+(1-\beta_n)(\gamma_n d(v,x_0)^2+(1-\gamma_n)d(w,x_0)^2),\\
b_n &= \beta_n d(u,Rx_n)^2+(1-\beta_n)(\gamma_n d(v,Sx_n)^2+(1-\gamma_n)d(w,Tx_n)^2),\\
c_n &= f_n-(1-\alpha_n)b_n,
\end{aligned}$$

we show that

$$\limsup_{n\to\infty} c_n \le 0.$$

Since $\beta_n\to\beta$ and $\gamma_n\to\gamma$, we get $|f_n-g(x_0)|\to 0$. Moreover, since $\{d(x_n,x_{n+1})\}$ converges to 0 by Lemma 3.6, $\{x_n\}$ belongs to $\tilde{F}(R)\cap\tilde{F}(S)\cap\tilde{F}(T)$ from Lemma 3.7. Thus, we get $|b_n-g(x_n)|\to 0$. Therefore, we obtain that

$$\begin{aligned}
|c_n-(g(x_0)-g(x_n))| &= |(f_n-(1-\alpha_n)b_n)-(g(x_0)-g(x_n))|\\
&\le |f_n-g(x_0)|+|b_n-g(x_n)|+\alpha_n|b_n|\\
&\to 0,
\end{aligned}$$

and hence

$$\limsup_{n\to\infty} c_n=\limsup_{n\to\infty}(g(x_0)-g(x_n)).$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(g(x_0)-g(x_n))=\lim_{i\to\infty}(g(x_0)-g(x_{n_i})),$$

and $\{x_{n_i}\}$ has a Δ-convergent subsequence by Theorem 2.5. Without loss of generality, we may assume that $\{x_{n_i}\}$ is Δ-convergent to some $x\in X$. From

Theorem 2.4, we have that

$$\begin{aligned}
&\lim_{i\to\infty}(g(x_0) - g(x_{n_i})) \\
&\le g(x_0) \\
&\quad - \beta \liminf_{i\to\infty} d(u, x_{n_i})^2 \\
&\quad - (1-\beta)(\gamma \liminf_{i\to\infty} d(v, x_{n_i})^2 + (1-\gamma)\liminf_{i\to\infty} d(w, x_{n_i})^2) \\
&\le g(x_0) - (\beta d(u,x)^2 + (1-\beta)(\gamma d(v,x)^2 + (1-\gamma) d(w,x)^2)) \\
&= g(x_0) - g(x).
\end{aligned}$$

Since $\{x_n\} \in \tilde{F}(R) \cap \tilde{F}(S) \cap \tilde{F}(T)$, x is an element of F by Theorem 2.11. Moreover, since x_0 is a minimizer of g on F, we have that

$$\limsup_{n\to\infty} c_n \le g(x_0) - g(x) \le 0.$$

Hence, we obtain that

$$\begin{aligned}
&d(x_{n+1}, x_0)^2 \\
&= d(\beta_n r_n \oplus (1-\beta_n) y_n, x_0)^2 \\
&\le \beta_n d(r_n, x_0)^2 + (1-\beta_n) d(\gamma_n s_n \oplus (1-\gamma_n) t_n, x_0)^2 \\
&\le \beta_n d(r_n, x_0)^2 + (1-\beta_n)(\gamma_n d(s_n, x_0)^2 + (1-\gamma_n) d(t_n, x_0)^2) \\
&\le (1-\alpha_n)(\beta_n d(Rx_n, x_0)^2 + (1-\beta_n)(\gamma_n d(Sx_n, x_0)^2 + (1-\gamma_n) d(Tx_n, x_0)^2)) \\
&\quad + \alpha_n f_n - \alpha_n(1-\alpha_n) b_n \\
&\le (1-\alpha_n) d(x_n, x_0)^2 + \alpha_n c_n,
\end{aligned}$$

and by the condition (ii) and Lemma 3.1, $\{x_n\}$ converges to x_0 in F. □

We can extend this theorem to that for a finite family of mappings. Since the technique is the same as Theorem 3.8, we omit the proof.

Theorem 3.9. *Let X be a Hadamard space and $T_1, T_2, \dots, T_r : X \to X$ nonexpansive mappings with $F = \bigcap_{i=1}^r F(T_i) \ne \emptyset$. Let $u_1, u_2, \dots, u_r, x_1$ be arbitrary points in X and let $\{x_n\}$ be iteratively generated by*

$$\begin{cases}
t_n^i = \alpha_n u_i \oplus (1-\alpha_n) T_i x_n, \quad i = 1, 2, \dots, r, \\
y_n^1 = t_n^1, \\
y_n^j = \beta_n^{j-1} t_n^j \oplus (1-\beta_n^{j-1}) y_n^{j-1}, \quad j = 2, 3, \dots, r, \\
x_{n+1} = y_n^r
\end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $]0,1[$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

and, for all $k = 1, 2, \dots, r-1$, $\{\beta_n^k\}$ are sequences in $[a,b] \subset]0,1[$ such that

$$\sum_{n=1}^{\infty} |\beta_{n+1}^k - \beta_n^k| < \infty.$$

Then $\{x_n\}$ converges to a unique minimizer of $g(x) = \sum_{i=1}^{r} \gamma_i d(u_i, x)^2$ on F, where $\gamma_k = \beta^{k-1} \prod_{j=k}^{r-1}(1-\beta^j)$ for $k = 1, 2, \dots, r-1$ and $\gamma_r = \beta^{r-1}$ for $\beta^0 = 1$ and $\beta^i = \lim_{n\to\infty} \beta_n^i$ for $i = 1, 2, \dots, r-1$.

By Theorem 2.3 and Corollary 2.9, for each point there exists a unique nearest fixed point of a nonexpansive mapping. Then, we also get the following result as a corollary of Theorem 3.9.

Corollary 3.10. *Let X be a Hadamard space and $T_1, T_2, \dots, T_r : X \to X$ nonexpansive mappings with $F = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Suppose u, x_1 are arbitrary points in X and $\{x_n\}$ is iteratively generated by*

$$\begin{cases} t_n^i = \alpha_n u \oplus (1-\alpha_n) T_i x_n, \quad i = 1, 2, \dots, r, \\ y_n^1 = t_n^1, \\ y_n^j = \beta_n^{j-1} t_n^j \oplus (1-\beta_n^{j-1}) y_n^{j-1}, \quad j = 2, 3, \dots, r, \\ x_{n+1} = y_n^r \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $]0,1[$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

and, for all $k = 1, 2, \dots, r-1$, $\{\beta_n^k\}$ are sequences in $[a,b] \subset]0,1[$ such that

$$\sum_{n=1}^{\infty} |\beta_{n+1}^k - \beta_n^k| < \infty.$$

Then $\{x_n\}$ converges to a nearest point of F to u.

Proof. Let $x_0 \in F$ be the nearest point to u. Then we have that

$$d(u, x_0)^2 = \inf_{x \in F} d(u, x)^2.$$

From Theorem 3.9, $\{x_n\}$ converges to the minimizer of $g(x) = d(u, x)^2$ on F. Therefore, $\{x_n\}$ is convergent to x_0. $\square$

3.3 Convergence theorems extending a range of coefficients

We assume that $\{\alpha_n\}, \{\beta_n\}$ are sequences in $]0,1[$ and $\{\alpha_n\}$ satisfies the following.

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

We consider a iteration $\{x_n\}$ generated by

$$(**) \begin{cases} x_1 \in X, \\ r_n = \alpha_n u \oplus (1-\alpha_n) R x_n, \\ s_n = \alpha_n v \oplus (1-\alpha_n) S x_n, \\ x_{n+1} = \beta_n r_n \oplus (1-\beta_n) s_n \end{cases}$$

for all $n \geq 1$, where u, v are arbitrary points in a Hadamard space X and R, S are nonexpansive mappings in X with $F(R) \cap F(T) \neq \emptyset$.

Since $s_n = \gamma_n s_n \oplus (1-\gamma_n) s_n$ for any $\{\gamma_n\} \subset]0,1[$, it can be seen that $\{x_n\}$ satisfies the construction $(*)$. Thus, we obtain the following result by Theorem 3.8.

Theorem 3.11. *Let $\{\beta_n\}$ belong to $[a,b] \subset]0,1[$. Then, if $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\{x_n\}$ converges to x_0 which is a minimizer of $g(x) = \beta d(u,x)^2 + (1-\beta) d(v,x)^2$ on $F(R) \cap F(T)$ where $\beta = \lim_{n\to\infty} \beta_n$.*

We consider the limit point x_0 of $\{x_n\}$ where $\{\beta_n\}$ does not belong to $[a,b]$ for any $a, b \in]0,1[$.

Theorem 3.12. *Let $\{\beta_n\}$ is convergent to 1. Then, if $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, $\{x_n\}$ converges to x_0 which is the nearest point of $F(R)$ to u.*

Proof. By assumption, we have

$$\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| \leq \sum_{n=1}^{\infty}(1-\beta_n) + \sum_{n=1}^{\infty}(1-\beta_{n+1}) < \infty,$$

so $\{x_n\}$, $\{Rx_n\}$, and $\{Sx_n\}$ are bounded and $\{d(x_{n+1}, x_n)\}$ converges to 0 by Lemma 3.5 and Lemma 3.6. Letting

$$\begin{aligned} f_n &= \beta_n d(u, x_0)^2 + (1-\beta_n) d(v, x_0)^2, \\ b_n &= \beta_n d(u, Rx_n)^2 + (1-\beta_n) d(v, Sx_n)^2, \\ c_n &= f_n - (1-\alpha_n) b_n, \end{aligned}$$

we show that

$$\limsup_{n\to\infty} c_n \le 0.$$

Since $\beta_n \to 1$, we get

$$|f_n - d(u,x_0)^2| = (1-\beta_n)\left|d(v,x_0)^2 - d(u,x_0)^2\right| \to 0.$$

Further, since $\{Rx_n\}$ is bounded and $d(x_{n+1},x_n) \to 0$, we obtain that

$$\begin{aligned} d(Rx_n,x_n) &\le d(Rx_n,x_{n+1}) + d(x_{n+1},x_n) \\ &\le \beta_n d(Rx_n,r_n) + (1-\beta_n)d(Rx_n,s_n) + d(x_{n+1},x_n) \\ &= \beta_n\alpha_n d(u,Rx_n) + (1-\beta_n)d(Rx_n,s_n) + d(x_{n+1},x_n) \\ &\to 0. \end{aligned}$$

Thus we have $\{x_n\} \in \tilde{F}(R)$, so we get

$$\begin{aligned} &|b_n - d(u,x_n)^2| \\ &\le \beta_n|d(u,Rx_n)^2 - d(u,x_n)^2| + (1-\beta_n)|d(v,Sx_n)^2 - d(u,x_n)^2| \\ &\le \beta_n(d(u,Rx_n) + d(u,x_n))d(Rx_n,x_n) + (1-\beta_n)|d(v,Sx_n)^2 - d(u,x_n)^2| \\ &\to 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |c_n - (d(u,x_0)^2 - d(u,x_n)^2)| &\le \left|f_n - d(u,x_0)^2\right| + \left|b_n - d(u,x_n)^2\right| + \alpha_n|b_n| \\ &\to 0, \end{aligned}$$

and so

$$\limsup_{n\to\infty} c_n = \limsup_{n\to\infty}(d(u,x_0)^2 - d(u,x_n)^2).$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(d(u,x_0)^2 - d(u,x_n)^2) = \lim_{i\to\infty}(d(u,x_0)^2 - d(u,x_{n_i})^2),$$

and $\{x_{n_i}\}$ has a Δ-convergent subsequence by Theorem 2.5. Assume that $\{x_{n_i}\}$ is Δ-convergent to $x \in X$. From Theorem 2.4, we have that

$$\begin{aligned} \lim_{i\to\infty}(d(u,x_0)^2 - d(u,x_{n_i})^2) &= d(u,x_0)^2 - \liminf_{i\to\infty} d(u,x_{n_i})^2 \\ &\le d(u,x_0)^2 - d(u,x)^2. \end{aligned}$$

Since $\{x_n\} \in \tilde{F}(R)$, x belongs to $F(R)$ by Theorem 2.11. Moreover, since x_0 is a nearest point of $F(R)$ to u, we have that

$$\limsup_{n\to\infty} c_n \leq d(u, x_0)^2 - d(u, x)^2 \leq 0.$$

Then, since x_0 is a point of $F(R)$, it follows that

$$\begin{aligned}
&d(x_{n+1}, x_0)^2 \\
&= d(\beta_n r_n \oplus (1-\beta_n)s_n, x_0)^2 \\
&\leq \beta_n d(r_n, x_0)^2 + (1-\beta_n)d(s_n, x_0)^2 \\
&\leq \beta_n(\alpha_n d(u, x_0)^2 + (1-\alpha_n)d(Rx_n, x_0)^2 - \alpha_n(1-\alpha_n)d(u, Rx_n)^2) \\
&\quad + (1-\beta_n)(\alpha_n d(v, x_0)^2 + (1-\alpha_n)d(Sx_n, x_0)^2 - \alpha_n(1-\alpha_n)d(v, Sx_n)^2) \\
&\leq (1-\alpha_n)d(x_n, x_0)^2 + \alpha_n c_n + (1-\beta_n)d(Sx_n, x_0)^2.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, $\{x_n\}$ is convergent to x_0 by Lemma 3.1. □

Similarly, it follows that the result in the case $\{\beta_n\}$ converges to 0. The method of proof is the same, so we omit it.

Theorem 3.13. *Let $\{\beta_n\}$ is convergent to 0. Then, if $\sum_{n=1}^{\infty}\beta_n < \infty$, $\{x_n\}$ converges to x_0 which is the nearest point of $F(S)$ to v.*

Suppose $\beta = \lim_{n\to\infty}\beta_n$. By those results, x_0 changes to the following according to β.

$$x_0 = \begin{cases} \displaystyle\operatorname*{arg\,min}_{x\in F(R)} d(u, x)^2 & (\beta = 1) \\ \displaystyle\operatorname*{arg\,min}_{x\in F(R)\cap F(S)} (\beta d(u, x)^2 + (1-\beta)d(v, x)^2) & (\beta \in]0, 1[) \\ \displaystyle\operatorname*{arg\,min}_{x\in F(S)} d(v, x)^2 & (\beta = 0) \end{cases}$$

Now, we consider a mapping $\beta R \oplus (1-\beta)S$. From Theorem 2.7, $F(\beta R \oplus (1-\beta)S) = F(R) \cap F(S)$ where $\beta \in]0, 1[$. If $\beta = 1$, $F(\beta R \oplus (1-\beta)S) = F(R)$, and if $\beta = 0$, $F(\beta R \oplus (1-\beta)S) = F(S)$. Thus, we can write x_0 as follows:

$$x_0 = \operatorname*{arg\,min}_{x\in F(\beta R\oplus(1-\beta)S)} (\beta d(u, x)^2 + (1-\beta)d(v, x)^2).$$

On the other hand, $\{\beta_n\}$ needs to satisfy the following.

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty}(1-\beta_n) < \infty \quad (\beta = 1) \\ \displaystyle\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty \quad (\beta \in \,]0,1[) \\ \displaystyle\sum_{n=1}^{\infty}\beta_n < \infty \quad (\beta = 0) \end{cases}$$

Then, we show the significant lemma.

Lemma 3.14. *Let $\{\beta_n\}$ be a real sequence with $\sum_{n=1}^{\infty}|\beta_n-\beta| < \infty$ for some $\beta \in \mathbb{R}$. Then, $\{\beta_n\}$ is convergent to β, and satisfies $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$.*

Proof. For any $n \in \mathbb{N}$, we have

$$|\beta_n-\beta| = \sum_{k=1}^{n}|\beta_k-\beta| - \sum_{k=1}^{n-1}|\beta_k-\beta| \to 0. \quad (n \to \infty)$$

Thus $\beta_n \to \beta$. Further,

$$|\beta_{n+1}-\beta_n| \le |\beta_{n+1}-\beta| + |\beta-\beta_n|$$

for any $n \in \mathbb{N}$. Hence, we get the desired results. □

Let $\{\beta_n\}$ satisfy $\sum_{n=1}^{\infty}|\beta_n-\beta| < \infty$ for some $\beta \in [0,1]$. Then, we have

$$\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty.$$

Moreover, if $\lim_{n\to\infty}\beta_n = 1$, since $\beta = \lim_{n\to\infty}\beta_n$, we have

$$\sum_{n=1}^{\infty}(1-\beta_n) = \sum_{n=1}^{\infty}(\beta-\beta_n) = \sum_{n=1}^{\infty}|\beta_n-\beta| < \infty,$$

and if $\lim_{n\to\infty}\beta_n = 0$, we also get

$$\sum_{n=1}^{\infty}\beta_n = \sum_{n=1}^{\infty}(\beta_n-\beta) = \sum_{n=1}^{\infty}|\beta_n-\beta| < \infty.$$

By these facts, we get the convergence theorem with a weaker condition of β.

Theorem 3.15. *Let X be a Hadamard space and $R, S : X \to X$ nonexpansive mappings with a common fixed point. Suppose u, v are arbitrary points in X and we iteratively generate $\{x_n\}$ by $(**)$ for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $]0,1[$ such that*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

and, for some $\beta \in [0,1]$,

$$\sum_{n=1}^{\infty} |\beta_n - \beta| < \infty.$$

Then $\{x_n\}$ converges to a minimizer of $g(x) = \beta d(u,x)^2 + (1-\beta)d(v,x)^2$ on $F(\beta R \oplus (1-\beta)S)$.

Chapter 4

Strong convergence theorems with mixed type

In this chapter, we introduce iterations with the combination of Mann and Halpern type. This iteration is used in Theorem 1.2, and in this result, we assume the mappings are nonexpansive, and either mapping is strongly nonexpansive. In our result, we define a strongly quasinonexpansive mapping instead of strongly nonexpansive, and assume a mapping in the Halpern side is a quasinonexpansive and Δ-demiclosed. Moreover, we extend a Halpern type construction of the iteration, which is the convergence theorem to a common fixed point of triple mappings.

4.1 Strongly quasinonexpansive mappings

We define a new mapping in a metric space X. If a quasinonexpansive mapping T in X satisfies

$$d(x_n, Tx_n) \to 0$$

whenever a bounded sequence $\{x_n\}$ of X and a fixed point z satisfy $d(x_n, z) - d(Tx_n, z) \to 0$, T is said to be strongly quasinonexpansive.

The following is a significant result. See [1].

Lemma 4.1. *Let X be a metric space with a metric d. Suppose $S : X \to X$ is a nonexpansive mapping, $T : X \to X$ is a quasinonexpansive mapping, and $F(S) \cap F(T) \neq \emptyset$. Then, if S or T is strongly quasinonexpansive, $\tilde{F}(S) \cap \tilde{F}(T) = \tilde{F}(ST)$.*

Proof. Let $\{x_n\} \in \tilde{F}(S) \cap \tilde{F}(T)$. Then, we have

$$d(x_n, STx_n) \leq d(x_n, Sx_n) + d(Sx_n, STx_n) \leq d(x_n, Sx_n) + d(x_n, Tx_n) \to 0,$$

and so we get $\{x_n\} \in \tilde{F}(ST)$, that is $\tilde{F}(S) \cap \tilde{F}(T) \subset \tilde{F}(ST)$.

Next, we show $\tilde{F}(S) \cap \tilde{F}(T) \supset \tilde{F}(ST)$. Let $\{x_n\} \in \tilde{F}(ST)$. Assume S is strongly quasinonexpansive. Putting $p \in F(S) \cap F(T)$, we have

$$d(Tx_n, p) \leq d(x_n, p) \leq d(x_n, STx_n) + d(STx_n, p).$$

Then, we get

$$d(Tx_n, p) - d(STx_n, p) \leq d(x_n, STx_n) \to 0,$$

so $d(Tx_n, STx_n) \to 0$. Thus, we get

$$d(x_n, Tx_n) \leq d(x_n, STx_n) + d(STx_n, Tx_n) \to 0,$$

and

$$d(x_n, Sx_n) \leq d(x_n, STx_n) + d(STx_n, Sx_n) \leq d(x_n, STx_n) + d(Tx_n, x_n) \to 0.$$

On the other hand, if T is strongly quasinonexpansive, we have

$$d(x_n, p) \leq d(x_n, STx_n) + d(Tx_n, p),$$

and so

$$d(x_n, p) - d(Tx_n, p) \leq d(x_n, STx_n) \to 0.$$

Thus $\{x_n\} \in \tilde{F}(T)$. Moreover, we have

$$d(x_n, Sx_n) \leq d(x_n, STx_n) + d(Tx_n, x_n) \to 0,$$

that is $\{x_n\} \in \tilde{F}(S)$. Therefore, we get the desired result. □

Lemma 4.2. *Let X be a Hadamard space and $S, T : X \to X$ quasinonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. For any $\beta \in]0, 1[$, define a mapping U by $U = \beta S \oplus (1 - \beta)T$. Then, if S or T is a strongly quasinonexpansive mapping, then so is U.*

Proof. Suppose $\{x_n\} \subset X$ is a bounded sequence such that $d(x_n, z) - d(Ux_n, z) \to 0$ for some $z \in F(U)$. We get $F(U) = F(S) \cap F(T)$ by Theorem 2.7, so we have

$$d(Ux_n, z)^2 \leq d(x_n, z)^2 - \beta(1-\beta)d(Sx_n, Tx_n)^2$$

by Theorem 2.1. Thus, we get

$$\beta(1-\beta)d(Sx_n, Tx_n)^2 \leq d(x_n, z)^2 - d(Ux_n, z)^2 \to 0.$$

Since $\beta \in]0, 1[$, we have $d(Sx_n, Tx_n) \to 0$. If S is strongly quasinonexpanisve, then we get

$$\begin{aligned} d(x_n, z) - d(Sx_n, z) &= (d(x_n, z) - d(Ux_n, z)) + (d(Ux_n, z) - d(Sx_n, z)) \\ &\leq (d(x_n, z) - d(Ux_n, z)) + d(Ux_n, Sx_n) \\ &= (d(x_n, z) - d(Ux_n, z)) + (1-\beta)d(Tx_n, Sx_n) \\ &\to 0. \end{aligned}$$

Therefore we have $d(x_n, Sx_n) \to 0$, and we get

$$d(x_n, Ux_n) \leq d(x_n, Sx_n) + (1-\beta)d(Sx_n, Tx_n) \to 0.$$

If T is strongly quasinonexpansive, we obtain the result using the same technique. □

4.2 Iterations with the combination of Mann and Halpern types

To prove the result, we need the following lemma.

Lemma 4.3 (Saejung-Yotkaew [20]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence in $]0, 1[$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{t_n\}$ a sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n t_n$$

for all $n \geq 1$. If $\limsup_{k\to\infty} t_{n_k} \leq 0$ for every subsequence $\{s_{n_k}\} \subset \{s_n\}$ satisfying $\limsup_{k\to\infty}(s_{n_k} - s_{n_k+1}) \leq 0$, then $\lim_{n\to\infty} s_n = 0$.

Now, we show the main result.

Theorem 4.4. *Let X be a Hadamard space. Assume that $R : X \to X$ is a nonexpansive mapping, $S, T : X \to X$ are quasinonexpansive and Δ-demiclosed mappings, and $F = F(R) \cap F(S) \cap F(T) \neq \emptyset$. Suppose any one of R, S, and T is strongly quasinonexpansive, and u, x_1 are arbitrary points in X. We define an iterative scheme $\{x_n\}$ by*

$$\begin{cases} s_n = \alpha_n u \oplus (1-\alpha_n) S x_n, \\ t_n = \alpha_n u \oplus (1-\alpha_n) T x_n, \\ x_{n+1} = \gamma_n x_n \oplus (1-\gamma_n) R(\beta_n s_n \oplus (1-\beta_n) t_n) \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $]0,1[$ such that

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

and $\{\beta_n\}, \{\gamma_n\}$ belong to $[a,b] \subset]0,1[$. Then, $\{x_n\}$ converges to the nearest point of F to u.

Proof. By Theorem 2.6 and Corollary 2.9, F is closed and convex, and there exists a unique nearest point x_0 of F to u by Theorem 2.3. We put $y_n = \beta_n s_n \oplus (1-\beta_n) t_n$ and $c_n = d(u,x_0)^2 - (1-\alpha_n)(\beta_n d(u,Sx_n)^2 + (1-\beta_n) d(u,Tx_n)^2)$. Then

$$\begin{aligned}
&d(x_{n+1},x_0)^2 \\
&= d(\gamma_n x_n \oplus (1-\gamma_n) R y_n, x_0)^2 \\
&\leq \gamma_n d(x_n,x_0)^2 + (1-\gamma_n) d(y_n,x_0)^2 \\
&\leq \gamma_n d(x_n,x_0)^2 + (1-\gamma_n)(\beta_n d(s_n,x_0)^2 + (1-\beta_n) d(t_n,x_0)^2) \\
&\leq \gamma_n d(x_n,x_0)^2 + (1-\gamma_n)(1-\alpha_n)(\beta_n d(Sx_n,x_0)^2 + (1-\beta_n) d(Tx_n,x_0)^2) \\
&\quad + (1-\gamma_n)\beta_n(\alpha_n d(u,x_0)^2 - \alpha_n(1-\alpha_n) d(u,Tx_n)^2) \\
&\quad + (1-\gamma_n)(1-\beta_n)(\alpha_n d(u,x_0)^2 - \alpha_n(1-\alpha_n) d(u,Sx_n)^2) \\
&\leq (1-(1-\gamma_n)\alpha_n) d(x_n,x_0)^2 + (1-\gamma_n)\alpha_n c_n.
\end{aligned}$$

Thus,

$$\begin{aligned}
d(x_{n+1},x_0)^2 &\leq (1-(1-\gamma_n)\alpha_n) d(x_n,x_0)^2 + (1-\gamma_n)\alpha_n d(u,x_0)^2 \\
&\leq \max\{d(x_n,x_0), d(u,x_0)^2\},
\end{aligned}$$

so we obtain that $\{x_n\}$ is bounded. Since S, T are quasinonexpansive, $\{Sx_n\}$ and $\{Tx_n\}$ are also bounded.

For any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ satisfying $\limsup_{i\to\infty}(d(x_{n_i},x_0)^2 - d(x_{n_i+1},x_0)^2) \leq 0$, we show that

$$\limsup_{i\to\infty} c_{n_i} \leq 0.$$

We first show $d(x_{n_i}, Ry_{n_i}) \to 0$. It follows that

$$\begin{aligned}
&d(y_{n_i},x_0)^2 - d(x_{n_i+1},x_0)^2 \\
&= d(\beta_{n_i}s_{n_i} \oplus (1-\beta_{n_i})t_{n_i}, x_0)^2 - d(x_{n_i+1},x_0)^2 \\
&\leq \beta_{n_i} d(s_{n_i},x_0)^2 + (1-\beta_{n_i})d(t_{n_i},x_0)^2 - d(x_{n_i+1},x_0)^2 \\
&\leq \beta_{n_i}(\alpha_{n_i} d(u,x_0)^2 + (1-\alpha_{n_i})d(Sx_{n_i},x_0)^2) \\
&\quad + (1-\beta_{n_i})(\alpha_{n_i} d(u,x_0)^2 + (1-\alpha_{n_i})d(Tx_{n_i},x_0)^2) - d(x_{n_i+1},x_0)^2 \\
&\leq \alpha_{n_i}(d(u,x_0)^2 - d(x_{n_i},x_0)^2) + d(x_{n_i},x_0)^2 - d(x_{n_i+1},x_0)^2.
\end{aligned}$$

We also have

$$\begin{aligned}
&d(x_{n_i+1},x_0)^2 \\
&= d(\gamma_{n_i}x_{n_i} \oplus (1-\gamma_{n_i})Ry_{n_i}, x_0)^2 \\
&\leq \gamma_{n_i} d(x_{n_i},x_0)^2 + (1-\gamma_{n_i})d(Ry_{n_i},x_0)^2 - \gamma_{n_i}(1-\gamma_{n_i})d(x_{n_i},Ry_{n_i})^2 \\
&\leq \gamma_{n_i} d(x_{n_i},x_0)^2 + (1-\gamma_{n_i})d(y_{n_i},x_0)^2 - \gamma_{n_i}(1-\gamma_{n_i})d(x_{n_i},Ry_{n_i})^2.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\gamma_{n_i}(1-\gamma_{n_i})d(x_{n_i},Ry_{n_i})^2 \\
&\leq \gamma_{n_i}(d(x_{n_i},x_0)^2 - d(x_{n_i+1},x_0)^2) + (1-\gamma_{n_i})(d(y_{n_i},x_0)^2 - d(x_{n_i+1},x_0)^2) \\
&\leq \gamma_{n_i}(d(x_{n_i},x_0)^2 - d(x_{n_i+1},x_0)^2) \\
&\quad + (1-\gamma_{n_i})(\alpha_{n_i}(d(u,x_0)^2 - d(x_{n_i},x_0)^2) + (d(x_{n_i},x_0)^2 - d(x_{n_i+1},x_0)^2)) \\
&= d(x_{n_i},x_0)^2 - d(x_{n_i+1},x_0)^2 + \alpha_{n_i}(1-\gamma_{n_i})(d(u,x_0)^2 - d(x_{n_i},x_0)^2),
\end{aligned}$$

and, since $\alpha_n \to 0$, we have

$$\limsup_{i\to\infty} \gamma_{n_i}(1-\gamma_{n_i})d(x_{n_i},Ry_{n_i})^2 \leq \limsup_{i\to\infty}(d(x_{n_i},x_0)^2 - d(x_{n_i+1},x_0)^2) \leq 0.$$

Since $\{\gamma_n\}$ belongs to $[a,b] \subset]0,1[$, we obtain that

$$\limsup_{i\to\infty} d(x_{n_i},Ry_{n_i})^2 \leq 0,$$

and so we have $d(x_{n_i}, Ry_{n_i}) \to 0$.

By the definition of superior limit, there exists natural numbers $\{m_j\} \subset \{n_i\}$ such that

$$\lim_{j\to\infty} c_{m_j} = \limsup_{i\to\infty} c_{n_i}.$$

Then, there exists a convergent subsequence of $\{\beta_{m_j}\}$. Without loss of generality, we may suppose that $\{\beta_{m_j}\}$ is convergent to β in $]0,1[$. Let $U = \beta S \oplus (1-\beta)T$. Considering a geodesic triangle $\triangle(u, Sx_{m_j}, Tx_{m_j})$, we get

$$\begin{aligned} &d(y_{m_j}, Ux_{m_j}) \\ &\leq \alpha_{m_j}(\beta_{m_j} d(u, Sx_{m_j}) + (1-\beta_{m_j})d(u, Tx_{m_j})) + \left|\beta_{m_j} - \beta\right| d(Sx_{m_j}, Tx_{m_j}). \end{aligned}$$

Since $\alpha_{m_j} \to 0$ and $\beta_{m_j} \to \beta$, we have $d(y_{m_j}, Ux_{m_j}) \to 0$. Therefore, we obtain that

$$\begin{aligned} d(x_{m_j}, RUx_{m_j}) &\leq d(x_{m_j}, Ry_{m_j}) + d(Ry_{m_j}, RUx_{m_j}) \\ &\leq d(x_{m_j}, Ry_{m_j}) + d(y_{m_j}, Ux_{m_j}) \\ &\to 0, \end{aligned}$$

that is, $\{x_{m_j}\} \in \tilde{F}(RU)$. By Lemma 2.7, U is a quasinonexpansive mapping with $F(R) \cap F(U) = F \neq \emptyset$. Moreover, if S or T is strongly quasinonexpansive, U becomes a strongly quasinonexpansive mapping from Lemma 4.2. In the other case, R is strongly quasinonexpansive. Hence, by Lemma 4.1, we get $\tilde{F}(RU) = \tilde{F}(R) \cap \tilde{F}(U)$. Furthermore, from Theorem 2.8, $\tilde{F}(U) = \tilde{F}(S) \cap \tilde{F}(T)$. Thus we have

$$\{x_{m_j}\} \in \tilde{F}(RU) = \tilde{F}(R) \cap \tilde{F}(U) = \tilde{F}(R) \cap \tilde{F}(S) \cap \tilde{F}(T).$$

Since $\{x_{m_j}\}, \{Sx_{m_j}\}$ and $\{Tx_{m_j}\}$ are bounded and $\{x_{m_j}\} \in \tilde{F}(S) \cap \tilde{F}(T)$, we have

$$\begin{aligned} &\left|d(u, x_{m_j})^2 - (\beta_{m_j}(d(u, Sx_{m_j})^2 + (1-\beta_{m_j})d(u, Tx_{m_j})^2)\right| \\ &\leq \beta_{m_j}(d(u, x_{m_j}) + d(u, Sx_{m_j}))d(x_{m_j}, Sx_{m_j}) \\ &\quad + (1-\beta_{m_j})(d(u, x_{m_j}) + d(u, Tx_{m_j}))d(x_{m_j}, Tx_{m_j}) \\ &\to 0. \end{aligned}$$

Thus we obtain that

$$\begin{aligned}
&\left|c_{m_j} - (d(u,x_0)^2 - d(u,x_{m_j})^2)\right| \\
&\le \left|d(u,x_{m_j})^2 - (\beta_{m_j}(d(u,Sx_{m_j})^2 + (1-\beta_{m_j})d(u,Tx_{m_j})^2)\right| \\
&\quad + \alpha_{m_j}\left|\beta_{m_j}(d(u,Sx_{m_j})^2 + (1-\beta_{m_j})d(u,Tx_{m_j})^2\right| \\
&\to 0,
\end{aligned}$$

and hence

$$\lim_{j\to\infty} c_{m_j} = \limsup_{j\to\infty}(d(u,x_0)^2 - d(u,x_{m_j})^2).$$

There exists a Δ-converget subsequence of $\{x_{m_j}\}$ by Theorem 2.5. Suppose $\{x_{m_j}\}$ is Δ-convergence to some x in X. From Theorem 2.4, we get

$$\begin{aligned}
\limsup_{i\to\infty} c_{n_i} &= \lim_{j\to\infty} c_{m_j} \\
&= \limsup_{j\to\infty}(d(u,x_0)^2 - d(u,x_{m_j})^2) \\
&= d(u,x_0)^2 - \liminf_{j\to\infty} d(u,x_{m_j})^2 \\
&\le d(u,x_0)^2 - d(u,x)^2
\end{aligned}$$

Since $\{x_{m_j}\} \in \tilde{F}(R) \cap \tilde{F}(S) \cap \tilde{F}(T)$ and each mapping is Δ-demiclosed by Theorem 2.11, x is an element of F. Moreover, since x_0 is the nearest point of F to u, we have that

$$\limsup_{i\to\infty} c_{n_i} \le d(u,x_0)^2 - d(u,x)^2 \le 0.$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, we get $d(x_{n+1},x_0) \to 0$ by Lemma 4.3, so we have $x_n \to x_0$. $\square$

Chapter 5

Conclusion

In this chapter, we summarize the results proved in this thesis. Our main theorems are Theorem 3.9, Theorem 3.15, and Theorem 4.4.

Since Hadamard spaces contain Hilbert spaces, we can show a convergence theorem generated by a Halpern type iteration with multiple anchor points in a Hilbert space by Theorem 3.9. However, in Hilbert spaces, we can write a limit point of the iteration without a function g. We show the convergence theorem generated by a iteration with double anchor points in a Hilbert space.

Corollary 5.1. *Let H be a Hilbert space and R, S nonexpansive mappings in H with $F = F(R) \cap F(S) \neq \emptyset$. Suppose u, v, x_1 are arbitrary points in H and we define a iteration $\{x_n\}$ by*

$$x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n)Rx_n) + (1-\beta_n)(\alpha_n v + (1-\alpha_n)Sx_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $]0,1[$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

and $\{\beta_n\}$ is sequence in $[a,b] \subset]0,1[$ such that

$$\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Then, $\{x_n\}$ converges to a nearest point of F to $\beta u + (1-\beta)v$.

Proof. By Theorem 3.9, $\{x_n\}$ is convergent to x_0 which is a minimizer of $g(x) = \beta \|u - x\|^2 + (1-\beta) \|v - x\|^2$ on F, where $\beta = \lim_{n\to\infty} \beta_n$. Then, for any $y \in H$, we have

$$\begin{aligned}
&\|(\beta u + (1-\beta)v) - x_0\|^2 \\
&= \beta \|u - x_0\|^2 + (1-\beta) \|v - x_0\|^2 - \beta(1-\beta) \|u - v\|^2 \\
&\leq \beta \|u - y\|^2 + (1-\beta) \|v - y\|^2 - \beta(1-\beta) \|u - v\|^2 \\
&= \|(\beta u + (1-\beta)v) - y\|^2 .
\end{aligned}$$

Thus, x_0 is a nearest point of F to $\beta u + (1-\beta)v$. □

For this corollary, it is a characteristic result of Hadamard spaces that the limit point is written with the function g, this is exactly a result of Theorem 3.9.

Considering Theorem 1.1 and Corollary 5.1, we know that a coefficient sequence $\{\beta_n\}$ must be contained in $[a, b] \subset]0, 1[$. Considering an iteration in Theorem 1.1 in any spaces, we expect $\{\beta_n\}$ need to belong to $[a, b]$. If we apply a technique of Theorem 3.15, the range of $\{\beta_n\}$ of each result will be extended.

Next, we show that Theorem 4.4 is a generalization of Theorem 1.2. Since Hadamard spaces are generalization of Hilbert spaces, the underlying space is extended. Moreover, nonexpansive mappings with a fixed point are quasinonexpansive and Δ-demiclosed, and strongly nonexpanisive mappings with a fixed point are strongly quasinonexpanisve. Thus, mappings in Theorem 4.4 extend those of Theorem 1.2. Therefore, Theorem 4.4 is a generalization of Theorem 1.2. Furthermore, the following corollary of Theorem 4.4 is a new result even in Hilbert space.

Corollary 5.2. *Let H be a Hilbert space. Suppose $S : H \to H$ is a nonexpansive mapping, $T : H \to H$ is a quasinonexpansive mapping and, for identity mapping I, $I - T$ is demiclosed at* 0*, and $F(S) \cap F(T) \neq \emptyset$. Moreover, we assume that S or T is a strongly quasinonexpansive mapping. Let $u, x_1 \in H$ and $\{x_n\}$ is iteratively generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S(\alpha_n u + (1 - \alpha_n) T x_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \subset]0, 1[$ satisfies $\alpha_n \to 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{\beta_n\} \subset [a, b] \subset]0, 1[$. Then, $\{x_n\}$ converges to the nearest point of $F(S) \cap F(T)$ to u.

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