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Equilibrium problems and convex functions on geodesic spaces with curvature bounded above

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Chapter 1

Introduction

Equilibrium problem is a crucial topic for nonlinear analysis. An equilibrium problem for a bifunction $f\colon K \times K \to \mathbb{R}$ is defined by to find a point $z \in K$ such that for all $y \in K$, an inequality $f(z,y) \geq 0$ holds. This problem was proposed by Blum and Oettli [1] in 1994, and this includes various crucial nonlinear problems such as convex minimization problem, fixed point problem, minimax problem, variational inequality problem, saddle point problem, Nash equilibria, and so on.

Various researchers have studied the equilibrium problem on complete CAT(κ) spaces. A CAT(κ) space is a metric space which has a unique geodesic for each pair of two points and has a curvature bounded above by κ. We know that the class of complete CAT(κ) spaces includes the class of Hilbert spaces.

In 2005, Combettes and Hirstoaga [3] studied the equilibrium problem on Hilbert spaces. They found that an operator R_f defined by f, which is called a *resolvent* operator, plays an important role to solve the equilibrium problem. They show that the set of all solutions to the equilibrium problem for f coincides with the set of all fixed points of R_f. Therefore, that equilibrium problem can be reduces to the fixed point problem. Note that they consider that f satisfies the following basic conditions to well-define a resolvent R_f.

(E1) $f(z,z) = 0$ for all $z \in K$;
(E2) $f(z,y) + f(y,z) \leq 0$ for all $z, y \in K$;
(E3) $f(z,\cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex for all $z \in K$;
(E4$^+$) $f(\cdot,y)\colon K \to \mathbb{R}$ is upper hemicontinuous for all $y \in K$.

We often call the mapping which serves to reduce a problem to a fixed point problem a resolvent of that problem.

In recent years, some researchers showed that resolvent operators for the equilibrium problem can be defined in complete CAT(0) spaces [12], complete CAT(1) spaces [11], and complete CAT(-1) spaces [20]. For instance, Kimura and Kishi [12] define a resolvent R_f by a formula

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(f(z,y) + \frac{1}{2} d(x,y)^2 - \frac{1}{2} d(x,z)^2 \right) \geq 0 \right\},$$

where d is a metric. This formula has the form of adding a perturbation $d^2/2$ to the set of all solutions to the equilibrium problem for f:

$$\operatorname{Equil} f = \left\{ z \in K \;\middle|\; \inf_{y \in K} f(z,y) \geq 0 \right\} = \left\{ z \in K \;\middle|\; \text{for all } y \in K,\ f(z,y) \geq 0 \right\}.$$

They also show the following result.

Theorem 1.1 (Kimura and Kishi [12]). *Let X be a complete* CAT(0) *space and suppose that X has the convex hull finite property. Let K be a nonempty closed convex subset of X, and $f: K \times K \to \mathbb{R}$ be a bifunction satisfying* (E1)–(E4$^+$). *Define a resolvent $R_f: X \to K$ by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(f(z,y) + \frac{1}{2} d(x,y)^2 - \frac{1}{2} d(x,z)^2 \right) \geq 0 \right\}$$

for $x \in X$. Then R_f is well-defined as a single-valued mapping. Moreover, $F(R_f) = \operatorname{Equil} f$, where $F(R_f)$ stands for the set of all fixed points of R_f.

In 2021, Kimura [11] showed that a resolvent S_f defined by

$$S_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(f(z,y) - \log\cos d(x,y) + \log\cos d(x,z) \right) \geq 0 \right\}$$

is well-defined as a single-valued mapping, and this satisfies $F(S_f) = \operatorname{Equil} f$ on admissible complete CAT(1) spaces. They also proved a Δ-convergence theorem as follows.

Theorem 1.2 (Kimura [11]). *Let X be an admissible complete* CAT(1) *space, and suppose that X has the convex hull finite property and $\sup_{u,v \in X} d(u,v) < \pi/2$. Let K be a nonempty closed convex subset of X, and $f: K \times K \to \mathbb{R}$ satisfies* (E1)–(E4) *as follows:*

(E1) $f(z,z) = 0$ *for all $z \in K$;*
(E2) $f(z,y) + f(y,z) \leq 0$ *for all $z, y \in K$;*
(E3) $f(z,\cdot): K \to \mathbb{R}$ *is lower semicontinuous and convex for all $z \in K$;*
(E4) $\limsup_{t \searrow 0} f(ty \oplus (1-t)z, y) \leq f(z,y)$ *for all $z, y \in K$.*

Let x be an arbitrary point on X and S_f a resolvent defined in above. Then a sequence $\{S_f^n x\}$ Δ-converges to some element in $\operatorname{Equil} f$.

Later, Kimura and Ogihara [20] proved a well-definedness of a resolvent T_f defined by

$$T_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(f(z,y) + \cosh d(x,y) - \cosh d(x,z) \right) \geq 0 \right\}$$

on complete CAT(-1) spaces in 2023.

Resolvents R_f, S_f and T_f use perturbation functions $d^2/2$, $-\log(\cos d)$ and $\cosh d$, respectively. We know that we cannot use a perturbation $-\log(\cos d)$ to define a resolvent on CAT(-1) spaces generally. Similarly, we cannot use $\cosh d$ as a perturbation function to define a resolvent on CAT(1) spaces. This means that available perturbations depend on the curvature of the space.

In admissible complete CAT(1) spaces, we know that we can use perturbations other than $-\log(\cos d)$, such as $\tan d \sin d$, which will be shown in this thesis. By choosing an appropriate perturbation Φ, we can $R_f: X \to 2^K$ defined by $R_f x = \{z \in K \mid \inf_{y \in K} (f(z,y) + \Phi(d(x,y)) - \Phi(d(x,z))) \geq 0\}$ for each $x \in X$ to be a single-valued mapping from X into K, and $F(R_f)$ can be identical to $\operatorname{Equil} f$.

In this thesis, we show that we can use another perturbations such as $\tan d \sin d$ in admissible complete CAT(1) spaces, $\tanh d \sinh d$ and $\log\cosh d$ in complete CAT(-1) spaces, and others. We also give sufficient conditions for perturbations to well-define the resolvent in general admissible complete CAT(κ) spaces.

We know that not all resolvents have exactly the same properties. In fact, the properties of resolvents depend on perturbations. For instance, a resolvent defined by using a perturbation $-\log(\cos d)$ has a property named *spherically nonspreading of sum type*, and a resolvent defined by using a perturbation $\tan d \sin d$ has a property named *firmly spherically nonspreading*; these two nonspreadingness are independent. This implies that there exists a different behavior of an approximation sequence to a fixed point of a resolvent for each perturbation. In this thesis, we prove an approximation theorem of a solution to the equilibrium problem using a resolvent defined by generalized perturbations by focusing on the characteristics that resolvents have in common. We also consider a resolvent of convex functions.

In addition, we consider a special type of convex functions.

Chapter 2

Preliminaries

Let X be a nonempty set and T a mapping from X into itself. Then $F(T)$ denotes a set of all fixed points of T. Let f be a real function on a set X. Then we define an epigraph of f by $\operatorname{epi} f = \{(x, y) \in X \times \mathbb{R} \mid y \geq f(x)\}$. We write $\operatorname{argmax}_{x \in X} f(x)$ for the set of all maximizers of f. Similarly, $\operatorname{argmin}_{x \in X} f(x)$ stands for the set of all minimizers of f. In addition, if a maximizer (resp. minimizer) of f is unique, then $\operatorname{argmax}_{x \in X} f(x)$ (resp. $\operatorname{argmin}_{x \in X} f(x)$) directly denotes such a unique maximizer (resp. minimizer). Two sets $\operatorname{argmax}_{x \in X} f(x)$ and $\operatorname{argmin}_{x \in X} f(x)$ are often abbreviated to $\operatorname{argmax} f$ and $\operatorname{argmin} f$, respectively.

Let X, Y be nonempty sets and A a subset of X. For a function g from X into Y, we write $g|_A \colon A \to Y$ for the restriction of g to A.

Let (X, d) be a metric space and f a mapping from X into $]-\infty, \infty]$. Let us denote an effective domain of f by $\operatorname{dom}(f) = \{x \in X \mid f(x) \neq \infty\}$. A function f is said to be *proper* if $\operatorname{dom}(f) \neq \emptyset$.

Let (X, d) be a metric space and T a mapping from X into itself. Then T is said to be *asymptotically regular* if $\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0$ for every $x \in X$. We say that T is *quasinonexpansive* if $F(T) \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for any $x \in X$ and $p \in F(T)$.

Let (X, d) be a metric space and $\{x_n\}$ a bounded sequence on X. A point $z \in X$ is called an *asymptotic center* of $\{x_n\}$ if z is a minimizer of a function $\limsup_{n\to\infty} d(x_n, \cdot)$ on X. $\{x_n\}$ is said to Δ-converge to a point $z \in X$ if z is the unique asymptotic center of any subsequence of $\{x_n\}$. We call such a point z a Δ*-limit* of $\{x_n\}$. A mapping $T \colon X \to X$ is said to be Δ*-demiclosed* if a Δ-limit of any Δ-convergent sequences $\{x_n\}$ on X satisfying $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ belongs to $F(T)$.

2.1 Geodesic spaces

Let (X, d) be a metric space. For two points $x, y \in X$, a mapping $\gamma_{x,y} \colon [0,1] \to X$ is called a *geodesic* joining x and y if $\gamma_{x,y}(0) = y$, $\gamma_{x,y}(1) = x$, and $d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = |s-t| d(x, y)$ hold for any $s, t \in [0,1]$. For $D \in]0, \infty]$, a metric space (X, d) is called a *uniquely* D*-geodesic space* if for any two points $x, y \in X$ with $d(x, y) < D$, there exists a unique geodesic joining x and y. In particular, a uniquely ∞-geodesic space is simply called a *uniquely geodesic space*.

Let (X, d) be a uniquely D-geodesic space. Let $x, y \in X$ such that $d(x, y) < D$ and $\gamma_{x,y}$ a unique geodesic joining x and y. Then we write a point $\gamma_{x,y}(t)$ by $tx \oplus (1-t)y$ for every $t \in [0,1]$. We call this point a *convex combination* of x and y. It follows that $d(x, tx \oplus (1-t)y) = (1-t)d(x, y)$ and $d(y, tx \oplus (1-t)y) = td(x, y)$. Furthermore, we can show that $tx \oplus (1-t)y = (1-t)y \oplus tx$ for all $t \in [0,1]$. Put $[x, y] = [y, x] = \gamma_{x,y}([0,1]) = \{tx \oplus (1-t)y \mid t \in [0,1]\}$. We call it a *geodesic segment* joining x and y. Moreover, define partial segments $]x, y[$, $[x, y[$, and $]x, y]$ by $\gamma_{x,y}(]0,1[)$, $\gamma_{x,y}(]0,1])$, and $\gamma_{x,y}([0,1[)$, respectively.

Let X be a uniquely D-geodesic space and C a subset of X such that $d(u, v) < D$ for every

$u, v \in C$. Then C is said to be *convex* if $[x, y] \subset C$ for every $x, y \in C$. It is equivalent to $tx \oplus (1-t)y \in C$ for every $x, y \in C$ and $t \in]0,1[$.

Let X be a uniquely D-geodesic space and C a subset of X such that $d(u, v) < D$ for every $u, v \in C$. Then a convex hull of C, which is written by $\operatorname{co} C$, is defined by $\bigcup_{n=1}^{\infty} C_n$, where $C_1 = C$ and $C_{n+1} = \{tx \oplus (1-t)y \mid x, y \in C_n, t \in [0,1]\}$ for every $n \in \mathbb{N}$. We know that $\operatorname{co} C$ is convex. Moreover, $\operatorname{cl} C$ denotes a closure of C.

Let X be a uniquely D-geodesic space and f a function from X into $]-\infty, \infty]$. Then f is said to be *convex* if for any $x, y \in X$ such that $d(x, y) < D$ and $t \in]0,1[$, $f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$ holds. Note that the inequality $f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$ always holds if $x \notin \operatorname{dom}(f)$ or $y \notin \operatorname{dom}(f)$. f is said to be *upper hemicontinuous* if $\limsup_{t \searrow 0} f(tx \oplus (1-t)y) \leq f(y)$ for any $x, y \in X$ such that $d(x, y) < D$.

Let M_κ be a 2-dimensional model space with a metric ρ and a constant curvature $\kappa \in \mathbb{R}$ defined by

$$M_\kappa = \begin{cases} \dfrac{1}{\sqrt{\kappa}} \mathbb{S}^2 & (\text{if } \kappa > 0); \\ \mathbb{R}^2 & (\text{if } \kappa = 0); \\ \dfrac{1}{\sqrt{-\kappa}} \mathbb{H}^2 & (\text{if } \kappa < 0), \end{cases}$$

where $\mathbb{S}^2$ is the 2-dimensional unit sphere, $\mathbb{R}^2$ is the 2-dimensional Euclidean space, and $\mathbb{H}^2$ is the 2-dimensional hyperbolic space. Let us denote a diameter of M_κ by D_κ, which coincides with

$$D_\kappa = \begin{cases} \infty & (\text{if } \kappa \leq 0); \\ \dfrac{\pi}{\sqrt{\kappa}} & (\text{if } \kappa > 0). \end{cases}$$

Then M_κ is a complete uniquely D_κ-geodesic space.

Let X be a metric space and f a function from X into $]-\infty, \infty]$. For $D \in]0, \infty]$, f is said to be *D-coercive* if $f(y) \to \infty$ whenever $d(x, y) \nearrow D$ for some $x \in X$. We call a ∞-coercive function simply a *coercive* function.

Let $\kappa \in \mathbb{R}$ and X a uniquely D_κ-geodesic space. For $x, y, z \in X$ such that $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, define a *geodesic triangle* with vertices x, y and z by $\triangle(x, y, z) = [x, y] \cup [y, z] \cup [z, x]$. Then for each geodesic triangle $\triangle(x, y, z)$ on X, there exists $\overline{x}, \overline{y}, \overline{z} \in M_\kappa$ such that $d(x, y) = \rho(\overline{x}, \overline{y})$, $d(y, z) = \rho(\overline{y}, \overline{z})$, and $d(z, x) = \rho(\overline{z}, \overline{x})$. Thus we define a *comparison triangle* $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$ of $\triangle(x, y, z)$ by $[\overline{x}, \overline{y}] \cup [\overline{y}, \overline{z}] \cup [\overline{z}, \overline{x}]$. For an arbitrary point $p \in \triangle(x, y, z)$, there exists a corresponding point $\overline{p} \in \overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$ to p such that the distances from two adjacent vertices are identical. We call such a point $\overline{p}$ a *comparison point* of p.

For $\kappa \in \mathbb{R}$, let (X, d) be a uniquely D_κ-geodesic space and (M_κ, ρ) a model space. We call X a CAT(κ) space if for any $\triangle := \triangle(x, y, z)$, $\overline{\triangle} := \overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$, and for any two points $p, q \in \triangle$ and these comparison points $\overline{p}, \overline{q} \in \overline{\triangle}$, an inequality $d(p, q) \leq \rho(\overline{p}, \overline{q})$ always holds. The inequality $d(p, q) \leq \rho(\overline{p}, \overline{q})$ is called a CAT(κ) inequality. We remark that every CAT(κ) space is also a CAT(κ') space if $\kappa < \kappa'$, see [2].

A Hilbert space is an example of the complete CAT(0) space, and therefore is a complete CAT(κ) space for any $\kappa \geq 0$. It yields that the class of the complete CAT(0) spaces includes Hilbert spaces, but not Banach spaces in general. Moreover, a model space M_κ is a complete CAT(κ) space.

A CAT(κ) space X is said to be *admissible* if $d(x, y) < D_\kappa/2$ for every $x, y \in X$. The admissibility of CAT(κ) spaces only makes sense when $\kappa > 0$, because $D_\kappa = \infty$ for all $\kappa \leq 0$.

Lemma 2.1 (Mayer [24])**.** *Let X be a complete* CAT(0) *space and f a lower semicontinuous convex function from X into $\mathbb{R}$. Then there exists $L \in]-\infty, 0]$ such that for any $u \in X$,*

$$\liminf_{d(u,z)\to\infty} \frac{f(z)}{d(u,z)} \geq L.$$

Lemma 2.2 (Kimura and Kohsaka [13])**.** *For $\kappa > 0$, let X be an admissible complete* CAT(κ) *space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Then f is bounded below.*

A sequence $\{x_n\}$ on a CAT(κ) space X is said to be *κ-bounded* if an inequality

$$\limsup_{n\to\infty} d(x_n, u) < \frac{D_\kappa}{2}$$

holds for some $u \in X$.

Let X be a complete CAT(κ) space. If $\{x_n\} \subset X$ is κ-bounded, then an asymptotic center of $\{x_n\}$ is always unique, see [4, 5]. In this thesis, $AC(\{x_n\})$ denotes the unique asymptotic center of a κ-bounded sequence $\{x_n\}$. Moreover, the Δ-limit of any Δ-convergent sequence $\{x_n\}$ on X is also unique, and therefore we write $\Delta\text{-}\lim_{n\to\infty} x_n$ for such a point. We also use a notation $x_n \overset{\Delta}{\rightharpoonup} x$ if $\{x_n\}$ Δ-converges to x. In addition, if $\{x_n\}$ is convergent, then we get $\lim_{n\to\infty} x_n = \Delta\text{-}\lim_{n\to\infty} x_n$.

Theorem 2.3 ([4, 5])**.** *Let X be a complete* CAT(κ) *space and $\{x_n\}$ a κ-bounded sequence on X. Then there exists a Δ-convergent subsequence of $\{x_n\}$.*

Let X be a complete CAT(κ) space. A subset $C \subset X$ is said to be Δ-*compact* if every sequence $\{x_n\}$ on C has a Δ-convergent subsequence to a point in C. A subset $C \subset X$ is said to be Δ-*closed* if a Δ-limit of every Δ-convergent sequence on C belongs to C.

Lemma 2.4 (Kirk and Panyanak [21])**.** *Let X be a complete* CAT(0) *space and M a bounded closed convex subset of X. Then M is Δ-compact.*

Lemma 2.5 (Kirk and Panyanak [21])**.** *Let X be a complete* CAT(0) *space and M a closed convex subset of X. Then M is Δ-closed.*

For the sake of completeness, we give the proof of Lemma 2.5 at Section 2.3.

Lemma 2.6 (He, Fang, Lopez and Li [6])**.** *Let X be a complete* CAT(κ) *space and $\{x_n\}$ a κ-bounded sequence on X such that $x_n \overset{\Delta}{\rightharpoonup} z \in X$. Then for any $u \in X$ with $\limsup_{n\to\infty} d(u, x_n) < D_\kappa/2$,*

$$d(u,z) \leq \liminf_{n\to\infty} d(u, x_n).$$

Corollary 2.7. *Let X be an admissible complete* CAT(κ) *space and $\{x_n\}$ a κ-bounded sequence on X such that $x_n \overset{\Delta}{\rightharpoonup} z \in X$. Let $u \in X$ and suppose that there exists a limit $\lim_{n\to\infty} d(u, x_n)$. Then*

$$d(u,z) \leq \lim_{n\to\infty} d(u, x_n).$$

Proof. If $\lim_{n\to\infty} d(u, x_n) < D_\kappa/2$, then we have the conclusion from Lemma 2.6. We may consider the case where $\lim_{n\to\infty} d(u, x_n) = D_\kappa/2$. Since $\{x_n\}$ is κ-bounded, there exists $p \in X$ such that $\sup_{n\in\mathbb{N}} d(x_n, p) < D_\kappa/2$. Assume that $\kappa \leq 0$. Then $d(u, x_n) \to \infty$ and therefore $d(x_n, p) \geq |d(u, x_n) - d(u, p)| \to \infty$, which is a contradiction. Hence we have $\kappa > 0$. It follows from the admissibility of X that $d(u, z) < D_\kappa/2 = \lim_{n\to\infty} d(u, x_n)$. This is the desired result. □

A complete CAT(κ) space X is said to have the *convex hull finite property* if for any nonempty finite subset E of X and every continuous mapping T from $\operatorname{cl}\operatorname{co} E$ into itself, T has a fixed point. The convex hull finite property is defined by [28] for CAT(0) spaces originally. Notice that all Hilbert spaces have the convex hull finite property.

Let X be a CAT(κ) space and let $t \in [0,1]$. Then, the following inequalities hold for any $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$:

- If $\kappa < 0$,

$$\begin{aligned}&\cosh(\sqrt{-\kappa}\, d(tx \oplus (1-t)y, z)) \sinh(\sqrt{-\kappa} D)\\ &\quad \le \cosh(\sqrt{-\kappa}\, d(x,z)) \sinh(t\sqrt{-\kappa} D) + \cosh(\sqrt{-\kappa}\, d(y,z)) \sinh((1-t)\sqrt{-\kappa} D);\end{aligned}$$

- if $\kappa = 0$,

$$d(tx \oplus (1-t)y, z)^2 \le t d(x,z)^2 + (1-t) d(y,z)^2 - t(1-t) d(x,y)^2;$$

- if $\kappa > 0$,

$$\begin{aligned}&\cos(\sqrt{\kappa}\, d(tx \oplus (1-t)y, z)) \sin(\sqrt{\kappa} D)\\ &\quad \ge \cos(\sqrt{\kappa}\, d(x,z)) \sin(t\sqrt{\kappa} D) + \cos(\sqrt{\kappa}\, d(y,z)) \sin((1-t)\sqrt{\kappa} D),\end{aligned}$$

where $D = d(x,y)$. These hold as an equation if CAT(κ) space X is just a model space M_κ. Therefore, we call these inequalities *Stewart's theorem* on CAT(κ) spaces.

The following are easily obtained by Stewart's theorem on a CAT(κ) space X:

- If $\kappa < 0$,

$$\cosh(\sqrt{-\kappa}\, d(tx \oplus (1-t)y, z)) \le t \cosh(\sqrt{-\kappa}\, d(x,z)) + (1-t) \cosh(\sqrt{-\kappa}\, d(y,z));$$

- if $\kappa = 0$,

$$d(tx \oplus (1-t)y, z)^2 \le t d(x,z)^2 + (1-t) d(y,z)^2;$$

- if $\kappa > 0$,

$$\cos(\sqrt{\kappa}\, d(tx \oplus (1-t)y, z)) \ge t \cos(\sqrt{\kappa}\, d(x,z)) + (1-t) \cos(\sqrt{\kappa}\, d(y,z))$$

for any $t \in [0,1]$ and $x, y, z \in X$ with $d(x,z) < D_\kappa/2$, $d(y,z) < D_\kappa/2$, and $d(x,y) < D_\kappa$. In this thesis, we call these inequalities the *corollaries of Stewart's theorem* on CAT(κ) spaces.

2.2 A function c_κ

For $\kappa \in \mathbb{R}$, define a function $c_\kappa \colon]-\infty, \infty] \to]-\infty, \infty]$ by

$$c_\kappa(d) = \sum_{n=1}^{\infty} \frac{\kappa^{n-1}(-1)^{n-1} d^{2n}}{(2n)!} = \begin{cases} \dfrac{1}{-\kappa}(\cosh(\sqrt{-\kappa}\, d) - 1) = \dfrac{2}{-\kappa} \sinh^2 \dfrac{\sqrt{-\kappa}\, d}{2} & (\text{if } \kappa < 0);\\ \dfrac{1}{2} d^2 & (\text{if } \kappa = 0);\\ \dfrac{1}{\kappa}(1 - \cos(\sqrt{\kappa}\, d)) = \dfrac{2}{\kappa} \sin^2 \dfrac{\sqrt{\kappa}\, d}{2} & (\text{if } \kappa > 0)\end{cases}$$

for $d \in \mathbb{R}$ and $c_\kappa(\infty) = \infty$, where $\kappa^0 := 1$ if $\kappa = 0$. The function c_κ is infinitely differentiable on $\mathbb{R}$. The first and second derivative c'_κ and c''_κ are represented as

$$c'_\kappa(d) = \sum_{n=1}^{\infty} \frac{\kappa^{n-1}(-1)^{n-1} d^{2n-1}}{(2n-1)!} = \begin{cases} \dfrac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\, d) & (\text{if } \kappa < 0); \\ d & (\text{if } \kappa = 0); \\ \dfrac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\, d) & (\text{if } \kappa > 0) \end{cases}$$

and

$$c''_\kappa(d) = \sum_{n=0}^{\infty} \frac{\kappa^{n}(-1)^{n} d^{2n}}{(2n)!} = 1 - \kappa c_\kappa(d) = \begin{cases} \cosh(\sqrt{-\kappa}\, d) & (\text{if } \kappa < 0); \\ 1 & (\text{if } \kappa = 0); \\ \cos(\sqrt{\kappa}\, d) & (\text{if } \kappa > 0) \end{cases}$$

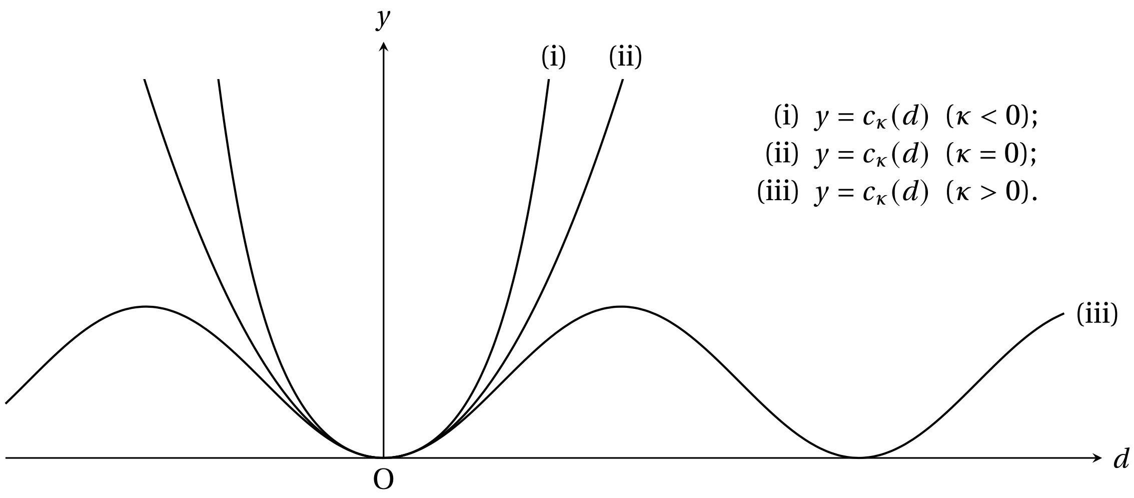
for $d \in \mathbb{R}$, respectively. Furthermore, for any $\kappa \in \mathbb{R}$, we get the following:

- $c_\kappa(0) = 0$, and $c_\kappa(d) > 0$ for any $d \in \,]0, D_\kappa]$;
- $c'_\kappa(0) = 0$, and $c'_\kappa(d) > 0$ for any $d \in \,]0, D_\kappa[$;
- $c''_\kappa(0) = 1$, and $c''_\kappa(d) > 0$ for any $d \in \,]0, D_\kappa/2[$;
- if $\kappa > 0$, then $c''_\kappa(d) \leq 0$ for any $d \in [D_\kappa/2, D_\kappa[$;
- c_κ is an odd function, and c''_κ is an even function;
- c_κ is strictly increasing on $[0, D_\kappa]$;
- c_κ is convex on $[0, D_\kappa/2]$;
- $c''_\kappa(d)^2 + \kappa c'_\kappa(d)^2 = 1$ for any $d \in \mathbb{R}$;
- $c'_\kappa(d_1 + d_2) = c'_\kappa(d_1)c''_\kappa(d_2) + c''_\kappa(d_1)c'_\kappa(d_2)$ for any $d_1, d_2 \in \mathbb{R}$;
- $c''_\kappa(d_1 + d_2) = c''_\kappa(d_1)c''_\kappa(d_2) - \kappa c'_\kappa(d_1)c'_\kappa(d_2)$ for any $d_1, d_2 \in \mathbb{R}$;
- $c''_\kappa(d_1)c''_\kappa(d_2) = (c''_\kappa(d_1 + d_2) + c''_\kappa(d_1 - d_2))/2$ for any $d_1, d_2 \in \mathbb{R}$.

Notice that $\lim_{d\to\infty} c_\kappa(d)/d = \infty$ if $\kappa \leq 0$. Moreover, we have

$$c_\kappa\left(\frac{D_\kappa}{2}\right) = \begin{cases} \infty & (\text{if } d \leq 0); \\ \dfrac{1}{\kappa} & (\text{if } d > 0). \end{cases}$$

The following figure describes the graphs of c_κ for $\kappa < 0$, $\kappa = 0$, and $\kappa > 0$.



For every $t \in [0,1]$, $d \in [0, D_\kappa[$, and $\kappa \in \mathbb{R}$, put

$$(t)_d^\kappa = \begin{cases} \dfrac{c'_\kappa(td)}{c'_\kappa(d)} & (\text{if } d \neq 0); \\ t & (\text{if } d = 0). \end{cases}$$

Then we have

$$(t)_d^\kappa = \begin{cases} \sinh(t\sqrt{-\kappa}\, d)/\sinh(\sqrt{-\kappa}\, d) & (\text{if } \kappa < 0); \\ t & (\text{if } \kappa = 0); \\ \sin(t\sqrt{\kappa}\, d)/\sin(\sqrt{\kappa}\, d) & (\text{if } \kappa > 0) \end{cases}$$

if $d \neq 0$. Kimura and Sudo [19] discovered that we can write all Stewart's theorems on CAT(κ) spaces in the same formula as follows:

$$c_\kappa(d(tx \oplus (1-t)y, z)) \le (t)_D^\kappa (c_\kappa(d(x,z)) - c_\kappa((1-t)D)) + (1-t)_D^\kappa (c_\kappa(d(y,z)) - c_\kappa(tD))$$

for any $t \in [0,1]$ and $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$, where $D = d(x,y)$. Similarly, the corollaries of Stewart's theorem on a CAT(κ) space X can be expressed by

$$c_\kappa(d(tx \oplus (1-t)y, z)) \le t c_\kappa(d(x,z)) + (1-t) c_\kappa(d(y,z))$$

for any $t \in [0,1]$ and $x, y, z \in X$ with $d(x,z) < D_\kappa/2$, $d(y,z) < D_\kappa/2$, and $d(x,y) < D_\kappa$.

Now we show several natures of functions c_κ, c'_κ, and c''_κ.

Lemma 2.8. *For any $\kappa \in \mathbb{R}$ and $d_1, d_2 \in \mathbb{R}$, an equation*

$$c_\kappa(d_1) + c''_\kappa(d_1) c_\kappa(d_2) = c_\kappa(d_2) + c''_\kappa(d_2) c_\kappa(d_1)$$

holds. In addition, these are equal to $\dfrac{1 - c''_\kappa(d_1) c''_\kappa(d_2)}{\kappa}$ *if* $\kappa \neq 0$.

Proof. For any $\kappa \in \mathbb{R}$ and $d_1, d_2 \in \mathbb{R}$, we get

$$\begin{aligned} c_\kappa(d_1) + c''_\kappa(d_1) c_\kappa(d_2) &= c_\kappa(d_1) + (1 - \kappa c_\kappa(d_1)) c_\kappa(d_2) \\ &= c_\kappa(d_2) + (1 - \kappa c_\kappa(d_2)) c_\kappa(d_1) \\ &= c_\kappa(d_2) + c''_\kappa(d_2) c_\kappa(d_1). \end{aligned}$$

Moreover, if $\kappa \neq 0$, then

$$c_\kappa(d_1) + c''_\kappa(d_1) c_\kappa(d_2) = \frac{1 - c''_\kappa(d_1)}{\kappa} + \frac{c''_\kappa(d_1)(1 - c''_\kappa(d_2))}{\kappa} = \frac{1 - c''_\kappa(d_1) c''_\kappa(d_2)}{\kappa}.$$

Thus we get the conclusion. □

Lemma 2.9. *Let X be a* CAT(κ) *space. Then an inequality*

$$c_\kappa\Big(\frac{1}{2} d(x,y)\Big) + c''_\kappa\Big(\frac{1}{2} d(x,y)\Big) c_\kappa\Big(d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, z\Big)\Big) \le \frac{1}{2} c_\kappa(d(x,z)) + \frac{1}{2} c_\kappa(d(y,z))$$

holds for any $x, y, z \in X$ such that $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$.

Proof. It holds as an equation if $x = y$, thus we suppose that $x \neq y$. Put $D = d(x, y)$. Then we obtain $0 < D < D_\kappa$ since $d(x, y) < 2D_\kappa - (d(y, z) + d(z, x)) \le 2D_\kappa - d(x, y)$. Therefore, by Stewart's theorem on CAT(κ) spaces, we get

$$\begin{aligned} c_\kappa\Big(d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, z\Big)\Big) &\le \Big(\frac{1}{2}\Big)_D^\kappa \Big(c_\kappa(d(x, z)) - c_\kappa\Big(\frac{D}{2}\Big)\Big) + \Big(\frac{1}{2}\Big)_D^\kappa \Big(c_\kappa(d(y, z)) - c_\kappa\Big(\frac{D}{2}\Big)\Big) \\ &= \frac{c_\kappa'(D/2)}{c_\kappa'(D)} \Big(c_\kappa(d(x, z)) + c_\kappa(d(y, z)) - 2c_\kappa\Big(\frac{D}{2}\Big)\Big). \end{aligned}$$

Since $c_\kappa'(D) = 2c_\kappa'(D/2)c_\kappa''(D/2)$ and $c_\kappa''(d) > 0$ for every $d \in]0, D_\kappa/2[$, we obtain

$$c_\kappa''\Big(\frac{D}{2}\Big) c_\kappa\Big(d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, z\Big)\Big) \le \frac{1}{2}\Big(c_\kappa(d(x, z)) + c_\kappa(d(y, z)) - 2c_\kappa\Big(\frac{D}{2}\Big)\Big).$$

This is the conclusion. □

Lemma 2.10. *Let X be a* CAT(κ) *space and $T: X \to X$ a quasinonexpansive mapping. Then $F(T)$ is closed. Moreover, if $d(z_1, z_2) < D_\kappa$ for every $z_1, z_2 \in F(T)$, then $F(T)$ is convex.*

Proof. Let $\{z_n\}$ be a sequence on $F(T)$ converging to $z_0 \in X$. Then for $n \in \mathbb{N}$, we get $0 \le d(z_0, Tz_0) \le d(z_0, z_n) + d(z_n, Tz_0) \le 2d(z_0, z_n) \to 0$ as $n \to \infty$. This means that $z_0 \in F(T)$, and hence $F(T)$ is closed.

Suppose that $d(z_1, z_2) < D_\kappa$ for every $z_1, z_2 \in F(T)$. Let $x, y \in F(T)$ and $t \in]0, 1[$. Then we can take a point $u = tx \oplus (1 - t)y$. Put $D = d(x, y) < D_\kappa$. Then since c_κ is increasing on $[0, D_\kappa]$, we obtain $c_\kappa(d(x, Tu)) \le c_\kappa(d(x, u))$ and $c_\kappa(d(y, Tu)) \le c_\kappa(d(y, u))$. Therefore, since

$$d(x, y) + d(y, Tu) + d(Tu, x) \le d(x, y) + d(y, u) + d(u, x) = D + tD + (1 - t)D = 2D < 2D_\kappa,$$

we obtain

$$\begin{aligned} 0 \le c_\kappa(d(u, Tu)) &= c_\kappa(d(tx \oplus (1 - t)y, Tu)) \\ &\le (t)_D^\kappa \big(c_\kappa(d(x, Tu)) - c_\kappa((1 - t)D)\big) + (1 - t)_D^\kappa \big(c_\kappa(d(y, Tu)) - c_\kappa(tD)\big) \\ &= (t)_D^\kappa \big(c_\kappa(d(x, Tu)) - c_\kappa(d(x, u))\big) + (1 - t)_D^\kappa \big(c_\kappa(d(y, Tu)) - c_\kappa(d(y, u))\big) \le 0. \end{aligned}$$

This means that $u = Tu$, and hence $F(T)$ is convex. □

2.3 A metric projection

Let X be an admissible complete CAT(κ) space and K a nonempty closed convex subset of X. Then for each $x \in X$, there exists a unique point p_x such that $p_x \in K$ and $d(x, p_x) = \inf_{y \in K} d(x, y)$. It derives a mapping P_K from X onto K by $x \longmapsto p_x$ for every $x \in X$. Such a mapping P_K is called a *metric projection* onto K. Then we obtain $F(P_K) = K$.

Lemma 2.11. *Let X be an admissible complete* CAT(κ) *space and K a nonempty closed convex subset of X. Then inequalities*

$$c_\kappa(d(x, P_K x)) c_\kappa''(d(P_K x, z)) \le c_\kappa(d(x, z)) - c_\kappa(d(P_K x, z)) \tag{i}$$

and

$$c_\kappa(d(P_K x, z)) c_\kappa''(d(x, P_K x)) \le c_\kappa(d(x, z)) - c_\kappa(d(x, P_K x)) \tag{ii}$$

hold for any $x \in X$ and $z \in K$.

Proof. Put $u = P_K x$ and $D = d(u, z)$. Then since $tz \oplus (1-t)u \in K$, we get

$$\begin{aligned} c_\kappa(d(x,u)) &\leq c_\kappa(d(x, tz \oplus (1-t)u)) \\ &\leq (t)_D^\kappa (c_\kappa(d(x,z)) - c_\kappa((1-t)D)) + (1-t)_D^\kappa (c_\kappa(d(x,u)) - c_\kappa(tD)) \end{aligned}$$

for any $t \in]0,1[$. Hence we have

$$\begin{aligned} \frac{1-(1-t)_D^\kappa}{(t)_D^\kappa} c_\kappa(d(x,u)) &\leq c_\kappa(d(x,z)) - c_\kappa((1-t)D) - \frac{(1-t)_D^\kappa}{(t)_D^\kappa} c_\kappa(tD) \\ &\leq c_\kappa(d(x,z)) - c_\kappa((1-t)D) \end{aligned}$$

for any $t \in]0,1[$. Letting $t \to 0$, we obtain (i). Indeed, we get

$$\lim_{t\to 0} \frac{1-(1-t)_D^\kappa}{(t)_D^\kappa} = c_\kappa''(D).$$

Furthermore, using (i) and Lemma 2.8, we have (ii). □

Corollary 2.12. *Let X be an admissible complete* CAT(κ) *space and K a nonempty closed convex subset of X. Then the metric projection P_K is quasinonexpansive.*

Proof. From Lemma 2.11 (i), we obtain $0 \leq c_\kappa(d(x,z)) - c_\kappa(d(P_K x, z))$ for any $x \in X$ and $z \in K$, which implies the conclusion. □

Using Corollary 2.12, we can proof Lemma 2.5.

Proof of Lemma 2.5. Let $\{x_n\}$ be a sequence on M such that $x_n \xrightarrow{\Delta} x_0 \in X$. Then x_0 is the unique asymptotic center of $\{x_n\}$. Assume that $x_0 \notin M$, and let P_M be a metric projection from X onto M. Then since P_M is quasinonexpansive, we obtain

$$\limsup_{n\to\infty} d(x_n, x_0) < \limsup_{n\to\infty} d(x_n, P_M x_0) \leq \limsup_{n\to\infty} d(x_n, x_0),$$

which is a contradiction. This follows the conclusion. □

2.4 Equilibrium problems

Let X be a uniquely D-geodesic space and K a nonempty closed convex subset of X. The equilibrium problem for a bifunction $f\colon K^2 \to \mathbb{R}$ is a problem to find a point $z \in K$ satisfying $\inf_{y\in K} f(z,y) \geq 0$. Then let us denote the set of all solutions to the equilibrium problem by $\operatorname{Equil} f$. That is, $\operatorname{Equil} f = \{z \in K \mid \inf_{y\in K} f(z,y) \geq 0\}$.

In this thesis, we always assume that f satisfies the following conditions (E1)–(E4):

(E1) $f(z,z) = 0$ for all $z \in K$;
(E2) $f(z,y) + f(y,z) \leq 0$ for all $z, y \in K$;
(E3) $f(z,\cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex for all $z \in K$;
(E4) $\limsup_{t\searrow 0} f(ty \oplus (1-t)z, y) \leq f(z,y)$ for all $z, y \in K$.

The condition (E4) is true when (E4$^+$) is true, see Chapter 1. These conditions are required to define resolvent operators of equilibrium problems.

As described in Chapter 1, certain convex minimization problems can be attributed to an equilibrium problem. Let X be a uniquely D-geodesic space, K a nonempty closed convex subset of X, and g a lower semicontinuous convex function from K into $\mathbb{R}$. Define $f\colon K^2 \to \mathbb{R}$ by $f(z,y) = g(y) - g(z)$ for each $y, z \in K$. Then f satisfies conditions (E1)–(E4), and we have $\operatorname{Equil} f = \{z \in K \mid \inf_{y\in K}(g(y) - g(z)) \geq 0\} = \{z \in K \mid \inf_{y\in K} g(y) = g(z)\} = \operatorname{argmin} g$.

2.5 Other lemmas

Let $\tanh^{-1}\colon\,]-1,1[\ \to \mathbb{R}$ be the inverse of the hyperbolic tangent function. Similarly, let $\tan^{-1}\colon \mathbb{R}\cup\{\pm\infty\} \to [0,\pi[$ be the extended inverse of the extended trigonometric tangent function $\tan\colon [0,\pi[\ \to \mathbb{R}\cup\{\pm\infty\}$, where we set $\tan^{-1}(\pm\infty) = \pi/2$.

For $\kappa \in \mathbb{R}$ and $D \in [0, D_\kappa[$, define a function $\zeta_D^\kappa\colon [0,1] \to [0,1]$ by

$$\zeta_D^\kappa(t) = \begin{cases} \dfrac{c_\kappa'(tD)}{c_\kappa'(tD) + c_\kappa'((1-t)D)} & (\text{if } D > 0); \\ t & (\text{if } D = 0) \end{cases}$$

for $t \in [0,1]$. Then we know that $\lim_{D\to 0} \zeta_D^\kappa(t) = t$ for any $\kappa \in \mathbb{R}$ and $t \in [0,1]$.

Lemma 2.13. *For $\kappa \in \mathbb{R}$ and $D \in [0, D_\kappa[$, the function $\zeta_D^\kappa\colon [0,1] \to [0,1]$ is continuous, strictly increasing, and bijective. Moreover, the following hold:*

- *$\zeta_D^\kappa(0) = 0$, $\zeta_D^\kappa(1/2) = 1/2$, and $\zeta_D^\kappa(1) = 1$;*
- *$\zeta_D^\kappa(t) + \zeta_D^\kappa(1-t) = 1$ for any $t \in [0,1]$;*
- *$(\zeta_D^\kappa)''(1/2) = 0$, where $(\zeta_D^\kappa)''$ is the second derivative of ζ_D^κ.*

In addition, the following hold if $D > 0$:

- *If $\kappa > 0$, then $(\zeta_D^\kappa)''(t) < 0$ for any $t \in]0,1/2[$;*
- *if $\kappa > 0$, then $(\zeta_D^\kappa)''(t) > 0$ for any $t \in]1/2,1[$;*
- *if $\kappa < 0$, then $(\zeta_D^\kappa)''(t) > 0$ for any $t \in]0,1/2[$;*
- *if $\kappa < 0$, then $(\zeta_D^\kappa)''(t) < 0$ for any $t \in]1/2,1[$.*

Let $\kappa \in \mathbb{R}$ and $D \in [0, D_\kappa[$. Since ζ_D^κ is bijective, there exists the inverse of ζ_D^κ. It is obvious that $(\zeta_0^\kappa)^{-1}(\alpha) = \alpha$ for any $\alpha \in [0,1]$. We also have the following facts.

Lemma 2.14. *For $\kappa \in \mathbb{R}$ and $D \in]0, D_\kappa[$, the inverse of ζ_D^κ is expressed by*

$$\begin{aligned}
&(\zeta_D^\kappa)^{-1}(\alpha) \\
&= \begin{cases} \dfrac{1}{\sqrt{-\kappa}D}\tanh^{-1}\dfrac{\alpha\sinh\left(\sqrt{-\kappa}D\right)}{1-\alpha+\alpha\cosh\left(\sqrt{-\kappa}D\right)} = \dfrac{1}{\sqrt{-\kappa}D}\tanh^{-1}\dfrac{\sqrt{-\kappa}\,\alpha c_\kappa'(D)}{1-\alpha+\alpha c_\kappa''(D)} & (\text{if } \kappa < 0); \\ \alpha & (\text{if } \kappa = 0); \\ \dfrac{1}{\sqrt{\kappa}D}\tan^{-1}\dfrac{\alpha\sin\left(\sqrt{\kappa}D\right)}{1-\alpha+\alpha\cos\left(\sqrt{\kappa}D\right)} = \dfrac{1}{\sqrt{\kappa}D}\tan^{-1}\dfrac{\sqrt{\kappa}\alpha c_\kappa'(D)}{1-\alpha+\alpha c_\kappa''(D)} & (\text{if } \kappa > 0) \end{cases} \\
&= 1 - \begin{cases} \dfrac{1}{\sqrt{-\kappa}D}\tanh^{-1}\dfrac{(1-\alpha)\sinh\left(\sqrt{-\kappa}D\right)}{\alpha+(1-\alpha)\cosh\left(\sqrt{-\kappa}D\right)} = \dfrac{1}{\sqrt{-\kappa}D}\tanh^{-1}\dfrac{\sqrt{-\kappa}(1-\alpha)c_\kappa'(D)}{\alpha+(1-\alpha)c_\kappa''(D)} & (\text{if } \kappa < 0); \\ 1-\alpha & (\text{if } \kappa = 0); \\ \dfrac{1}{\sqrt{\kappa}D}\tan^{-1}\dfrac{(1-\alpha)\sin\left(\sqrt{\kappa}D\right)}{\alpha+(1-\alpha)\cos\left(\sqrt{\kappa}D\right)} = \dfrac{1}{\sqrt{\kappa}D}\tan^{-1}\dfrac{\sqrt{\kappa}(1-\alpha)c_\kappa'(D)}{\alpha+(1-\alpha)c_\kappa''(D)} & (\text{if } \kappa > 0) \end{cases}
\end{aligned}$$

for $\alpha \in [0,1]$. If $D = 0$, then $(\zeta_D^\kappa)^{-1}(\alpha) = \alpha$ for every $\alpha \in [0,1]$. Therefore, the following hold for any $\kappa \in \mathbb{R}$ and $D \in [0, D_\kappa[$:

- $(\zeta_D^\kappa)^{-1}\colon [0,1] \to [0,1]$ *is continuous and strictly increasing;*
- $(\zeta_D^\kappa)^{-1}(0) = 0$*,* $(\zeta_D^\kappa)^{-1}(1/2) = 1/2$*, and* $(\zeta_D^\kappa)^{-1}(1) = 1$*;*
- $(\zeta_D^\kappa)^{-1}(\alpha) + (\zeta_D^\kappa)^{-1}(1-\alpha) = 1$ *for any* $\alpha \in [0,1]$.

Moreover, the following hold if $D > 0$*:*

- *If* $\kappa > 0$*, then* $((\zeta_D^\kappa)^{-1})''(\alpha) > 0$ *for any* $\alpha \in]0,1/2[$*;*
- *if* $\kappa > 0$*, then* $((\zeta_D^\kappa)^{-1})''(\alpha) < 0$ *for any* $\alpha \in]1/2,1[$*;*
- *if* $\kappa < 0$*, then* $((\zeta_D^\kappa)^{-1})''(\alpha) < 0$ *for any* $\alpha \in]0,1/2[$*;*
- *if* $\kappa < 0$*, then* $((\zeta_D^\kappa)^{-1})''(\alpha) > 0$ *for any* $\alpha \in]1/2,1[$.

The reason we set the codomain of $\tan^{-1}$ as $[0,\pi[$ instead of $]-\pi/2,\pi/2[$ is to ensure consistency of Lemma 2.14 when $D_\kappa/2 \le D < D_\kappa$.

Corollary 2.15. *For* $\kappa > 0$ *and* $D \in]0, D_\kappa[$*, the following hold.*

- $\zeta_D^\kappa(t) > t$ *for any* $t \in]0,1/2[$*;*
- $\zeta_D^\kappa(t) < t$ *for any* $t \in]1/2,1[$*;*
- $(\zeta_D^\kappa)^{-1}(\alpha) < \alpha$ *for any* $\alpha \in]0,1/2[$*;*
- $(\zeta_D^\kappa)^{-1}(\alpha) > \alpha$ *for any* $\alpha \in]1/2,1[$.

Corollary 2.16. *For* $\kappa < 0$ *and* $D \in]0,\infty[$*, the following hold.*

- $\zeta_D^\kappa(t) < t$ *for any* $t \in]0,1/2[$*;*
- $\zeta_D^\kappa(t) > t$ *for any* $t \in]1/2,1[$*;*
- $(\zeta_D^\kappa)^{-1}(\alpha) > \alpha$ *for any* $\alpha \in]0,1/2[$*;*
- $(\zeta_D^\kappa)^{-1}(\alpha) < \alpha$ *for any* $\alpha \in]1/2,1[$.

For $D > 0$, the first derivative of ζ_D^κ is expressed by

$$(\zeta_D^\kappa)'(t) = \frac{Dc_\kappa'(D)}{(c_\kappa'(tD) + c_\kappa'((1-t)D))^2}$$

for $t \in]0,1[$. Then there exist limits $\lim_{t\to 0}(\zeta_D^\kappa)'(t)$ and $\lim_{t\to 1}(\zeta_D^\kappa)'(t)$. Thus we have

$$\lim_{t\to 0}(\zeta_D^\kappa)'(t) = \lim_{t\to 1}(\zeta_D^\kappa)'(t) = \frac{D}{c_\kappa'(D)}.$$

By noting Lemma 2.13, we also get $(\zeta_D^\kappa)'(t) = (\zeta_D^\kappa)'(1-t)$ for all $t \in]0,1[$. Therefore, we obtain the following facts.

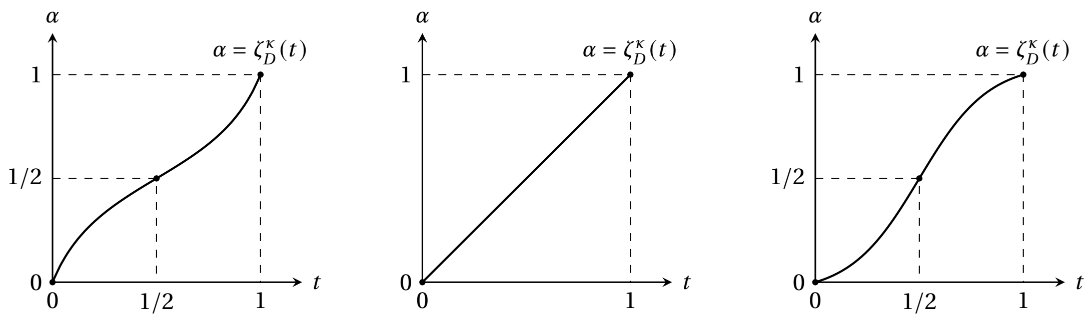
Lemma 2.17. *For* $\kappa > 0$ *and* $D \in]0, D_\kappa[$*, the following hold.*

- $\dfrac{1}{2}t < \zeta_D^\kappa(t) < \dfrac{D}{c_\kappa'(D)}t$ *for any* $t \in]0,1]$*;*
- $\dfrac{1}{2}(1-t) < 1 - \zeta_D^\kappa(t) < \dfrac{D}{c_\kappa'(D)}(1-t)$ *for any* $t \in [0,1[$*;*
- $\dfrac{c_\kappa'(D)}{D}\alpha < (\zeta_D^\kappa)^{-1}(\alpha) < 2\alpha$ *for any* $\alpha \in]0,1]$*;*
- $\dfrac{c_\kappa'(D)}{D}(1-\alpha) < 1 - (\zeta_D^\kappa)^{-1}(\alpha) < 2(1-\alpha)$ *for any* $\alpha \in [0,1[$.

Lemma 2.18. *For $\kappa < 0$ and $D \in \,]0,\infty[$, the following hold.*

- $\dfrac{D}{c'_\kappa(D)} t < \zeta^\kappa_D(t) < 2t$ *for any* $t \in \,]0,1]$*;*
- $\dfrac{D}{c'_\kappa(D)}(1-t) < 1 - \zeta^\kappa_D(t) < 2(1-t)$ *for any* $t \in [0,1[$*;*
- $\dfrac{1}{2}\alpha < (\zeta^\kappa_D)^{-1}(\alpha) < \dfrac{c'_\kappa(D)}{D}\alpha$ *for any* $\alpha \in \,]0,1]$*;*
- $\dfrac{1}{2}(1-\alpha) < 1 - (\zeta^\kappa_D)^{-1}(\alpha) < \dfrac{c'_\kappa(D)}{D}(1-\alpha)$ *for any* $\alpha \in [0,1[$.

The following are graphs of the function ζ^κ_D for some $D > 0$.



For $\kappa \in \mathbb{R}$ and $t \in [0,1]$, define a function $\eta^\kappa_t \colon [0, D_\kappa[\,\to [0,1]$ by

$$\eta^\kappa_t(D) = \zeta^\kappa_D(t) = \begin{cases} \dfrac{c'_\kappa(tD)}{c'_\kappa(tD) + c'_\kappa((1-t)D)} & \text{(if } D > 0\text{);} \\ t & \text{(if } D = 0\text{)} \end{cases}$$

for $D \in [0, D_\kappa[$. Furthermore, for $\kappa \in \mathbb{R}$ and $\alpha \in [0,1]$, define a function $\overline{\eta}^\kappa_\alpha \colon [0, D_\kappa[\,\to [0,1]$ by

$$\overline{\eta}^\kappa_\alpha(D) = (\zeta^\kappa_D)^{-1}(\alpha)$$

for $D \in [0, D_\kappa[$. Then we have $\eta^0_t(D) = t$, $\overline{\eta}^0_\alpha(D) = \alpha$, $\overline{\eta}^\kappa_\alpha(0) = \alpha$, $\eta^\kappa_t(D) + \eta^\kappa_{1-t}(D) = 1$, and $\overline{\eta}^\kappa_\alpha(D) + \overline{\eta}^\kappa_{1-\alpha}(D) = 1$ for every $\kappa \in \mathbb{R}$, $t \in [0,1]$, $\alpha \in [0,1]$, and $D \in [0, D_\kappa[$.

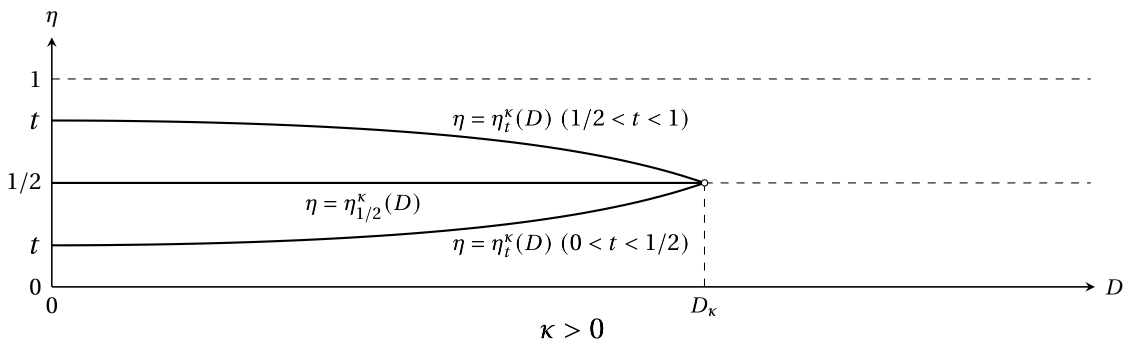
Lemma 2.19. *For $\kappa > 0$ and $t \in \,]0,1[$, the following hold:*

(i) $\lim_{d\to 0} \eta^\kappa_t(d) = t$*;*
(ii) $\lim_{d\to D_\kappa} \eta^\kappa_t(d) = 1/2$*;*
(iii) *if $t < 1/2$, then η^κ_t is strictly increasing;*
(iv) *if $t > 1/2$, then η^κ_t is strictly decreasing.*

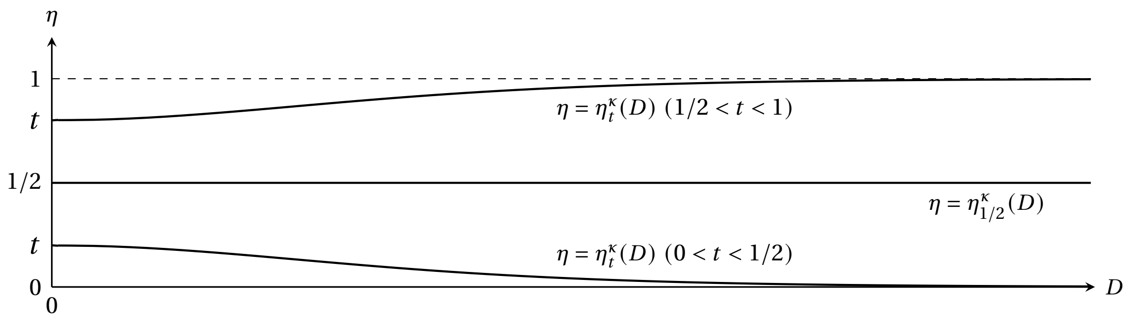
Lemma 2.20. *For $\kappa < 0$ and $t \in \,]0,1[$, the following hold:*

(i) $\lim_{d\to 0} \eta^\kappa_t(d) = t$*;*
(ii) *if $t < 1/2$, then* $\lim_{d\to\infty} \eta^\kappa_t(d) = 0$*;*
(iii) *if $t > 1/2$, then* $\lim_{d\to\infty} \eta^\kappa_t(d) = 1$*;*
(iv) *if $t < 1/2$, then η^κ_t is strictly decreasing;*
(v) *if $t > 1/2$, then η^κ_t is strictly increasing.*

The following figures show graphs of η_t^κ for several t.



$\kappa > 0$



$\kappa < 0$

We give several natures of the function $\overline{\eta}_\alpha^\kappa$ in Section 3.1.

Chapter 3

Convex combinations

3.1 κ-convex combination on geodesic spaces

As described in the Preliminaries, on a uniquely D-geodesic space, a convex combination $tx \oplus (1-t)y$ stands for the unique point on the geodesic segment $[x,y]$ such that $d(x,z) : d(z,y) = (1-t) : t$. Now we introduce the following fact.

Fact 3.1. *Let X be a uniquely D-geodesic space and take $x,y \in X$ such that $d(x,y) < D$. Then for any $t \in [0,1]$, $tx \oplus (1-t)y = \operatorname{argmin}_{z\in X}\bigl(td(x,z)^2 + (1-t)d(y,z)^2\bigr)$.*

For the sake of completeness, we show this. We prepare a lemma to prove the above fact.

Lemma 3.2. *Let X be a uniquely D-geodesic space. Take $x,y \in X$ such that $d(x,y) < D$. For a given $t \in [0,1]$ and a strictly increasing function $H\colon [0,\infty[\to [0,\infty[$, define $g\colon X \to [0,\infty[$ by*

$$g(\cdot) = tH(d(x,\cdot)) + (1-t)H(d(y,\cdot)).$$

Assume that there exists a unique minimizer z_0 of $g|_{[x,y]}\colon [x,y] \to [0,\infty[$. Then z_0 is a unique minimizer of g.

Proof of Lemma 3.2. If $x = y$, then we obtain $z_0 = x = \operatorname{argmin}_{z\in X} H(d(x,z)) = \operatorname{argmin}_{z\in X} g(z)$, which is the conclusion. Suppose that $x \neq y$ and take $w \in X \setminus \{z_0\}$ arbitrarily. From the assumption, we obtain $g(z_0) < g(w)$ if $w \in [x,y]$. In what follows, assume that $w \notin [x,y]$. Put $\sigma = d(y,w)/(d(x,w)+d(y,w))$ and $z_1 = \sigma x \oplus (1-\sigma)y$. Then we have $\sigma \in]0,1[$ and $d(x,z_1) : d(y,z_1) = d(x,w) : d(y,w)$. Moreover, we obtain $g(z_0) \le g(z_1)$, especially we get $g(z_0) < g(z_1)$ if $z_0 \neq z_1$.

Suppose that $z_0 = z_1$. Then we get $w \neq z_1$ and hence $w \notin [x,y]$. Thus we have $d(x,z_1) + d(y,z_1) = d(x,y) < d(x,w) + d(y,w)$. It implies that $d(x,z_1) < d(x,w)$ and $d(y,z_1) < d(y,w)$. Therefore we get $g(z_1) < g(w)$, and this follows $g(z_0) < g(w)$.

Next we assume $z_0 \neq z_1$. Then we obtain $d(x,z_1) \le d(x,w)$ and $d(y,z_1) \le d(y,w)$, and hence $g(z_1) \le g(w)$. It implies $g(z_0) < g(w)$ and thus we get the conclusion. □

Proof of Fact 3.1. Define a function $g\colon X \to [0,\infty[$ by $g(\cdot) = td(x,\cdot)^2 + (1-t)d(y,\cdot)^2$. Then for any $\alpha \in [0,1]$, we have

$$\begin{aligned} g(\alpha x \oplus (1-\alpha)y) &= td(x, \alpha x \oplus (1-\alpha)y)^2 + (1-t)d(y, \alpha x \oplus (1-\alpha)y)^2 \\ &= t((1-\alpha)d(x,y))^2 + (1-t)(\alpha d(x,y))^2 \\ &= \bigl(t(1-\alpha)^2 + (1-t)\alpha^2\bigr)d(x,y)^2 \\ &= \bigl((\alpha-t)^2 + t(1-t)\bigr)d(x,y)^2. \end{aligned}$$

This follows that a restriction $g|_{[x,y]}$ has the unique minimizer $z_0 = tx \oplus (1-t)y$. Therefore, from Lemma 3.2, z_0 is also the unique minimizer of g. □

Next, we consider a different type of internally dividing points of geodesic segments named *κ-convex combination.* We hereinafter assume that ζ_D^κ, η_t^κ, and $\overline{\eta}_\alpha^\kappa$ are functions defined in Section 2.5.

Lemma 3.3. *Let $\kappa \in \mathbb{R}$, $D \in \,]0, D_\kappa[$, and $\alpha \in [0,1]$. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by*

$$f(\lambda) = \alpha c_\kappa((1-\lambda)D) + (1-\alpha)c_\kappa(\lambda D)$$

for $\lambda \in \mathbb{R}$. Then a set $\operatorname{argmin}_{\lambda\in[0,1]} f(\lambda)$ *consists exactly one point t_0, and it satisfies $f'(t_0) = 0$.*

Proof. First, we consider the case where $\alpha = 0$. Then we have $f(\lambda) = c_\kappa(\lambda D)$ for every $\lambda \in [0,1]$ and hence $\operatorname{argmin}_{\lambda\in[0,1]} f(\lambda) = \{0\}$. Similarly, we get $\operatorname{argmin}_{\lambda\in[0,1]} f(\lambda) = \{1\}$ if $\alpha = 1$. We can verify easily that $f'(\operatorname{argmin}_{\lambda\in[0,1]} f(\lambda)) = 0$ holds.

Next, assume that $\alpha \in \,]0,1[$. Then we obtain

$$\begin{aligned} f'(\lambda)/D &= -\alpha c_\kappa'((1-\lambda)D) + (1-\alpha)c_\kappa'(\lambda D) \\ &= -\alpha(c_\kappa'(D)c_\kappa''(\lambda D) - c_\kappa''(D)c_\kappa'(\lambda D)) + (1-\alpha)c_\kappa'(\lambda D) \\ &= -\alpha c_\kappa'(D)c_\kappa''(\lambda D) + (1-\alpha+\alpha c_\kappa''(D))c_\kappa'(\lambda D) \end{aligned}$$

and

$$f''(\lambda)/D^2 = \alpha c_\kappa''((1-\lambda)D) + (1-\alpha)c_\kappa''(\lambda D)$$

for any $\lambda \in [0,1]$. It follows that $f'(0)/D = -\alpha c_\kappa'(D) < 0$ and $f'(1)/D = (1-\alpha)c_\kappa'(D) > 0$. Thus there exists $t \in [0,1]$ such that $f'(t) = 0$.

If $D < D_\kappa/2$, then $f''(\lambda)/D^2 > 0$ holds for every $\lambda \in [0,1]$. Hence the set $\operatorname{argmin}_{\lambda\in[0,1]} f(\lambda)$ consists exactly one point if $D < D_\kappa/2$.

Consider the case where $D_\kappa/2 \le D < D_\kappa$. Then we get $\kappa > 0$ by the definition of D_κ, and thus there exists $\theta_0 \in \mathbb{R}$ such that

$$\begin{aligned} \sqrt{\kappa} f'(\lambda)/D &= -\alpha \sin(\sqrt{\kappa}D)\cos(\sqrt{\kappa}\lambda D) + (1-\alpha+\alpha\cos(\sqrt{\kappa}D))\sin(\sqrt{\kappa}\lambda D) \\ &= \sqrt{\alpha^2 + 2\alpha(1-\alpha)\cos(\sqrt{\kappa}D) + (1-\alpha)^2}\,\sin(\sqrt{\kappa}\lambda D + \theta_0). \end{aligned}$$

This shows that the zeros of a function $\lambda \mapsto f'(\lambda)$ appear exactly $\pi/(\sqrt{\kappa}D)$ apart. Therefore, since $\pi/(\sqrt{\kappa}D) > \pi/(\sqrt{\kappa}D_\kappa) = 1$, there exists a unique $t_0 \in [0,1]$ such that $f'(t_0) = 0$, which is the unique element of the set $\operatorname{argmin}_{\lambda\in[0,1]} f(\lambda)$. □

Let $\kappa \in \mathbb{R}$ and X a uniquely D_κ-geodesic space. Define a function $\overline{c_\kappa}\colon [0,\infty[\,\to [0,\infty[$ by

$$\overline{c_\kappa}(d) = \begin{cases} c_\kappa(d) & (\text{if } \kappa \le 0); \\ c_\kappa(d) & (\text{if } \kappa > 0 \text{ and } d \le D_\kappa); \\ \dfrac{2}{\sqrt{\kappa}\pi} d & (\text{if } \kappa > 0 \text{ and } d > D_\kappa) \end{cases}$$

for $d \in [0,\infty[$. Then $\overline{c_\kappa}$ is strictly increasing on $[0,\infty[$. Fix $x, y \in X$ such that $d(x,y) < D_\kappa$, and define a function $g\colon X \to \mathbb{R}$ by

$$g(\cdot) = \alpha\overline{c_\kappa}(d(x,\cdot)) + (1-\alpha)\overline{c_\kappa}(d(y,\cdot)).$$

We show that g has the unique minimizer z_0. It is clearly concluded if $x = y$. Assume that $x \neq y$, and define a function $g_\gamma \colon [0,1] \to \mathbb{R}$ by

$$\begin{aligned} g_\gamma(\lambda) &= g(\lambda x \oplus (1-\lambda)y) \\ &= \alpha c_\kappa((1-\lambda)d(x,y)) + (1-\alpha)c_\kappa(\lambda d(x,y)) \end{aligned}$$

for $\lambda \in [0,1]$. Then a set $\operatorname{argmin}_{\lambda\in[0,1]} g_\gamma(\lambda)$ consists exactly one point t_0 from Lemma 3.3. This means that a restriction $g|_{[x,y]}$ has the unique minimizer $z_0 = t_0 x \oplus (1-t_0)y \in [x,y]$. We also have g has the same unique minimizer z_0 form Lemma 3.2. Then we give such a point z_0 a specific notation as follows:

Definition 3.4 ([17], [18], [19]). Let X be a uniquely D_κ-geodesic space and take $x, y \in X$ such that $d(x,y) < D_\kappa$. Then we say that the unique minimizer of a function $g \colon X \to [0,\infty[$ defined by

$$g(\cdot) = \alpha\overline{c_\kappa}(d(x,\cdot)) + (1-\alpha)\overline{c_\kappa}(d(y,\cdot))$$

is *κ-convex combination* of x and y with the ratio $\alpha \in [0,1]$, and write it by $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y$.

Then we can get the following easily.

- $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y \in [x,y]$;
- $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = tx \oplus (1-t)y$, where $t = \operatorname{argmin}_{\lambda\in[0,1]}\big(\alpha c_\kappa((1-\lambda)d(x,y)) + (1-\alpha)c_\kappa(\lambda d(x,y))\big)$;
- $1x \overset{\kappa}{\oplus} 0y = x$, $0x \overset{\kappa}{\oplus} 1y = y$;
- $\alpha x \overset{\kappa}{\oplus} (1-\alpha)x = x$ for every $\alpha \in [0,1]$;
- $\alpha x \overset{0}{\oplus} (1-\alpha)y = \alpha x \oplus (1-\alpha)y$;
- $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = (1-\alpha)y \overset{\kappa}{\oplus} \alpha x$.

Note that papers [17], [18] and [19] define a κ-convex combination only when $d(x,y) < D_\kappa/2$. But actually, as shown above, the definition can be extended to include the cases where $D_\kappa/2 \le d(x,y) < D_\kappa$.

Lemma 3.5. *Let $\kappa \in \mathbb{R}$, $D \in]0, D_\kappa[$, and $\alpha \in [0,1]$. Then the following conditions are equivalent:*

(a) $t = \operatorname{argmin}_{\lambda\in[0,1]}\big(\alpha c_\kappa((1-\lambda)D) + (1-\alpha)c_\kappa(\lambda D)\big)$;

(b) $(1-\alpha+\alpha c''_\kappa(D))c'_\kappa(tD) = \alpha c'_\kappa(D)c''_\kappa(tD)$;

(c) $\alpha = \zeta^\kappa_D(t)$;

(d) $t = (\zeta^\kappa_D)^{-1}(\alpha)$.

Proof. Define a function $f \colon \mathbb{R} \to \mathbb{R}$ by $f(\lambda) = \alpha c_\kappa((1-\lambda)D) + (1-\alpha)c_\kappa(\lambda D)$ for $\lambda \in \mathbb{R}$. Then we obtain from Lemma 3.3 that there exists a unique minimizer $t \in [0,1]$ of $f|_{[0,1]}$, and it satisfies $f'(t) = 0$. Since

$$f'(t)/D = -\alpha c'_\kappa(D)c''_\kappa(tD) + (1-\alpha+\alpha c''_\kappa(D))c'_\kappa(tD),$$

we obtain that (a) is equivalent to (b). Moreover, the condition (b) is equivalent to

$$c'_\kappa(tD) + (-c'_\kappa(tD) + c''_\kappa(D)c'_\kappa(tD))\alpha = c'_\kappa(D)c''_\kappa(tD)\alpha,$$

which is also equivalent to

$$\alpha = \frac{c'_\kappa(tD)}{c'_\kappa(tD) + c'_\kappa(D)c''_\kappa(tD) - c''_\kappa(D)c'_\kappa(tD)} = \frac{c'_\kappa(tD)}{c'_\kappa(tD) + c'_\kappa((1-t)D)} = \zeta^\kappa_D(t).$$

Thus (b) and (c) are equivalent. Since conditions (c) and (d) are equivalent obviously, we get the conclusion. ◻

Lemma 3.5 (a) and (d) follow that, for any $\kappa \in \mathbb{R}$, $\alpha \in [0,1]$, and $D \in]0, D_\kappa[$,

$$(\zeta_D^\kappa)^{-1}(\alpha) = \operatorname*{argmin}_{\lambda \in [0,1]} \big(\alpha c_\kappa((1-\lambda)D) + (1-\alpha)c_\kappa(\lambda D)\big).$$

We can describe a relationship between κ-convex combinations and usual convex combinations as follows.

Lemma 3.6. *For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space and take $x, y \in X$ such that $0 < d(x,y) < D_\kappa$. Let $\alpha \in [0,1]$, and take a unique $t \in [0,1]$ such that $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = tx \oplus (1-t)y$. Then the following hold, where $D = d(x,y)$.*

(i) $(1-\alpha+\alpha c_\kappa''(D))c_\kappa'(tD) = \alpha c_\kappa'(D)c_\kappa''(tD)$;

(ii) $\alpha = \zeta_D^\kappa(t)$;

(iii) $t = (\zeta_D^\kappa)^{-1}(\alpha)$.

Proof. We get $t = \operatorname{argmin}_{\lambda \in [0,1]}\big(\alpha c_\kappa((1-\lambda)D) + (1-\alpha)c_\kappa(\lambda D)\big)$ by the definition of κ-convex combination. Therefore, from Lemma 3.5, we get the conclusion. ◻

For every $x, y \in X$ such that $0 \le D = d(x,y) < D_\kappa$, we get from Lemma 3.6 that $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = (\zeta_D^\kappa)^{-1}(\alpha)x \oplus (\zeta_D^\kappa)^{-1}(1-\alpha)y$ for any $\alpha \in [0,1]$ and $tx \oplus (1-t)y = \zeta_D^\kappa(t)x \overset{\kappa}{\oplus} \zeta_D^\kappa(1-t)y$ for any $t \in [0,1]$.

Corollary 3.7. *For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space and take $x, y \in X$ such that $d(x,y) < D_\kappa$. Then*

$$\frac{1}{2}x \oplus \frac{1}{2}y = \frac{1}{2}x \overset{\kappa}{\oplus} \frac{1}{2}y.$$

Conversely, if $x \neq y$, $\alpha \in]0,1[$ and $\alpha x \oplus (1-\alpha)y = \alpha x \overset{\kappa}{\oplus} (1-\alpha)y$, then $\alpha = 1/2$.

Proof. An equation $\frac{1}{2}x \oplus \frac{1}{2}y = \frac{1}{2}x \overset{\kappa}{\oplus} \frac{1}{2}y$ is clear if $x = y$. Suppose that $x \neq y$. By Lemma 3.6, we get

$$\frac{1}{2}x \oplus \frac{1}{2}y = \left(\frac{c_\kappa'(D/2)}{c_\kappa'(D/2)+c_\kappa'(D/2)}\right)x \overset{\kappa}{\oplus} \left(\frac{c_\kappa'(D/2)}{c_\kappa'(D/2)+c_\kappa'(D/2)}\right)y = \frac{1}{2}x \overset{\kappa}{\oplus} \frac{1}{2}y.$$

Next, assume that $\alpha x \oplus (1-\alpha)y = \alpha x \overset{\kappa}{\oplus} (1-\alpha)y$. Then $\alpha = \zeta_D^\kappa(\alpha)$, and this implies $\alpha = 1/2$ by Lemma 2.13. ◻

The κ-convex combination has basic properties that make it worthy of the name convex combination as follows. From Lemma 3.6, we have $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = (\zeta_D^\kappa)^{-1}(\alpha)x \oplus (\zeta_D^\kappa)^{-1}(1-\alpha)y$ for any $\alpha \in [0,1]$. Since $(\zeta_D^\kappa)^{-1} \colon [0,1] \to [0,1]$ is surjective and strictly increasing, we get the following three lemmas.

Lemma 3.8. *For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space and take $x, y \in X$ such that $0 < d(x,y) < D_\kappa$. Then for any $t \in [0,1]$, there exists a unique $\alpha \in [0,1]$ such that $tx \oplus (1-t)y = \alpha x \overset{\kappa}{\oplus} (1-\alpha)y$, and then $\alpha = \zeta_D^\kappa(t)$.*

Lemma 3.9. *For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space and take $x, y \in X$ such that $d(x,y) < D_\kappa$. Then $\{\alpha x \overset{\kappa}{\oplus} (1-\alpha)y \mid \alpha \in [0,1]\} = \{tx \oplus (1-t)y \mid t \in [0,1]\} = [x,y]$.*

Lemma 3.10. *For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space and take $x, y \in X$ such that $0 < d(x,y) < D_\kappa$. Then $d(y, t_1 x \overset{\kappa}{\oplus} (1-t_1)y) < d(y, t_2 x \overset{\kappa}{\oplus} (1-t_2)y)$ holds if $0 \le t_1 < t_2 \le 1$.*

Now we will introduce a difference between two points $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y$ and $\alpha x \oplus (1-\alpha)y$ with respect to the distance from the midpoint $(1/2)x \oplus (1/2)y$.

Lemma 3.11. *For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space and take $x, y \in X$ such that $0 < d(x,y) < D_\kappa$. Take $\alpha \in]0,1[\setminus \{1/2\}$ and put $m = (1/2)x \oplus (1/2)y$, $u_\kappa = \alpha x \overset{\kappa}{\oplus} (1-\alpha)y$, and $u_0 = \alpha x \oplus (1-\alpha)y$. Then the following hold.*

- *If $\kappa > 0$, then a point u_κ is farther from the midpoint m than u_0.*
- *If $\kappa < 0$, then a point u_κ is closer to the midpoint m than u_0.*

Proof. Put $D = d(x,y)$. Then Lemma 3.6 deduces that $d(u_0, m) = |\alpha - 1/2|$ and

$$d(u_\kappa, m) = d((\zeta_D^\kappa)^{-1}(\alpha)x \oplus (\zeta_D^\kappa)^{-1}(1-\alpha)y, m) = |(\zeta_D^\kappa)^{-1}(\alpha) - 1/2|.$$

Therefore, Lemma 2.13 implies the conclusion. □

Lemma 3.12. *For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space and take $x, y \in X$ such that $0 < d(x,y) < D_\kappa$. Take $\alpha, t \in [0,1]$ and suppose that $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = tx \oplus (1-t)y$. Then*

$$\frac{c'_\kappa(D)}{c'_\kappa(tD) + c'_\kappa((1-t)D)} = \sqrt{\alpha^2 + 2\alpha(1-\alpha)c''_\kappa(D) + (1-\alpha)^2},$$

where $D = d(x,y)$.

Proof. By Lemma 3.6, we have $\alpha = c'_\kappa(tD)/(c'_\kappa(tD) + c'_\kappa((1-t)D))$. Thus we obtain

$$\sqrt{\alpha^2 + 2\alpha(1-\alpha)c''_\kappa(D) + (1-\alpha)^2} = \frac{\sqrt{c'_\kappa(tD)^2 + 2c'_\kappa(tD)c'_\kappa((1-t)D)c''_\kappa(D) + c'_\kappa((1-t)D)^2}}{c'_\kappa(tD) + c'_\kappa((1-t)D)}.$$

Since $1 - \kappa c'_\kappa(d)^2 = c''_\kappa(d)^2$ for any $d \in \mathbb{R}$, we get

$$\begin{aligned}
&c'_\kappa(tD)^2 + 2c'_\kappa(tD)c'_\kappa((1-t)D)c''_\kappa(D) + c'_\kappa((1-t)D)^2 \\
&\quad = c'_\kappa(tD)^2 + 2c'_\kappa(tD)c'_\kappa((1-t)D)c''_\kappa(tD + (1-t)D) + c'_\kappa((1-t)D)^2 \\
&\quad = c'_\kappa(tD)^2 + 2c'_\kappa(tD)c'_\kappa((1-t)D)c''_\kappa(tD)c''_\kappa((1-t)D) \\
&\qquad - \kappa 2 c'_\kappa(tD)^2 c'_\kappa((1-t)D)^2 + c'_\kappa((1-t)D)^2 \\
&\quad = c'_\kappa(tD)^2 + 2c'_\kappa(tD)c'_\kappa((1-t)D)c''_\kappa(tD)c''_\kappa((1-t)D) \\
&\qquad + (1 - \kappa 2 c'_\kappa(tD)^2)c'_\kappa((1-t)D)^2 \\
&\quad = c'_\kappa(tD)^2 + 2c'_\kappa(tD)c'_\kappa((1-t)D)c''_\kappa(tD)c''_\kappa((1-t)D) \\
&\qquad + (c''_\kappa(tD)^2 - \kappa c'_\kappa(tD)^2)c'_\kappa((1-t)D)^2 \\
&\quad = (1 - \kappa c'_\kappa((1-t)D)^2)c'_\kappa(tD)^2 + 2c'_\kappa(tD)c'_\kappa((1-t)D)c''_\kappa(tD)c''_\kappa((1-t)D) \\
&\qquad + c''_\kappa(tD)^2 c'_\kappa((1-t)D)^2 \\
&\quad = c''_\kappa((1-t)D)^2 c'_\kappa(tD)^2 + 2c'_\kappa(tD)c'_\kappa((1-t)D)c''_\kappa(tD)c''_\kappa((1-t)D) \\
&\qquad + c''_\kappa(tD)^2 c'_\kappa((1-t)D)^2 \\
&\quad = \big(c''_\kappa((1-t)D)c'_\kappa(tD) + c''_\kappa(tD)c'_\kappa((1-t)D)\big)^2
\end{aligned}$$

$$= c'_\kappa(tD + (1-t)D)^2$$
$$= c'_\kappa(D)^2.$$

Since $c'_\kappa(D) > 0$, we get the desired result. □

Lemma 3.13. *For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space and take $x, y \in X$ such that $0 < d(x,y) < D_\kappa$. Take $\alpha, t \in [0,1]$ and suppose that $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = tx \oplus (1-t)y$. Then two equations*

$$(t)^\kappa_D = \frac{\alpha}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)c''_\kappa(D) + (1-\alpha)^2}} \quad \textit{and} \quad (1-t)^\kappa_D = \frac{1-\alpha}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)c''_\kappa(D) + (1-\alpha)^2}}$$

hold, where $D = d(x,y)$.

Proof. From Lemma 3.6 (ii) and Lemma 3.12, we obtain the first equation by

$$(t)^\kappa_D = \frac{c'_\kappa(tD)}{c'_\kappa(D)} = \alpha \cdot \frac{c'_\kappa(tD) + c'_\kappa((1-t)D)}{c'_\kappa(D)} = \frac{\alpha}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)c''_\kappa(D) + (1-\alpha)^2}}.$$

Similarly, we can get the other by

$$(1-t)^\kappa_D = \frac{c'_\kappa((1-t)D)}{c'_\kappa(D)} = (1-\alpha) \cdot \frac{c'_\kappa(tD) + c'_\kappa((1-t)D)}{c'_\kappa(D)} = \frac{1-\alpha}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)c''_\kappa(D) + (1-\alpha)^2}}.$$

This is the conclusion. □

Theorem 3.14 (Stewart's theorem for $\overset{\kappa}{\oplus}$)**.** *For $\kappa \neq 0$, let X be a* CAT(κ) *space and take $x, y, z \in X$ such that $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$. Then*

$$\frac{1}{\kappa} c''_\kappa(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z)) \geq \frac{1}{\kappa} \cdot \frac{\alpha c''_\kappa(d(x,z)) + (1-\alpha)c''_\kappa(d(y,z))}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)c''_\kappa(d(x,y)) + (1-\alpha)^2}}$$

for any $\alpha \in [0,1]$.

Proof. Fix $\alpha \in [0,1]$ and put $D = d(x,y)$. It is enough to show the case where $D \neq 0$. Take $t \in [0,1]$ such that $tx \oplus (1-t)y = \alpha x \overset{\kappa}{\oplus} (1-\alpha)y$, and put $S = \sqrt{\alpha^2 + 2\alpha(1-\alpha)c''_\kappa(D) + (1-\alpha)^2}$. Then we have $(t)^\kappa_D = \alpha/S$ and $(1-t)^\kappa_D = (1-\alpha)/S$ from Lemma 3.13, and therefore

$$\begin{aligned}
&\frac{1}{\kappa}\left(1 - c''_\kappa(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z))\right) \\
&\quad = c_\kappa(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z)) \\
&\quad = c_\kappa(d(tx \oplus (1-t)y, z)) \\
&\quad \leq (t)^\kappa_D \left(c_\kappa(d(x,z)) - c_\kappa((1-t)D)\right) + (1-t)^\kappa_D \left(c_\kappa(d(y,z)) - c_\kappa(tD)\right) \\
&\quad = (t)^\kappa_D \cdot \frac{1}{\kappa}\left(c''_\kappa((1-t)D) - c''_\kappa(d(x,z))\right) + (1-t)^\kappa_D \cdot \frac{1}{\kappa}\left(c''_\kappa(tD) - c''_\kappa(d(y,z))\right) \\
&\quad = \frac{(t)^\kappa_D c''_\kappa((1-t)D) + (1-t)^\kappa_D c''_\kappa(tD)}{\kappa} - \frac{(t)^\kappa_D c''_\kappa(d(x,z)) + (1-t)^\kappa_D c''_\kappa(d(y,z))}{\kappa} \\
&\quad = \frac{\alpha c''_\kappa((1-t)D) + (1-\alpha)c''_\kappa(tD)}{\kappa S} - \frac{\alpha c''_\kappa(d(x,z)) + (1-\alpha)c''_\kappa(d(y,z))}{\kappa S}.
\end{aligned}$$

Using Lemma 3.6 (ii) and Lemma 3.12, we get

$$\begin{aligned}\alpha c_\kappa''((1-t)D) + (1-\alpha)c_\kappa''(tD) &= \frac{c_\kappa'(tD)c_\kappa''((1-t)D) + c_\kappa'((1-t)D)c_\kappa''(tD)}{c_\kappa'(tD) + c_\kappa'((1-t)D)} \\ &= \frac{c_\kappa'(tD + (1-t)D)}{c_\kappa'(tD) + c_\kappa'((1-t)D)} \\ &= \frac{c_\kappa'(D)}{c_\kappa'(tD) + c_\kappa'((1-t)D)} \\ &= S.\end{aligned}$$

This follows that

$$\frac{1}{\kappa} c_\kappa''(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z)) \geq \frac{1}{\kappa} \cdot \frac{\alpha c_\kappa''(d(x,z)) + (1-\alpha)c_\kappa''(d(y,z))}{S},$$

which is the conclusion. ◻

If a CAT(κ) space X coincides with M_κ, then the inequality in Theorem 3.14 holds as an equation.

Theorem 3.15. *For $\kappa \in \mathbb{R}$, let X be a* CAT(κ) *space and take $x, y, z \in X$ such that $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$. Then*

$$\begin{aligned}c_\kappa(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z)) &\leq \alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z)) \\ &\quad - \frac{2\alpha(1-\alpha)c_\kappa(d(x,y))}{1+S} \cdot \frac{\alpha c_\kappa''(d(x,z)) + (1-\alpha)c_\kappa''(d(y,z))}{S}\end{aligned}$$

for any $\alpha \in [0,1]$, where $S = \sqrt{\alpha^2 + 2\alpha(1-\alpha)c_\kappa''(d(x,y)) + (1-\alpha)^2}$.

Proof. Fix $\alpha \in [0,1]$. First, we suppose that $\kappa = 0$. Then we have $S = 1$ and thus

$$\frac{2\alpha(1-\alpha)c_\kappa(d(x,y))}{1+S} \cdot \frac{\alpha c_\kappa''(d(x,z)) + (1-\alpha)c_\kappa''(d(y,z))}{S} = \alpha(1-\alpha)d(x,y)^2.$$

Therefore, since $\alpha x \overset{0}{\oplus} (1-\alpha)y = \alpha x \oplus (1-\alpha)y$, we obtain the conclusion from Stewart's theorem on CAT(0) spaces. Next, we consider the case where $\kappa \neq 0$. Then we get

$$\begin{aligned}&c_\kappa(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z)) - \big(\alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))\big) \\ &\quad = \frac{1}{\kappa}\big(1 - c_\kappa''(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z))\big) - \big(\alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))\big) \\ &\quad \leq \frac{1}{\kappa}\left(1 - \frac{\alpha c_\kappa''(d(x,z)) + (1-\alpha)c_\kappa''(d(y,z))}{S}\right) - \big(\alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))\big) \\ &\quad = \frac{1}{\kappa}\left(1 - \frac{1 - \kappa\big(\alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))\big)}{S}\right) - \big(\alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))\big) \\ &\quad = \left(\frac{1}{\kappa} - \big(\alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))\big)\right)\left(1 - \frac{1}{S}\right) \\ &\quad = \left(\frac{1}{\kappa} - \big(\alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))\big)\right) \cdot \frac{-2\alpha(1-\alpha)(1 - c_\kappa''(d(x,y)))}{S(1+S)} \\ &\quad = -\frac{2\alpha(1-\alpha)c_\kappa(d(x,y))}{1+S} \cdot \frac{1 - \kappa\big(\alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))\big)}{S} \\ &\quad = -\frac{2\alpha(1-\alpha)c_\kappa(d(x,y))}{1+S} \cdot \frac{\alpha c_\kappa''(d(x,z)) + (1-\alpha)c_\kappa''(d(y,z))}{S},\end{aligned}$$

which is the desired result. □

Corollary 3.16. *For $\kappa \in \mathbb{R}$, let X be a* CAT(κ) *space. Take $x, y, z \in X$ and $\alpha \in [0,1]$ such that $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ and $\alpha c_\kappa''(d(x,z)) + (1-\alpha)c_\kappa''(d(y,z)) \geq 0$. Then*

$$c_\kappa(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z)) \leq \alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z)).$$

Proof. By the assumption, we get $d(x,y) < D_\kappa$ and hence $c_\kappa(d(x,y)) \geq 0$. Thus we get the conclusion by Theorem 3.15. □

If $d(x,z), d(y,z) \in [0, D_\kappa/2]$, then a condition

$$\alpha c_\kappa''(d(x,z)) + (1-\alpha)c_\kappa''(d(y,z)) \geq 0 \tag{$*$}$$

is always true. It means that, if $\kappa \leq 0$, then $(*)$ always holds. However, in the case where $\kappa > 0$, if $d(x,z) > D_\kappa/2$ and $d(y,z) \leq D_\kappa/2$, then $(*)$ does not always hold. Indeed, $(*)$ is true only when $0 \leq \alpha \leq c_\kappa''(d(y,z))/(c_\kappa''(d(y,z)) - c_\kappa''(d(x,z)))$. Similarly, if $d(x,z) \leq D_\kappa/2$ and $d(y,z) > D_\kappa/2$, then $(*)$ if and only if $1 \geq \alpha \geq c_\kappa''(d(y,z))/(c_\kappa''(d(y,z)) - c_\kappa''(d(x,z)))$. Moreover, if $d(x,z)$ and $d(y,z)$ are both greater than $D_\kappa/2$, then $(*)$ is false for all $\alpha \in [0,1]$.

If a CAT(κ) space X is admissible, then $(*)$ is true for any $x, y, z \in X$ and $\alpha \in [0,1]$. That is, the following holds.

Corollary 3.17 (Sudo [29])**.** *For $\kappa \in \mathbb{R}$, let X be an admissible* CAT(κ) *space. Take $x, y, z \in X$ and $\alpha \in [0,1]$ arbitrarily. Then*

$$c_\kappa(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z)) \leq \alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z)).$$

Next, we consider natures of the function $\overline{\eta}_\alpha^\kappa$.

Lemma 3.18. *Let $k \in]0,1[$ and define $f\colon]0,\pi[\to \mathbb{R}$ by $f(x) = (\sin kx)/\sin x$ for $x \in]0,\pi[$. Then f is strictly increasing.*

Lemma 3.19. *For $\kappa > 0$ and $\alpha \in]0,1[$, the following hold:*

(i) $\lim_{d\to 0} \overline{\eta}_\alpha^\kappa(d) = \alpha$;
(ii) *if $\alpha < 1/2$, then* $\lim_{d\to D_\kappa} \overline{\eta}_\alpha^\kappa(d) = 0$;
(iii) *if $\alpha > 1/2$, then* $\lim_{d\to D_\kappa} \overline{\eta}_\alpha^\kappa(d) = 1$;
(iv) *if $\alpha < 1/2$, then $\overline{\eta}_\alpha^\kappa$ is strictly decreasing;*
(v) *if $\alpha > 1/2$, then $\overline{\eta}_\alpha^\kappa$ is strictly increasing.*

Proof. It suffices to show the case where $\kappa = 1$. Hence we hereinafter assume that $\kappa = 1$.

Let $\tan^{-1}\colon \mathbb{R} \cup \{\pm\infty\} \to [0,\pi[$ be the inverse of $\tan\colon [0,\pi[\to \mathbb{R} \cup \{\pm\infty\}$, see Section 2.5. Define a function $g\colon]-\pi,\pi[\to]-\pi,\pi[$ by

$$g(d) = \begin{cases} \tan^{-1} \dfrac{\alpha \sin d}{1-\alpha+\alpha\cos d} & (\text{if } d \geq 0); \\ -\pi + \tan^{-1} \dfrac{\alpha \sin d}{1-\alpha+\alpha\cos d} & (\text{if } d < 0) \end{cases}$$

for $d \in]-\pi,\pi[$. Then g is differentiable on $]-\pi,\pi[$, and we get $\overline{\eta}_\alpha^\kappa(d) = g(d)/d$ and

$$g'(d) = \frac{\alpha(\alpha + (1-\alpha)\cos d)}{\alpha^2 + 2\alpha(1-\alpha)\cos d + (1-\alpha)^2}$$

for every $d \in]-\pi, \pi[$. Therefore, the limit (i) is obtained by

$$\lim_{d\to 0} \overline{\eta}^{\kappa}_{\alpha}(d) = \lim_{d\searrow 0} \frac{g(d)}{d} = \lim_{d\searrow 0} \frac{g(d)-g(0)}{d-0} = g'(0) = \alpha.$$

Next, we show (ii) and (iii). We know that

$$\lim_{d\nearrow \pi} \frac{\alpha \sin d}{1-\alpha+\alpha\cos d} = 0.$$

for every $\alpha \in]0,1[\setminus \{1/2\}$.

Assume that $\alpha < 1/2$. Then since $1-\alpha+\alpha\cos d > 0$ for every $d \in]0,\pi[$, we obtain

$$\lim_{d\nearrow \pi} g(d) = \lim_{d\nearrow \pi} \tan^{-1} \frac{\alpha \sin d}{1-\alpha+\alpha\cos d} = 0.$$

This means that $\lim_{d\to\pi} \overline{\eta}^{\kappa}_{\alpha}(d) = 0$, therefore we get (ii).

To show (iii), assume that $\alpha > 1/2$. Let $\cos^{-1}\colon [-1,1] \to [0,\pi]$ be the inverse of the trigonometric cosine function. Then, $1-\alpha+\alpha\cos d < 0$ for every $d \in]\cos^{-1}(-(1-\alpha)/\alpha), \pi[$. Therefore we have

$$\lim_{d\nearrow \pi} g(d) = \lim_{d\nearrow \pi} \tan^{-1} \frac{\alpha \sin d}{1-\alpha+\alpha\cos d} = \pi.$$

Hence $\lim_{d\to\pi} \overline{\eta}^{\kappa}_{\alpha}(d) = 1$, which concludes that (iii) holds.

We show (iv). Let $\kappa = 1$, $\alpha \in]0,1/2[$, $d_1, d_2 \in]0,\pi[$ and suppose $d_1 < d_2$. Put $\sigma_1 = \overline{\eta}^{\kappa}_{\alpha}(d_1)$ and $\sigma_2 = \overline{\eta}^{\kappa}_{\alpha}(d_2)$. Then we get $\sigma_1, \sigma_2 \in]0,1/2[$ by Lemma 2.14. Furthermore, we obtain from the definition of $\overline{\eta}^{\kappa}_{\alpha}$ that

$$\alpha = \zeta^{\kappa}_{d_2}(\sigma_2) = \frac{\sin(\sigma_2 d_2)}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}. \tag{$**$}$$

Define a strictly concave function $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = \alpha\cos((1-t)d_1) + (1-\alpha)\cos(td_1)$$

for $t \in [0,1]$. Then σ_1 is a unique maximizer of g by Lemma 3.5. We also have

$$g(t) = \frac{\sin(\sigma_2 d_2)\cos((1-t)d_1) + \sin((1-\sigma_2)d_2)\cos(td_1)}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}$$

for any $t \in [0,1]$ from the formula $(**)$. Hence

$$g'(t) = \frac{d_1\big(\sin(\sigma_2 d_2)\sin((1-t)d_1) - \sin((1-\sigma_2)d_2)\sin(td_1)\big)}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}$$

for any $t \in]0,1[$. Put $C = d_1/(\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2))$. Then we obtain $C > 0$ and

$$\frac{1}{C} g'(\sigma_2) = \sin(\sigma_2 d_2)\sin((1-\sigma_2)d_1) - \sin((1-\sigma_2)d_2)\sin(\sigma_2 d_1).$$

Put $p = (d_1+d_2)/2$, $q = (d_2-d_1)/2$, and $k = 1-2\sigma_2$. Then, since

$$\begin{aligned}\sin((a+b)(c-d))\sin((a-b)(c+d)) - \sin((a+b)(c+d))\sin((a-b)(c-d)) \\ = -\sin 2ac \sin 2bd + \sin 2ad \sin 2bc\end{aligned}$$

for any $a, b, c, d \in \mathbb{R}$, we get

$$\begin{aligned}\frac{1}{C}g'(\sigma_2) &= \sin\Big((p+q)\Big(\frac{1}{2}-\frac{1}{2}k\Big)\Big)\sin\Big((p-q)\Big(\frac{1}{2}+\frac{1}{2}k\Big)\Big)\\ &\quad -\sin\Big((p+q)\Big(\frac{1}{2}+\frac{1}{2}k\Big)\Big)\sin\Big((p-q)\Big(\frac{1}{2}-\frac{1}{2}k\Big)\Big)\\ &= -\sin kp \sin q + \sin kq \sin p\\ &= \sin p \sin q\left(\frac{\sin kq}{\sin q} - \frac{\sin kp}{\sin p}\right).\end{aligned}$$

Since $0 < q < p < \pi$ and $0 < k < 1$, we have $g'(\sigma_2) > 0$ from Lemma 3.18. Therefore we obtain $\sigma_1 > \sigma_2$. This implies $\overline{\eta}_\alpha^\kappa(d_1) > \overline{\eta}_\alpha^\kappa(d_2)$, and hence we get (iv). Furthermore, from (iv) and $\overline{\eta}_\alpha^\kappa(d) + \overline{\eta}_{1-\alpha}^\kappa(d) = 1$ for every $d \in [0, \pi[$, we also have (v). □

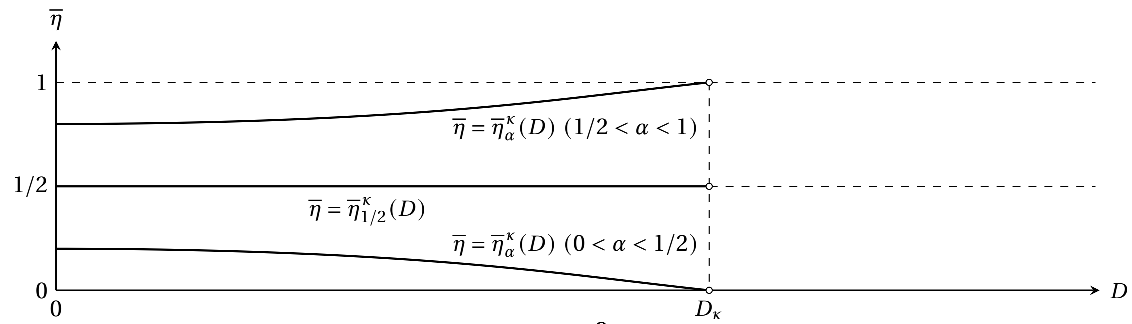
For $\kappa > 0$, Lemma 3.19 implies that the greater the distance between two points x and y, the farther the point $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y$ is from the midpoint of x and y as a ratio than the point $\alpha x \oplus (1-\alpha)y$. In the same fashion, we get natures of $\overline{\eta}_\alpha^\kappa$ for $\kappa < 0$ as follows.

Lemma 3.20. *For $\kappa < 0$ and $\alpha \in]0, 1[$, the following hold:*

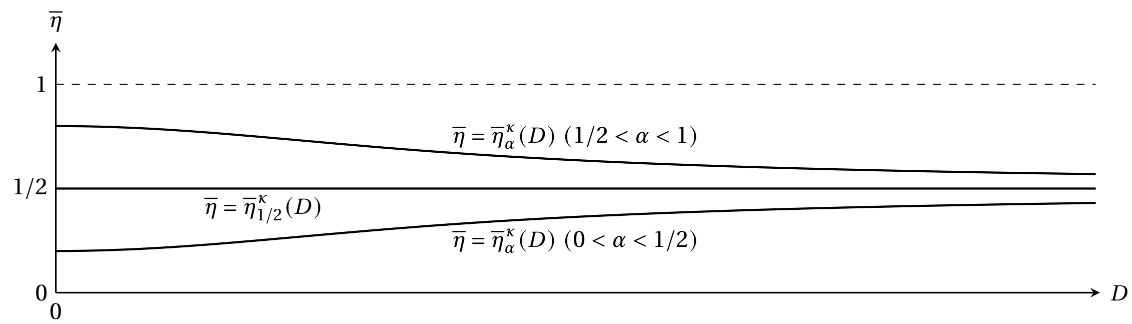
(i) $\lim_{d\to 0} \overline{\eta}_\alpha^\kappa(d) = \alpha$;
(ii) $\lim_{d\to \infty} \overline{\eta}_\alpha^\kappa(d) = 1/2$;
(iii) *if* $\alpha < 1/2$, *then* $\overline{\eta}_\alpha^\kappa$ *is strictly increasing;*
(iv) *if* $\alpha > 1/2$, *then* $\overline{\eta}_\alpha^\kappa$ *is strictly decreasing.*

This implies the following fact: For $\kappa < 0$, the greater the distance between two points x and y, the closer the point $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y$ is from the midpoint of x and y as a ratio than the point $\alpha x \oplus (1-\alpha)y$.

The following figures show graphs of $\overline{\eta}_\alpha^\kappa$ for several α.



$\kappa > 0$



$\kappa < 0$

Lemma 3.21. *Let X be a uniquely D_κ-geodesic space and take $x, y \in X$ such that $d(x,y) < D_\kappa$. Then* $\lim_{\alpha\to 0}(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y) = y$. *Similarly,* $\lim_{\alpha\to 1}(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y) = x$.

Proof. It is obvious if $x = y$, thus suppose that $x \neq y$. By symmetry, it suffices to show that $\lim_{\alpha\to 0}(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y) = y$. We assume that $\kappa \neq 0$ since the case where $\kappa = 0$ is obvious. Put $D = d(x,y)$. Then for every $\alpha \in]0,1[$, we have $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = (\zeta_D^\kappa)^{-1}(\alpha)x \oplus (\zeta_D^\kappa)^{-1}(1-\alpha)y$. Therefore we obtain

$$d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, y) = d((\zeta_D^\kappa)^{-1}(\alpha)x \oplus (\zeta_D^\kappa)^{-1}(1-\alpha)y, y) = (\zeta_D^\kappa)^{-1}(\alpha)D$$

for every $\alpha \in]0,1[$. If $\kappa < 0$, then we obtain from Lemma 2.18 that

$$(\zeta_D^\kappa)^{-1}(\alpha) < \frac{c'_\kappa(D)}{D}\alpha \to 0$$

as $\alpha \to 0$. Otherwise, if $\kappa > 0$, then we have $(\zeta_D^\kappa)^{-1}(\alpha) < \alpha$ for any $\alpha \in]0,1/2[$ by Corollary 2.15, and hence $(\zeta_D^\kappa)^{-1}(\alpha) \to 0$ as $\alpha \to 0$. Thus we get the conclusion. □

Lemma 3.22. *Let X be a uniquely D_κ-geodesic space and $x, y \in X$ such that $0 < d(x,y) < D_\kappa$. Take $\{\alpha_n\} \subset [0,1]$ and define a sequence $\{y_n\}$ on X by $y_n = \alpha_n x \overset{\kappa}{\oplus} (1-\alpha_n)y$ for every $n \in \mathbb{N}$. Then $y_n \to y$ if and only if $\alpha_n \to 0$.*

Proof. The if part is immediately obtained by Lemma 3.21, thus we show the only if part. Suppose that $y_n \to y$. Put $D = d(x,y)$ and $\beta_n = (\zeta_D^\kappa)^{-1}(\alpha_n)$ for every $n \in \mathbb{N}$. Then we have $y_n = \beta_n x \oplus (1-\beta_n)y$ for every $n \in \mathbb{N}$. Since $y_n \to y$, we get $\beta_n = d(y_n, y)/d(x,y) \to 0$ as $n \to \infty$. This implies $\alpha_n = \zeta_D^\kappa(\beta_n) \to \zeta_D^\kappa(0) = 0$. □

Lemma 3.23. *Let X be a uniquely D_κ-geodesic space and $x, y \in X$ such that $x \neq y$. Take $\{\alpha_n\} \subset [0,1]$. Suppose that a sequence $\{y_n\}$ on X satisfies $d(x, y_n) < D_\kappa$ for all $n \in \mathbb{N}$, $\limsup_{n\to\infty} d(x, y_n) < D_\kappa$, and $y = \alpha_n x \overset{\kappa}{\oplus} (1-\alpha_n)y_n$ for all $n \in \mathbb{N}$. Then $y_n \to y$ if and only if $\alpha_n \to 0$.*

Proof. It is obvious if $\kappa = 0$; hence we assume that $\kappa \neq 0$. Put $d_n = d(x, y_n)$ and $\beta_n = \left(\zeta_{d_n}^\kappa\right)^{-1}(\alpha_n)$ for every $n \in \mathbb{N}$. Then $y = \beta_n x \oplus (1-\beta_n)y_n$ for any $n \in \mathbb{N}$. Note that $0 < d(x,y) \le d(x,y_n)$ and $d(y, y_n) = d(x, y_n)\beta_n$ hold for all $n \in \mathbb{N}$. Thus $y_n \to y$ if and only if $\beta_n \to 0$. Therefore, we prove $\beta_n \to 0$ if and only if $\alpha_n \to 0$.

First, we consider the case where $\kappa > 0$. Assume that $\alpha_n \to 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\sup_{n \ge n_0} \alpha_n < 1/2$. Hence, we obtain from Corollary 2.15 that $0 \le \beta_n \le \alpha_n$ for $n \ge n_0$, which implies $\beta_n \to 0$.

Conversely, suppose that $\beta_n \to 0$. Then, since $\limsup_{n\to\infty} d_n < D_\kappa$, we get

$$0 \le \alpha_n = \zeta_{d_n}^\kappa(\beta_n) \le \frac{d_n}{c'_\kappa(d_n)}\beta_n = \frac{\sqrt{\kappa}\, d_n}{\sin(\sqrt{\kappa}\, d_n)}\beta_n \to 0$$

by Lemma 2.17.

Next, we consider the case where $\kappa < 0$. Suppose that $\alpha_n \to 0$. Then, from Lemma 2.18 and $\limsup_{n\to\infty} d_n < \infty$, we have

$$0 \le \beta_n = \left(\zeta_{d_n}^\kappa\right)^{-1}(\alpha_n) < \frac{c'_\kappa(d_n)}{d_n}\alpha_n = \frac{\sinh(\sqrt{-\kappa}\, d_n)}{\sqrt{-\kappa}\, d_n}\alpha_n \to 0,$$

which implies $\beta_n \to 0$. Conversely, assume that $\beta_n \to 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\sup_{n \ge n_0} \beta_n < 1/2$. It implies from Corollary 2.16 that $0 \le \alpha_n = \zeta_{d_n}^\kappa(\beta_n) \le \beta_n$ for any $n \ge n_0$, which is the conclusion. □

Lemma 3.24. *Let X be a uniquely D_κ-geodesic space and $\{x_n\}$, $\{y_n\}$ sequences on X such that $d(x_n, y_n) < D_\kappa$ for all $n \in \mathbb{N}$. In addition, suppose that* $\limsup_{n\to\infty} d(x_n, y_n) < D_\kappa$*. Take $\{\alpha_n\} \subset [0,1]$ such that* $\lim_{n\to\infty} \alpha_n = 0$*. Then* $\lim_{n\to\infty}(\alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) y_n) = y$ *if $y_n \to y \in X$. Similarly,* $\lim_{n\to\infty}(\alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) y_n) = x$ *if $x_n \to x \in X$.*

Proof. Suppose that $y_n \to y$. Then

$$\begin{aligned} d(\alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) y_n, y) &\le d(\alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) y_n, y_n) + d(y_n, y) \\ &= \left(\zeta^\kappa_{d(x_n,y_n)}\right)^{-1}(\alpha_n) d(x_n, y_n) + d(y_n, y) \end{aligned}$$

for any $n \in \mathbb{N}$. Put $S = \limsup_{n\to\infty} d(x_n, y_n) < D_\kappa$.

First, consider the case where $\kappa > 0$. Then there exists $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $\sup_{n\ge n_0} \alpha_n < 1/2$ and $\sup_{n\ge n_0} d(x_n, y_n) < S + \varepsilon < D_\kappa$. Thus we get $\left(\zeta^\kappa_{d(x_n,y_n)}\right)^{-1}(\alpha_n) \le \alpha_n$ for any $n \ge n_0$ by Corollary 2.15. It deduces

$$\limsup_{n\to\infty} \left(\left(\zeta^\kappa_{d(x_n,y_n)}\right)^{-1}(\alpha_n) d(x_n, y_n) + d(y_n, y) \right) \le \lim_{n\to\infty} \left(\alpha_n d(x_n, y_n) + d(y_n, y) \right) = 0,$$

which implies $\lim_{n\to\infty} d(\alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) y_n, y) = 0$.

Next, we assume that $\kappa \le 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\sup_{n\ge n_0} \alpha_n < 1/2$ and $\sup_{n\ge n_0} d(x_n, y_n) < \infty$. Put $M = \sup_{n\ge n_0} d(x_n, y_n) < \infty$. We divide into the following cases: (i) $\kappa = 0$; (ii) $\kappa < 0$.

(i) If $\kappa = 0$, then we have

$$\limsup_{n\to\infty} \left(\left(\zeta^\kappa_{d(x_n,y_n)}\right)^{-1}(\alpha_n) d(x_n, y_n) + d(y_n, y) \right) = \lim_{n\to\infty} \left(\alpha_n d(x_n, y_n) + d(y_n, y) \right) = 0$$

and hence we get $\lim_{n\to\infty} d(\alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) y_n, y) = 0$.

(ii) Let $\kappa < 0$. From Lemma 3.20 (iii), we get

$$\left(\zeta^\kappa_{d(x_n,y_n)}\right)^{-1}(\alpha_n) = \overline{\eta}^\kappa_{\alpha_n}(d(x_n, y_n)) \le \overline{\eta}^\kappa_{\alpha_n}(M) = (\zeta^\kappa_M)^{-1}(\alpha_n)$$

for any $n \ge n_0$. We also obtain $(\zeta^\kappa_M)^{-1}(\alpha_n) \to (\zeta^\kappa_M)^{-1}(0) = 0$ as $n \to \infty$ by Lemma 2.14. Hence we have $\left(\zeta^\kappa_{d(x_n,y_n)}\right)^{-1}(\alpha_n) \to 0$ as $n \to \infty$. Therefore,

$$0 \le \limsup_{n\to\infty} d(\alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) y_n, y) \le \limsup_{n\to\infty} \left((\zeta^\kappa_M)^{-1}(\alpha_n) M + d(y_n, y) \right) = 0.$$

This implies the conclusion. □

The next lemma and its corollary are used so as to prove a Mann type fixed point approximation theorem.

Lemma 3.25. *For $\kappa \in \mathbb{R}$, let $\{d_n\}$ be a nonnegative real sequence such that* $M = \sup_{n\in\mathbb{N}} d_n < D_\kappa$*. Let $\{\alpha_n\}$ be a real sequence on $[0,1]$, and put $\beta_n = \zeta^\kappa_{d_n}(\alpha_n)$ for every $n \in \mathbb{N}$. Then* $\liminf_{n\to\infty} \alpha_n > 0$ *if and only if* $\liminf_{n\to\infty} \beta_n > 0$.

Proof. Assume that $M > 0$ since it is clear if $M = 0$. If $\kappa = 0$, then we have $\beta_n = \alpha_n$ for all $n \in \mathbb{N}$ and thus we assume $\kappa \neq 0$.

First, we consider the case where $\kappa < 0$. Put $\varepsilon = \liminf_{n\to\infty} \alpha_n \in]0,1]$. Then there exists $n_0 \in \mathbb{N}$ such that $\alpha_n \ge \varepsilon/2$ for all $n \ge n_0$. We also have $\eta^\kappa_{\alpha_n}(d) \ge 1/2$ if and only if $\alpha_n \ge 1/2$ for each $d \in]0, D_\kappa[$. Hence

$$\begin{aligned}\beta_n = \eta^\kappa_{\alpha_n}(d_n) &\geq \min\left\{\alpha_n, \eta^\kappa_{\alpha_n}(d_n), \frac{1}{2}\right\}\\ &\geq \min\left\{\alpha_n, \eta^\kappa_{\alpha_n}(M), \frac{1}{2}\right\} = \min\left\{\alpha_n, \zeta^\kappa_M(\alpha_n), \frac{1}{2}\right\} \geq \min\left\{\frac{\varepsilon}{2}, \zeta^\kappa_M\left(\frac{\varepsilon}{2}\right)\right\} > 0\end{aligned}$$

for any $n \geq n_0$ from Lemma 2.20 (iv) and the strict increasingness of ζ^κ_M. Thus $\liminf_{n\to\infty} \beta_n > 0$ holds.

Conversely, suppose that $\varepsilon' := \liminf_{n\to\infty} \beta_n > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\beta_n \geq \varepsilon'/2$ for all $n \geq n_0$. It implies

$$\alpha_n = (\zeta^\kappa_{d_n})^{-1}(\beta_n) \geq \min\left\{\beta_n, \frac{1}{2}, (\zeta^\kappa_{d_n})^{-1}(\beta_n)\right\} = \min\left\{\beta_n, \frac{1}{2}\right\} \geq \frac{\varepsilon'}{2} > 0$$

for any $n \geq n_0$ by using Lemma 3.20 (iii) and hence $\liminf_{n\to\infty} \alpha_n > 0$ holds.

Next, we consider the case where $\kappa > 0$. Suppose that $\varepsilon := \liminf_{n\to\infty} \alpha_n > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\alpha_n \geq \varepsilon/2$ for all $n \geq n_0$. It follows that

$$\beta_n \geq \min\left\{\alpha_n, \zeta^\kappa_{d_n}(\alpha_n), \frac{1}{2}\right\} \geq \min\left\{\alpha_n, \frac{1}{2}\right\} \geq \frac{\varepsilon}{2} > 0$$

for any $n \in \mathbb{N}$ by Corollary 2.15. This concludes $\liminf_{n\to\infty} \beta_n > 0$.

Finally, we suppose $\varepsilon' := \liminf_{n\to\infty} \beta_n > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\beta_n \geq \varepsilon/2$ for all $n \geq n_0$. Thus, using Lemma 2.17, we obtain

$$\begin{aligned}\alpha_n = (\zeta^\kappa_{d_n})^{-1}(\beta_n) &\geq \begin{cases}\beta_n & (\text{if } d_n = 0);\\ \dfrac{c'_\kappa(d_n)}{d_n}\beta_n & (\text{if } d_n \neq 0)\end{cases}\\ &\geq \frac{c'_\kappa(M)}{M}\beta_n \geq \frac{c'_\kappa(M)}{2M}\varepsilon > 0\end{aligned}$$

for any $n \geq n_0$ since $c'_\kappa(d)/d = \sin(\sqrt{\kappa}d)/(\sqrt{\kappa}d) > 0$ for $d \in]0, D_\kappa[$. Therefore we get $\liminf_{n\to\infty} \alpha_n > 0$, which is the desired result. □

Corollary 3.26. *For $\kappa \in \mathbb{R}$, let $\{d_n\}$ be a nonnegative real sequence such that $M = \sup_{n\in\mathbb{N}} d_n < D_\kappa$. Let $\{\alpha_n\}$ be a real sequence on $[0,1]$, and put $\beta_n = \zeta^\kappa_{d_n}(\alpha_n)$ for every $n \in \mathbb{N}$. Then $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ if and only if $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.*

Proof. We obtain that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ is equivalent to the conjunction of $\liminf_{n\to\infty} \alpha_n > 0$ and $\liminf_{n\to\infty}(1-\alpha_n) > 0$, and so is $\{\beta_n\}$. Therefore, by Lemma 3.25, we get the conclusion. □

3.2 κ-convex combination on model spaces

In this section, we consider a behavior of the κ-convex combination on geodesic spaces with a constant curvature 1 or -1.

3.2.1 1-convex combination on the unit sphere in Hilbert spaces

We observe the nature of the 1-convex combination on a unit sphere of a Hilbert space to know a relation between $\oplus$ and $\overset{1}{\oplus}$.

Let $S_{\mathcal{H}}$ be a unit sphere embedded in a real Hilbert space $\mathcal{H}$, that is, $S_{\mathcal{H}} = \{x \in \mathcal{H} \mid \|x\| = 1\}$. Define $d\colon S_{\mathcal{H}} \to [0,\pi]$ by $d(x,y) = \cos^{-1}\langle x, y\rangle$ for each $x, y \in S_{\mathcal{H}}$, where $\cos^{-1}\colon [-1,1] \to [0,\pi]$. Then $(S_{\mathcal{H}}, d)$ is a metric space. Moreover, for any two points $x, y \in S_{\mathcal{H}}$ such that $d(x,y) < \pi$, there exists a unique geodesic joining x and y. Indeed, for $x, y \in S_{\mathcal{H}}$ such that $d(x,y) < \pi$, a function $\gamma_{x,y}\colon [0,1] \to S_{\mathcal{H}}$ defined by

$$\gamma_{x,y}(t) = (t)^1_{d(x,y)}x + (1-t)^1_{d(x,y)}y = \begin{cases} \dfrac{\sin(td(x,y))}{\sin d(x,y)}x + \dfrac{\sin((1-t)d(x,y))}{\sin d(x,y)}y & (\text{if } x \neq y); \\ x & (\text{if } x = y) \end{cases}$$

for $t \in [0,1]$ is a unique geodesic joining x and y. Thus $(S_{\mathcal{H}}, d)$ is a uniquely π-geodesic space.

We also know that $S_{\mathcal{H}}$ is a complete CAT(1) space. If $\mathcal{H} = \mathbb{R}^3$, then $S_{\mathcal{H}}$ represents a model of the unit sphere $\mathbb{S}^2$, which has a constant curvature 1.

In what follows, a symbol $\oplus$ denotes a convex combination on $S_{\mathcal{H}}$, $[x,y]$ denotes a geodesic segment on $S_{\mathcal{H}}$ joining $x, y \in S_{\mathcal{H}}$, and $[x,y]_{\mathcal{H}}$ denotes a geodesic segment on $\mathcal{H}$ joining $x, y \in \mathcal{H}$. That is, $[x,y] = \{tx \oplus (1-t)y \in S_{\mathcal{H}} \mid t \in [0,1]\}$, and $[x,y]_{\mathcal{H}} = \{tx + (1-t)y \in \mathcal{H} \mid t \in [0,1]\}$. Furthermore, $0_{\mathcal{H}}$ stands for the origin of $\mathcal{H}$.

Now we consider the 1-convex combination on $S_{\mathcal{H}}$. Suppose that $\alpha x \overset{1}{\oplus} (1-\alpha)y = tx \oplus (1-t)y$ for some $x, y \in S_{\mathcal{H}}$, $t \in [0,1]$, and $\alpha \in [0,1]$. Then from Lemma 3.13, we get

$$\begin{aligned} \alpha x \overset{1}{\oplus} (1-\alpha)y &= (t)^1_D x + (1-t)^1_D y \\ &= \frac{\alpha x + (1-\alpha)y}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cos D + (1-\alpha)^2}}, \end{aligned}$$

where $D = d(x,y)$. We also have

$$\begin{aligned} \|\alpha x + (1-\alpha)y\|^2 &= \alpha^2\|x\|^2 + 2\alpha(1-\alpha)\langle x, y\rangle + (1-\alpha)^2\|y\|^2 \\ &= \alpha^2 + 2\alpha(1-\alpha)\cos D + (1-\alpha)^2. \end{aligned}$$

Therefore, we get the following.

Theorem 3.27. *Let $x, y \in S_{\mathcal{H}}$ such that $d(x,y) < \pi$. Then for any $\alpha \in [0,1]$,*

$$\alpha x \overset{1}{\oplus} (1-\alpha)y = \frac{\alpha x + (1-\alpha)y}{\|\alpha x + (1-\alpha)y\|}.$$

Actually, it is also verified by the definition of 1-convex combination (Definition 3.4). Indeed, putting $p = tx + (1-t)y$ and $w = p/\|p\|$, we have

$$\langle tx + (1-t)y, w\rangle = \|p\| \geq \langle p, z\rangle = \langle tx + (1-t)y, z\rangle$$

for any $z \in S_{\mathcal{H}}$, and hence

$$\alpha x \overset{1}{\oplus} (1-\alpha)y = \operatorname*{argmax}_{z \in S_{\mathcal{H}}} \big(t\cos d(x,z) + (1-t)\cos d(y,z)\big) = \operatorname*{argmax}_{z \in S_{\mathcal{H}}} \langle tx + (1-t)y, z\rangle = w.$$

Corollary 3.28. *Take $x, y \in S_{\mathcal{H}}$ such that $d(x,y) < \pi$. For $\alpha \in [0,1]$, let $u = \alpha x + (1-\alpha)y \in \mathcal{H}$ and $v = \alpha x \overset{1}{\oplus} (1-\alpha)y \in [x,y]$. Then three points u, v, and $0_{\mathcal{H}}$ are on a straight line.*

Theorem 3.27 implies that a point $\alpha x \overset{1}{\oplus} (1-\alpha)y \in S_{\mathcal{H}}$ is a projection of $\alpha x + (1-\alpha)y \in \mathcal{H}$ onto the unit sphere $S_{\mathcal{H}}$.

Lemma 3.29. *Take $x, y \in S_{\mathcal{H}}$ such that $d(x,y) < \pi$. Let $k, l \in \,]0,1]$ and put $x' = kx$, $y' = ly$. Then the geodesic segment $[x,y] \subset S_{\mathcal{H}}$ is expressed by*

$$[x,y] = \left\{ \frac{tx' + (1-t)y'}{\|tx' + (1-t)y'\|} \;\middle|\; t \in [0,1] \right\} = \left\{ \frac{p}{\|p\|} \;\middle|\; p \in [x', y']_{\mathcal{H}} \right\}.$$

Proof. Take $u \in [x,y]$ arbitrarily. Then there exists $t \in [0,1]$ such that $u = tx \overset{1}{\oplus} (1-t)y$ by Lemma 3.9. Thus, putting $t' = tl/(tl + (1-t)k)$, we get

$$u = \frac{tx + (1-t)y}{\|tx + (1-t)y\|} = \frac{t'x' + (1-t')y'}{\|t'x' + (1-t')y'\|}.$$

On the other hand, take $s \in [0,1]$ and let $u' = (sx' + (1-s)y')/\|sx' + (1-s)y'\|$. Then putting $s' = sk/(sk + (1-s)l)$, we obtain

$$u' = \frac{sx' + (1-s)y'}{\|sx' + (1-s)y'\|} = \frac{s'x + (1-s')y}{\|s'x + (1-s')y\|} = s'x \overset{1}{\oplus} (1-s')y \in [x,y],$$

which implies the conclusion. □

Corollary 3.30. *Take $x, y \in S_{\mathcal{H}}$ such that $d(x,y) < \pi$. Let $k, l \in \,]0,1]$ and put $x' = kx$, $y' = ly$. Then $v/\|v\| \in [x,y]$ for any $v \in [x', y']_{\mathcal{H}}$.*

Using the 1-convex combination and the fact above, we can get a result which can be said to be Ceva's theorem on the unit sphere.

Theorem 3.31. *Let S be a nonempty convex subspace of $S_{\mathcal{H}}$ such that $d(u,v) < \pi$ for any $u, v \in S$. Let $\triangle(x,y,z)$ be a geodesic triangle on S such that $[x,y] \cap [y,z] \cap [z,x] = \emptyset$. For $\alpha, \beta, \gamma \in \,]0,1[$, take $p = (1-\alpha)x \overset{1}{\oplus} \alpha y$, $q = (1-\beta)y \overset{1}{\oplus} \beta z$, and $r = (1-\gamma)z \overset{1}{\oplus} \gamma x$. Then the following are equivalent:*

- $[x,q] \cap [y,r] \cap [z,p] \neq \emptyset$;
- $\alpha\beta\gamma/((1-\alpha)(1-\beta)(1-\gamma)) = 1$.

To prove this theorem, we prepare some lemmas.

Lemma 3.32. *Let S be a nonempty convex subspace of $S_{\mathcal{H}}$ such that $d(u,v) < \pi$ for any $u, v \in S$. Let $x, y, z \in S$. Suppose that there exist $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1x + k_2y + k_3z = 0$ and $(k_1, k_2, k_3) \neq (0,0,0)$. Then $[x,y] \cap [y,z] \cap [z,x] \neq \emptyset$.*

Proof. Assume that $k_1 = 0$. Then we have $k_2y = -k_3z$. Since $\|y\| = \|z\| = 1$, we obtain $|k_2| = |k_3|$. If $k_2 = k_3$, then we have $y = -z$. It follows $d(y,z) = \cos^{-1}\langle y, z\rangle = \pi$, which is a contradiction. Therefore we get $k_2 = -k_3$. Thus we obtain $y = z$, which implies $y \in [x,y] \cap [y,z] \cap [z,x]$.

In the same way, we have $x \in [x,y] \cap [y,z] \cap [z,x]$ in the case where $k_2 = 0$ or $k_3 = 0$.

Next, we assume that $k_1, k_2, k_3 > 0$. Then we have $z = -(k_1/k_3)x - (k_2/k_3)y$. Since $\|z\| = 1$, we get $\|(k_1/k_3)x + (k_2/k_3)y\| = 1$. Hence we obtain

$$z = -\frac{(k_1/k_3)x + (k_2/k_3)y}{\|(k_1/k_3)x + (k_2/k_3)y\|} = -\frac{\lambda x + (1-\lambda)y}{\|\lambda x + (1-\lambda)y\|} = -(\lambda x \overset{1}{\oplus} (1-\lambda)y),$$

where $\lambda = k_1/(k_1 + k_2)$. It means that $d(\lambda x \overset{1}{\oplus} (1-\lambda)y, z) = \cos^{-1}\langle -z, z\rangle = \pi$. This implies a contradiction since $\lambda x \overset{1}{\oplus} (1-\lambda)y \in S$. Similarly, we also get a contradiction if $k_1, k_2, k_3 < 0$.

Finally, suppose that there exists $i, j \in \{1,2,3\}$ such that $k_i > 0$ and $k_j < 0$. Without loss of generality, we may assume that $k_1 = 1$ and $k_2 < 0$. We divide into the following cases:

(i) $k_3 < 0$;
(ii) $k_3 > 0$.

Assume that (i) holds. Put $l_2 := -k_2$, and $l_3 := -k_3$. Then we have $l_2, l_3 > 0$, $x = l_2 y + l_3 z$, and $\|l_2 y + l_3 z\| = 1$. Put $\mu = l_2/(l_2 + l_3)$. Then we obtain

$$x = \frac{l_2 y + l_3 z}{\|l_2 y + l_3 z\|} = \frac{\mu y + (1-\mu)z}{\|\mu y + (1-\mu)z\|} = \mu y \overset{1}{\oplus} (1-\mu)z$$

and thus $x \in [x, y] \cap [y, z] \cap [z, x]$. Next, we consider the case where (ii) holds. Put $l_2 := -k_2$. Then we have $l_2 > 0$ and $y = (1/l_2)x + (k_3/l_2)z$. Therefore, putting $\nu = 1/(1 + k_3)$, we get

$$y = \frac{\frac{1}{l_2}x + \frac{k_3}{l_2}z}{\left\|\frac{1}{l_2}x + \frac{k_3}{l_2}z\right\|} = \frac{\nu x + (1-\nu)z}{\|\nu x + (1-\nu)z\|} = \nu x \overset{1}{\oplus} (1-\nu)z.$$

This implies $y \in [x, y] \cap [y, z] \cap [z, x]$.

Consequently, we obtain the conclusion. □

Corollary 3.33. *Let S be a nonempty convex subspace of S_H such that $d(u, v) < \pi$ for any $u, v \in S$. Let $\triangle(x, y, z)$ be a geodesic triangle on S such that $[x, y] \cap [y, z] \cap [z, x] = \emptyset$. Suppose that there exist $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1 x + k_2 y + k_3 z = 0$. Then $(k_1, k_2, k_3) = (0, 0, 0)$.*

Fact 3.34 (Ceva's theorem in plane geometry)**.** *Let V be a real vector space and $x, y, z \in V$. For $\alpha, \beta, \gamma \in \,]0, 1[$, take $p = (1-\alpha)x + \alpha y$, $q = (1-\beta)y + \beta z$ and $r = (1-\gamma)z + \gamma x$. Put $[u, v]_V = \{tu + (1-t)v \mid t \in [0, 1]\}$ for each $u, v \in V$. Suppose that $[x, y]_V \cap [y, z]_V \cap [z, x]_V = \emptyset$. Then the following are equivalent:*

- $[x, q]_V \cap [y, r]_V \cap [z, p]_V \neq \emptyset$;
- $[x, q]_V \cap [y, r]_V \cap [z, p]_V$ *is a singleton;*
- $\alpha\beta\gamma/((1-\alpha)(1-\beta)(1-\gamma)) = 1$.

Now we show Theorem 3.31.

Proof of Theorem 3.31. Let $\triangle_{\mathcal{H}}(x, y, z) = [x, y]_{\mathcal{H}} \cup [y, z]_{\mathcal{H}} \cup [z, x]_{\mathcal{H}}$ be a geodesic triangle on $\mathcal{H}$. Take three points $\overline{p} = (1-\alpha)x + \alpha y$, $\overline{q} = (1-\beta)y + \beta z$, and $\overline{r} = (1-\gamma)z + \gamma x$. From Theorem 3.27, these points satisfy $p = \overline{p}/\|\overline{p}\|$, $q = \overline{q}/\|\overline{q}\|$, $r = \overline{r}/\|\overline{r}\|$, and $\overline{p}, \overline{q}, \overline{r} \in \triangle_{\mathcal{H}}(x, y, z)$. Then, the following are equivalent by Fact 3.34:

- $[x, \overline{q}]_{\mathcal{H}} \cap [y, \overline{r}]_{\mathcal{H}} \cap [z, \overline{p}]_{\mathcal{H}} \neq \emptyset$;
- $\alpha\beta\gamma/((1-\alpha)(1-\beta)(1-\gamma)) = 1$.

Thus, it is sufficient to show that (i) and (ii):

(i) If $[x, \overline{q}]_{\mathcal{H}} \cap [y, \overline{r}]_{\mathcal{H}} \cap [z, \overline{p}]_{\mathcal{H}} \neq \emptyset$, then $[x, q] \cap [y, r] \cap [z, p] \neq \emptyset$;
(ii) if $[x, q] \cap [y, r] \cap [z, p] \neq \emptyset$, then $\alpha\beta\gamma/((1-\alpha)(1-\beta)(1-\gamma)) = 1$.

First, assume that $u \in [x, \overline{q}]_{\mathcal{H}} \cap [y, \overline{r}]_{\mathcal{H}} \cap [z, \overline{p}]_{\mathcal{H}}$. Then there exist $\delta, \varepsilon, \zeta \in [0, 1]$ such that

$$\begin{aligned} u &= \delta x + (1-\delta)\overline{q} = \varepsilon y + (1-\varepsilon)\overline{r} = \zeta z + (1-\zeta)\overline{p} \\ &= \delta x + (1-\delta)\|q\| q = \varepsilon y + (1-\varepsilon)\|r\| r = \zeta z + (1-\zeta)\|p\| p \end{aligned}$$

Therefore, from Corollary 3.30, we get $u/\|u\| \in [x, q] \cap [y, r] \cap [z, p]$. Hence (i) holds.

Next, suppose that $v \in [x,q] \cap [y,r] \cap [z,p]$. Then there exist $\eta, \theta, \iota \in [0,1]$ such that

$$\begin{aligned} v &= \eta x \overset{1}{\oplus} (1-\eta) q = \theta y \overset{1}{\oplus} (1-\theta) r = \iota z \overset{1}{\oplus} (1-\iota) p \\ &= \frac{\eta x + (1-\eta) q}{\|\eta x + (1-\eta) q\|} = \frac{\theta y + (1-\theta) r}{\|\theta y + (1-\theta) r\|} = \frac{\iota z + (1-\iota) p}{\|\iota z + (1-\iota) p\|} \\ &= \frac{\eta \|\overline{q}\| x + (1-\eta) \overline{q}}{\left\|\eta \|\overline{q}\| x + (1-\eta) \overline{q}\right\|} = \frac{\theta \|\overline{r}\| y + (1-\theta) \overline{r}}{\left\|\theta \|\overline{r}\| y + (1-\theta) \overline{r}\right\|} = \frac{\iota \|\overline{p}\| z + (1-\iota) \overline{p}}{\left\|\iota \|\overline{p}\| z + (1-\iota) \overline{p}\right\|} \\ &= \frac{\eta \|\overline{q}\| x + (1-\eta)(1-\beta) y + (1-\eta) \beta z}{\left\|\eta \|\overline{q}\| x + (1-\eta)(1-\beta) y + (1-\eta) \beta z\right\|} \\ &= \frac{(1-\theta)\gamma x + \theta \|\overline{r}\| y + (1-\theta)(1-\gamma) z}{\left\|(1-\theta)\gamma x + \theta \|\overline{r}\| y + (1-\theta)(1-\gamma) z\right\|} \\ &= \frac{(1-\iota)(1-\alpha) x + (1-\iota)\alpha y + \iota \|\overline{p}\| z}{\left\|(1-\iota)(1-\alpha) x + (1-\iota)\alpha y + \iota \|\overline{p}\| z\right\|}. \end{aligned}$$

It follows that

$$\begin{aligned} &\eta \|\overline{q}\| : (1-\eta)(1-\beta) : (1-\eta)\beta \\ &\quad = (1-\theta)\gamma : \theta \|\overline{r}\| : (1-\theta)(1-\gamma) \\ &\quad = (1-\iota)(1-\alpha) : (1-\iota)\alpha : \iota \|\overline{p}\| \end{aligned}$$

by Corollary 3.33. Therefore we have

$$\frac{\alpha}{1-\alpha} \cdot \frac{\gamma}{1-\gamma} = \frac{(1-\iota)\alpha}{(1-\iota)(1-\alpha)} \cdot \frac{(1-\theta)\gamma}{(1-\theta)(1-\gamma)} = \frac{\eta \|\overline{q}\|}{(1-\eta)(1-\beta)} \cdot \frac{(1-\eta)\beta}{\eta \|\overline{q}\|} = \frac{1-\beta}{\beta},$$

which is the desired result. □

Remark 3.35. In $S_{\mathcal{H}}$, if we use a usual convex combination instead of 1-convex combination, then we cannot obtain the result like Ceva's theorem. We introduce an counterexample. We consider the case where $\mathcal{H} = \mathbb{R}^3$, and put $S = \{(x,y,z) \in \mathbb{R}^3 \mid x,y,z \geq 0\}$. Let $x,y,z \in S$ such that $x = (1,0,0)$, $y = (1/2, \sqrt{3}/2, 0)$, $z = (1/2, 0, \sqrt{3}/2)$. Then $d(x,y) = d(x,z) = \pi/3$, and $d(y,z) = \cos^{-1}(1/4) \approx 1.318$. Let $\alpha = 2/5$, $\beta = 3/8$, $\gamma = 5/7$, which satisfy $\alpha\beta\gamma/((1-\alpha)(1-\beta)(1-\gamma)) = 1$. Take $p = \alpha x \oplus (1-\alpha) y$, $q = \beta y \oplus (1-\beta) z$ and $r = \gamma z \oplus (1-\gamma) x$. Then geodesic segments $[x,q]$ and $[y,r]$ intersect at exactly one point u. However, the point u is not on $[z,p]$.

Let $x_1, x_2, \ldots, x_m \in \mathcal{H}$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1]$ such that $\sum_{i=1}^m \alpha_i = 1$. Then we have

$$\sum_{i=1}^m \alpha_i x_i = \operatorname*{argmin}_{z \in \mathcal{H}} \sum_{i=1}^m \alpha_i \|x_i - z\|^2.$$

Indeed, it is obtained by

$$\sum_{i=1}^m \alpha_i \|x_i - z\|^2 = \left\| z - \sum_{i=1}^m \alpha_i x_i \right\|^2 + \sum_{i=1}^m \alpha_i \|x_i\|^2 - \left\| \sum_{i=1}^m \alpha_i x_i \right\|^2$$

for $z \in \mathcal{H}$. Based on this fact, we generalize the 1-convex combination to be defined for a finite number of points. Let S be a nonempty convex subspace of $S_{\mathcal{H}}$ such that $d(u,v) < \pi$ for any $u, v \in S$. For $x_1, x_2, \ldots, x_m \in S$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0, \infty[$ such that $\sum_{i=1}^m \alpha_i > 0$, we define a point $B(x_1, x_2, \ldots, x_m; \alpha_1, \alpha_2 \ldots, \alpha_m)$ on S by

$$B(x_1, x_2, \ldots, x_m; \alpha_1, \alpha_2 \ldots, \alpha_m) = \operatorname*{argmax}_{z \in S} \sum_{i=1}^m \alpha_i \cos d(x_i, z).$$

We hereinafter write $B(\{x_i\}, \{\alpha_i\})$ for this point simply. We call the point $B(\{x_i\}, \{\alpha_i\})$ a *balanced* 1*-convex combination* of $x_1, x_2, \ldots, x_m$ on S. The 1-convex combination is the case where $m = 2$ and $\alpha_1 + \alpha_2 = 1$ for the balanced 1-convex combination. Namely, for each $x_1, x_2 \in S$ and $\alpha \in [0, 1]$,

$$\alpha x_1 \overset{1}{\oplus} (1 - \alpha) x_2 = B(x_1, x_2; \alpha, 1 - \alpha).$$

Theorem 3.36. *Let S be a nonempty convex subspace of $S_{\mathcal{H}}$ such that $d(u, v) < \pi$ for any $u, v \in S$. Take $x_1, x_2, \ldots, x_m \in S$ and let $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0, \infty[$ such that $\sum_{i=1}^m \alpha_i > 0$. Then a balanced 1-convex combination $B(\{x_i\}, \{\alpha_i\}) \in S$ is well-defined, and*

$$B(\{x_i\}, \{\alpha_i\}) = \sum_{i=1}^m \alpha_i x_i \bigg/ \left\| \sum_{i=1}^m \alpha_i x_i \right\|.$$

Proof. By the definition of $B(\{x_i\}, \{\alpha_i\})$, we have

$$B(\{x_i\}, \{\alpha_i\}) = \operatorname*{argmax}_{z \in S} \sum_{i=1}^m \alpha_i \cos d(x_i, z) = \operatorname*{argmax}_{z \in S} \left\langle \sum_{i=1}^m \alpha_i x_i, z \right\rangle.$$

Then putting $p = \sum_{i=1}^m \alpha_i x_i$ and $w = p / \|p\| \in S$, we obtain

$$\left\langle \sum_{i=1}^m \alpha_i x_i, w \right\rangle = \|p\| > \langle p, z \rangle = \left\langle \sum_{i=1}^m \alpha_i x_i, z \right\rangle$$

for any $z \in S \setminus \{p\}$. This is the conclusion. □

Theorem 3.36 is a generalization of Theorem 3.27.

Theorem 3.37. *Let S be a nonempty convex subspace of $S_{\mathcal{H}}$ such that $d(u, v) < \pi$ for any $u, v \in S$, and let $\triangle(x, y, z)$ be a geodesic triangle on S. Take $\alpha_1, \alpha_2, \alpha_3 \in]0, \infty[$ and put $\beta = \alpha_2 / (\alpha_2 + \alpha_3)$. Let $u = B(x, y, z; \alpha_1, \alpha_2, \alpha_3)$ and $w = \beta y \overset{1}{\oplus} (1 - \beta) z$. Then $u \in [x, w]$.*

Proof. Put $p = \beta y + (1 - \beta) z$ and $q = \alpha_1 x + \alpha_2 y + \alpha_3 z$. Then, from Theorems 3.27 and 3.36, we obtain $w = p / \|p\|$ and $u = q / \|q\|$. Since $1 - \alpha_1 = \alpha_2 + \alpha_3$, we also have $q = \alpha_1 x + (1 - \alpha_1) p$. Thus, putting $\gamma = \alpha_1 / (\alpha_1 + (1 - \alpha_1)\|p\|)$, we get $q = (\alpha_1 + (1 - \alpha_1)\|p\|)(\gamma x + (1 - \gamma) w)$. It implies

$$u = \frac{q}{\|q\|} = \frac{\gamma x + (1 - \gamma) w}{\|\gamma x + (1 - \gamma) w\|} = \gamma x \overset{1}{\oplus} (1 - \gamma) w \in [x, w]$$

from Lemma 3.9. □

We consider that Theorem 3.37 is a crucial result that shows the suitability of the 1-convex combination on the unit sphere. Indeed, if we only use the usual convex combination $\oplus$, then we do not obtain a simple result such as Theorem 3.37.

3.2.2 (-1)-convex combination on the hyperbolic plane

Next, we consider natures of the (-1)-convex combination on the hyperbolic plane. We consider the hyperboloid model of the hyperbolic plane. Define a function $B\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by $B(u,v) = z_1z_2 - x_1x_2 - y_1y_2$ for each $u = (z_1, x_1, y_1) \in \mathbb{R}^3$ and $v = (z_2, x_2, y_2) \in \mathbb{R}^3$, and define $Q\colon \mathbb{R}^3 \to \mathbb{R}$ by $Q(u) = B(u,u) = z^2 - x^2 - y^2$ for $u = (z,x,y) \in \mathbb{R}^3$. Let $H = \{(z,x,y) \in \mathbb{R}^3 \mid z^2 - x^2 - y^2 = 1,\ z > 0\}$ and $d\colon H \times H \to [0,\infty[$ by $d(u,v) = \cosh^{-1} B(u,v)$ for $u, v \in H$. Then (H,d) is a metric space, and it behaves as a two-dimensional hyperbolic space. (H,d) is also a uniquely geodesic space. Indeed, for every $u, v \in H$, a mapping $\gamma_{u,v}\colon [0,1] \to H$ defined by

$$\gamma(t) = (t)^{-1}_{d(u,v)} u + (1-t)^{-1}_{d(u,v)} v = \begin{cases} \dfrac{\sinh(td(u,v))}{\sinh d(u,v)} u + \dfrac{\sinh((1-t)d(u,v))}{\sinh d(u,v)} v & (\text{if } u \neq v); \\ u & (\text{if } u = v) \end{cases}$$

for $t \in [0,1]$ is a unique geodesic joining u and v. This means that a convex combination $tu \oplus (1-t)v$ on H is expressed by

$$tu \oplus (1-t)v = (t)^{-1}_{d(u,v)} u + (1-t)^{-1}_{d(u,v)} v$$

for any $u, v \in H$ and $t \in [0,1]$.

Functions B and Q are called the Minkowski bilinear form, and the Minkowski quadratic form, respectively. We know that these have the following properties.

- $B(u,v) = B(v,u)$ for any $u, v \in \mathbb{R}^3$;
- $B(su + tv, w) = sB(u,w) + tB(v,w)$ for any $u, v, w \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$;
- $Q(su + tv) = s^2Q(u) + 2stB(u,v) + t^2Q(v)$ for any $u, v \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$;
- $Q(u) = 1$ for any $u \in H$.

For $x, y \in H$, $t \in [0,1]$ and $\alpha \in [0,1]$, let $\alpha x \overset{-1}{\oplus} (1-\alpha)y = tx \oplus (1-t)y$. Then we have

$$\alpha x \overset{-1}{\oplus} (1-\alpha)y = \frac{\alpha x + (1-\alpha)y}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(x,y) + (1-\alpha)^2}}$$

by Lemma 3.13. We also obtain

$$\begin{aligned} Q(\alpha x + (1-\alpha)y) &= \alpha^2 Q(x) + 2\alpha(1-\alpha)B(x,y) + (1-\alpha)^2 Q(y) \\ &= \alpha^2 + 2\alpha(1-\alpha)\cosh d(x,y) + (1-\alpha)^2. \end{aligned}$$

Consequently, we get an explicit expression of the (-1)-convex combination on H as follows.

Theorem 3.38. *Let $x, y \in H$. Then for any $\alpha \in [0,1]$,*

$$\alpha x \overset{-1}{\oplus} (1-\alpha)y = \frac{\alpha x + (1-\alpha)y}{\sqrt{Q(\alpha x + (1-\alpha)y)}}.$$

Chapter 4

Fixed point problems

Int this chapter, we consider a fixed point problem for a quasinonexpansive mapping.

4.1 Natures of vicinal mappings

The notion of vicinal mappings is first proposed by Kohsaka [22]. Motivated by this study, Kajimura and Kimura [9] proposed the notion of vicinal mappings with ψ.

Definition 4.1 (Kajimura and Kimura [9]). Let X be an admissible CAT(κ) space and suppose $\psi\colon [0, D_\kappa/2[\to]0,\infty[$ is right continuous at 0. A mapping $T\colon X \to X$ is said to be *vicinal with* ψ if

$$\big(\psi(d(x,Tx)) + \psi(d(y,Ty))\big)c_\kappa(d(Tx,Ty)) \le \psi(d(x,Tx))c_\kappa(d(x,Ty)) + \psi(d(y,Ty))c_\kappa(d(y,Tx))$$

for any $x, y \in X$. A mapping $T\colon X \to X$ is said to be *firmly vicinal with* ψ if

$$\begin{aligned}\big(\psi(d(x,Tx))c_\kappa(d(x,Tx)) + \psi(d(y,Ty))c_\kappa(d(y,Ty))\big)c_\kappa''(d(Tx,Ty))&\\ + \big(\psi(d(x,Tx)) + \psi(d(y,Ty))\big)c_\kappa(d(Tx,Ty))&\\ \le \psi(d(x,Tx))c_\kappa(d(x,Ty)) + \psi(d(y,Ty))c_\kappa(d(y,Tx))&\end{aligned}$$

for any $x, y \in X$.

It can be easily obtained that every firmly vicinal mapping with ψ is vicinal with ψ.

Lemma 4.2 (Kajimura and Kimura [9]). *Let X be an admissible* CAT(κ) *space. Suppose that $T\colon X \to X$ is vicinal with ψ. Then T is Δ-demiclosed. Moreover, if $F(T)$ is nonempty, then T is quasinonexpansive.*

Lemma 4.3. *Let X be an admissible* CAT(κ) *space and let $\psi\colon [0, D_\kappa/2[\to]0,\infty[$ such that ψ is right continuous at* 0*. Then for a mapping $T\colon X \to X$, the following are equivalent:*

(i) *T is firmly vicinal with ψ;*
(ii) *for any $x, y \in X$,*

$$\begin{aligned}\big(\psi(d(x,Tx))c_\kappa''(d(x,Tx)) + \psi(d(y,Ty))c_\kappa''(d(y,Ty))\big)\,c_\kappa(d(Tx,Ty))&\\ \le \psi(d(x,Tx))\big(c_\kappa(d(x,Ty)) - c_\kappa(d(x,Tx))\big)&\\ + \psi(d(y,Ty))\big(c_\kappa(d(y,Tx)) - c_\kappa(d(y,Ty))\big);&\end{aligned}$$

(iii) *for any* $x, y \in X$,

$$\frac{1}{\kappa}\big(\psi(d(x,Tx))c''_\kappa(d(x,Tx)) + \psi(d(y,Ty))c''_\kappa(d(y,Ty))\big)c''_\kappa(d(Tx,Ty)) \\ \geq \frac{1}{\kappa}\big(\psi(d(x,Tx))c''_\kappa(d(x,Ty)) + \psi(d(y,Ty))c''_\kappa(d(y,Tx))\big),$$

where (iii) *is considered only when* $\kappa \neq 0$.

Proof. Using Lemma 2.8, we get

$$\begin{aligned}
& T \text{ is firmly vicinal with } \psi \\
\iff & \psi(d(x,Tx))\big(c_\kappa(d(x,Tx))c''_\kappa(d(Tx,Ty)) + c_\kappa(d(Tx,Ty))\big) \\
& + \psi(d(y,Ty))\big(c_\kappa(d(y,Ty))c''_\kappa(d(Tx,Ty)) + c_\kappa(d(Tx,Ty))\big) \\
& \quad \leq \psi(d(x,Tx))c_\kappa(d(x,Ty)) + \psi(d(y,Ty))c_\kappa(d(y,Tx)) \\
\iff & \psi(d(x,Tx))\big(c_\kappa(d(Tx,Ty))c''_\kappa(d(x,Tx)) + c_\kappa(d(x,Tx))\big) \\
& + \psi(d(y,Ty))\ \big(c_\kappa(d(Tx,Ty))c''_\kappa(d(y,Ty)) + c_\kappa(d(y,Ty))\big) \\
& \quad \leq \psi(d(x,Tx))c_\kappa(d(x,Ty)) + \psi(d(y,Ty))c_\kappa(d(y,Tx)) \\
\iff & \big(\psi(d(x,Tx))c''_\kappa(d(x,Tx)) + \psi(d(y,Ty))c''_\kappa(d(y,Ty))\big)c_\kappa(d(Tx,Ty)) \\
& \leq \psi(d(x,Tx))\ (c_\kappa(d(x,Ty)) - c_\kappa(d(x,Tx))) \\
& \quad + \psi(d(y,Ty))\ (c_\kappa(d(y,Tx)) - c_\kappa(d(y,Ty)))
\end{aligned}$$

for $x, y \in X$ and thus (i) and (ii) are equivalent. In addition, if $\kappa \neq 0$, then (ii) is equivalent to

$$\big(\psi(d(x,Tx))c''_\kappa(d(x,Tx)) + \psi(d(y,Ty))c''_\kappa(d(y,Ty))\big) \cdot \frac{1 - c''_\kappa(d(Tx,Ty))}{\kappa} \\ \leq \psi(d(x,Tx))\left(\frac{1 - c''_\kappa(d(x,Ty))}{\kappa} - \frac{1 - c''_\kappa(d(x,Tx))}{\kappa}\right) \\ + \psi(d(y,Ty))\left(\frac{1 - c''_\kappa(d(y,Tx))}{\kappa} - \frac{1 - c''_\kappa(d(y,Ty))}{\kappa}\right)$$

and so is

$$\frac{1}{\kappa}\big(\psi(d(x,Tx))c''_\kappa(d(x,Tx)) + \psi(d(y,Ty))c''_\kappa(d(y,Ty))\big)c''_\kappa(d(Tx,Ty)) \\ \geq \frac{1}{\kappa}\big(\psi(d(x,Tx))c''_\kappa(d(x,Ty)) + \psi(d(y,Ty))c''_\kappa(d(y,Tx))\big)$$

for $x, y \in X$. Hence we get (ii) and (iii) are equivalent if $\kappa \neq 0$. □

Lemma 4.4. *For* $\kappa \neq 0$, *let* X *be an admissible* CAT(κ) *space and let* $\psi\colon [0, D_\kappa/2[\to]0, \infty[$ *such that* ψ *is right continuous at* 0. *Then for a mapping* $T\colon X \to X$, *the following are equivalent:*

(i) T *is vicinal with* ψ;

(ii) *for any* $x, y \in X$,

$$\frac{1}{\kappa}(\psi(d(x,Tx)) + \psi(d(y,Ty)))c''_\kappa(d(Tx,Ty)) \\ \geq \frac{1}{\kappa}\big(\psi(d(x,Tx))c''_\kappa(d(x,Ty)) + \psi(d(y,Ty))c''_\kappa(d(y,Tx))\big).$$

Proof. We obtain the conclusion easily by using an equation $c_\kappa(d) = (1 - c''_\kappa(d))/\kappa$ for $d \in \mathbb{R}$. □

The notion of firm vicinity with ψ unifies a definition of some type of nonspreading mappings as follows. Let X be a CAT(0) space and T a mapping from X into itself. T is said to be *firmly metrically nonspreading* [23] if

$$2d(Tx, Ty)^2 \leq d(x, Ty)^2 - d(x, Tx)^2 + d(y, Tx)^2 - d(y, Ty)^2$$

for every $x, y \in X$. It is equivalent to the firm vicinity of T with $\psi\colon [0, \infty[\ni t \mapsto 1$.

Let X be an admissible CAT(1) space and T a mapping from X into itself. T is said to be *spherically nonspreading of sum type* [10] if

$$2\cos d(Tx, Ty) \geq \cos d(x, Ty) + \cos d(y, Tx)$$

for every $x, y \in X$. It is equivalent to the firm vicinity of T with $\psi\colon [0, \pi/2[\ni t \mapsto 1$.

4.2 Tightly quasinonexpansive mappings

In this section, we define a new notion of a special quasinonexpansive mapping.

Definition 4.5. Let X be an admissible CAT(κ) space and $T\colon X \to X$ a mapping such that $F(T) \neq \emptyset$. Then we call T a *tightly quasinonexpansive* mapping for κ if for any $x \in X$ and $z \in F(T)$, an inequality

$$c_\kappa(d(x, Tx))c''_\kappa(d(Tx, z)) \leq c_\kappa(d(x, z)) - c_\kappa(d(Tx, z)) \qquad (*_1)$$

holds. Or equivalently from Lemma 2.8, for any $x \in X$ and $z \in F(T)$,

$$c_\kappa(d(Tx, z))c''_\kappa(d(x, Tx)) \leq c_\kappa(d(x, z)) - c_\kappa(d(x, Tx)) \qquad (*_2)$$

holds.

Note that inequalities $(*_1)$ and $(*_2)$ always hold if $x \in F(T)$. Therefore, we obtain that T is tightly quasinonexpansive if and only if $(*_1)$ or $(*_2)$ holds for any $x \in X \setminus F(T)$ and $z \in F(T)$.

From Lemma 2.8, T is tightly quasinonexpansive for $\kappa \neq 0$ if and only if

$$\frac{1}{\kappa}c''_\kappa(d(x, Tx))c''_\kappa(d(Tx, z)) \geq \frac{1}{\kappa}c''_\kappa(d(x, z))$$

for any $x \in X \setminus F(T)$ and $z \in F(T)$; T is tightly quasinonexpansive for $\kappa = 0$ if and only if

$$d(x, Tx)^2 + d(Tx, z)^2 \leq d(x, z)^2$$

for any $x \in X \setminus F(T)$ and $z \in F(T)$.

We hereinafter omit words '*for* κ' if such κ is clear from context.

Example 4.6. Let X be an admissible complete CAT(κ) space. Then the identity mapping on X is tightly quasinonexpansive.

Example 4.7. Let X be an admissible complete CAT(κ) space and K a nonempty closed convex subset of X. Then a metric projection P_K from X onto K is tightly quasinonexpansive from Lemma 2.11.

Example 4.8. Let X be an admissible complete CAT(κ) space and K a nonempty closed convex subset of X. Let f be a function from X into $[0,1]$ such that $f(X \setminus K) \subset [0,1[$. Then a mapping $T\colon X \to X$ defined by $Tx = f(x)x \oplus (1 - f(x))P_K x$ is tightly quasinonexpansive.

Proof. It is obvious that $F(T) = K$. Take $x \in X \setminus F(T)$ and $z \in F(T) = K$ arbitrarily. Then we have $Tx \in]x, P_K x]$.

We consider the case where $Tx = P_K x$. Then, since P_K is tightly quasinonexpansive, we obtain

$$c_\kappa(d(x,Tx))c''_\kappa(d(Tx,z)) + c_\kappa(d(Tx,z)) = c_\kappa(d(x,P_Kx))c''_\kappa(d(P_Kx,z)) + c_\kappa(d(P_Kx,z)) \le c_\kappa(d(x,z)).$$

Henceforth, suppose that $Tx \ne P_K x$. Put $p = P_K x$. For a model space (M_κ, ρ), take a comparison triangle $\overline{\triangle}(\overline{x}, \overline{z}, \overline{p}) \subset M_\kappa$ of $\triangle(x, z, p)$ and a comparison point $\overline{m} \in]\overline{x}, \overline{p}[$ of $Tx \in [x, P_K x]$.

Assume that $\overline{m} \in]\overline{x}, \overline{z}[$. Then there exists $\alpha \in]0,1[$ such that $\overline{m} = \alpha\overline{x} \oplus (1-\alpha)\overline{z}$. This follows that

$$\begin{aligned} d(x,Tx)^2 + d(Tx,z)^2 &\le \rho(\overline{x},\overline{m})^2 + \rho(\overline{m},\overline{z})^2 \\ &= (1-\alpha)^2\rho(\overline{x},\overline{z})^2 + \alpha^2\rho(\overline{x},\overline{z})^2 \\ &= (1-2\alpha(1-\alpha))\rho(\overline{x},\overline{z})^2 \\ &\le \rho(\overline{x},\overline{z})^2 = d(x,z)^2 \end{aligned}$$

if $\kappa = 0$, and

$$\begin{aligned} \frac{1}{\kappa}c''_\kappa(d(x,Tx))c''_\kappa(d(Tx,z)) &\ge \frac{1}{\kappa}c''_\kappa(\rho(\overline{x},\overline{m}))c''_\kappa(\rho(\overline{m},\overline{z})) \\ &= \frac{1}{\kappa}c''_\kappa((1-\alpha)\rho(\overline{x},\overline{z}))c''_\kappa(\alpha\rho(\overline{x},\overline{z})) \\ &= \frac{1}{2\kappa}\big(c''_\kappa(\rho(\overline{x},\overline{z})) + c''_\kappa((1-2\alpha)\rho(\overline{x},\overline{z}))\big) \\ &\ge \frac{1}{2\kappa}\big(c''_\kappa(\rho(\overline{x},\overline{z})) + c''_\kappa(\rho(\overline{x},\overline{z}))\big) \\ &= \frac{1}{\kappa}c''_\kappa(\rho(\overline{x},\overline{z})) \\ &= \frac{1}{\kappa}c''_\kappa(d(x,z)) \end{aligned}$$

if $\kappa \ne 0$.

Finally, assume that $\overline{m} \notin]\overline{x}, \overline{z}[$. Then we easily have $\overline{p} \notin]\overline{x}, \overline{z}[$. Put $\theta = \angle\overline{x}\,\overline{p}\,\overline{z} \in [0,\pi[$ and $\varphi = \angle\overline{x}\,\overline{m}\,\overline{z} \in [0,\pi[$. The angle θ is determined by the formula

$$\begin{aligned} \cos\theta &= \frac{c_\kappa(\rho(\overline{x},\overline{p}))c''_\kappa(\rho(\overline{p},\overline{z})) + c_\kappa(\rho(\overline{p},\overline{z})) - c_\kappa(\rho(\overline{x},\overline{z}))}{c'_\kappa(\rho(\overline{x},\overline{p}))c'_\kappa(\rho(\overline{p},\overline{z}))} \\ &= \frac{c_\kappa(\rho(\overline{p},\overline{z}))c''_\kappa(\rho(\overline{x},\overline{p})) + c_\kappa(\rho(\overline{x},\overline{p})) - c_\kappa(\rho(\overline{x},\overline{z}))}{c'_\kappa(\rho(\overline{x},\overline{p}))c'_\kappa(\rho(\overline{p},\overline{z}))} \\ &= \begin{cases} \dfrac{\rho(\overline{x},\overline{p})^2 + \rho(\overline{p},\overline{z})^2 - \rho(\overline{x},\overline{z})^2}{2\rho(\overline{x},\overline{p})\rho(\overline{p},\overline{z})} & (\text{if } \kappa = 0); \\ \dfrac{c''_\kappa(\rho(\overline{x},\overline{z})) - c''_\kappa(\rho(\overline{x},\overline{p}))c''_\kappa(\rho(\overline{p},\overline{z}))}{\kappa c'_\kappa(\rho(\overline{x},\overline{p}))c'_\kappa(\rho(\overline{p},\overline{z}))} & (\text{if } \kappa \ne 0) \end{cases} \\ &= \begin{cases} \dfrac{\cosh(\sqrt{-\kappa}\rho(\overline{x},\overline{p}))\cosh(\sqrt{-\kappa}\rho(\overline{p},\overline{z})) - \cosh(\sqrt{-\kappa}\rho(\overline{x},\overline{z}))}{\sinh(\sqrt{-\kappa}\rho(\overline{x},\overline{p}))\sinh(\sqrt{-\kappa}\rho(\overline{p},\overline{z}))} & (\text{if } \kappa < 0); \\ \dfrac{\rho(\overline{x},\overline{p})^2 + \rho(\overline{p},\overline{z})^2 - \rho(\overline{x},\overline{z})^2}{2\rho(\overline{x},\overline{p})\rho(\overline{p},\overline{z})} & (\text{if } \kappa = 0); \\ \dfrac{\cos(\sqrt{\kappa}\rho(\overline{x},\overline{z})) - \cos(\sqrt{\kappa}\rho(\overline{x},\overline{p}))\cos(\sqrt{\kappa}\rho(\overline{p},\overline{z}))}{\sin(\sqrt{\kappa}\rho(\overline{x},\overline{p}))\sin(\sqrt{\kappa}\rho(\overline{p},\overline{z}))} & (\text{if } \kappa > 0). \end{cases} \end{aligned}$$

This formula is just consistent with the law of cosines. Similarly, the angle φ is determined by

$$\cos\varphi = \frac{c_\kappa(\rho(\overline{x},\overline{m}))c''_\kappa(\rho(\overline{m},\overline{z})) + c_\kappa(\rho(\overline{m},\overline{z})) - c_\kappa(\rho(\overline{x},\overline{z}))}{c'_\kappa(\rho(\overline{x},\overline{m}))c'_\kappa(\rho(\overline{m},\overline{z}))}.$$

We divide into the following cases: (i) $\kappa = 0$; (ii) $\kappa \neq 0$.

(i) Let $\kappa = 0$. Since P_K is tightly quasinonexpansive, we obtain

$$\rho(\overline{x},\overline{p})^2 + \rho(\overline{p},\overline{z})^2 - \rho(\overline{x},\overline{z})^2 = d(x,p)^2 + d(p,z)^2 - d(x,z)^2 \le 0,$$

which implies $\pi/2 \le \theta < \pi$. Therefore, since $\overline{m} \in]\overline{x},\overline{p}[$, we obtain $\theta \le \varphi < \pi$. This implies that

$$d(x,Tx)^2 + d(Tx,z)^2 - d(x,z)^2 \le \rho(\overline{x},\overline{m})^2 + \rho(\overline{m},\overline{z})^2 - \rho(\overline{x},\overline{z})^2 \le 0.$$

(ii) Assume that $\kappa \neq 0$. Then we get

$$\frac{1}{\kappa}c''_\kappa(\rho(\overline{x},\overline{p}))c''_\kappa(\rho(\overline{p},\overline{z})) = \frac{1}{\kappa}c''_\kappa(d(x,p))c''_\kappa(d(p,z)) \ge \frac{1}{\kappa}c''_\kappa(d(x,z)) = \frac{1}{\kappa}c''_\kappa(\rho(\overline{x},\overline{z})),$$

by tight quasinonexpansiveness of P_K. This follows that $\pi/2 \le \theta < \pi$. We also have $\theta \le \varphi < \pi$ from $\overline{m} \in]\overline{x},\overline{p}[$. Hence we obtain

$$\frac{1}{\kappa}c''_\kappa(d(x,Tx))c''_\kappa(d(Tx,z)) \ge \frac{1}{\kappa}c''_\kappa(\rho(\overline{x},\overline{m}))c''_\kappa(\rho(\overline{m},\overline{z})) \ge \frac{1}{\kappa}c''_\kappa(\rho(\overline{x},\overline{z})) = \frac{1}{\kappa}c''_\kappa(d(x,z)).$$

Consequently, we get the conclusion. □

Next, we show natures for tightly quasinonexpansive mappings.

Lemma 4.9. *Let X be an admissible* CAT(κ) *space. Then every firmly vicinal mapping with ψ such that $F(T) \neq \emptyset$ is tightly quasinonexpansive.*

Proof. Let $T\colon X \to X$ be a firmly vicinal mapping with ψ such that $F(T) \neq \emptyset$. Take $x \in X$ and $z \in F(T)$ arbitrarily. Then, using the definition of firmly vicinal mapping with ψ, we get

$$\begin{aligned}\psi(d(x,Tx))c_\kappa(d(x,Tx))c''_\kappa(d(Tx,z)) + \big(\psi(d(x,Tx)) + \psi(0)\big)c_\kappa(d(Tx,z))\\ \le \psi(d(x,Tx))c_\kappa(d(x,z)) + \psi(0)c_\kappa(d(Tx,z)).\end{aligned}$$

Dividing by $\psi(d(x,Tx)) > 0$, we get the conclusion. □

Lemma 4.10. *Let X be an admissible* CAT(κ) *space. Then every tightly quasinonexpansive mapping is quasinonexpansive.*

Proof. Let $T\colon X \to X$ be tightly quasinonexpansive, and take $x \in X$ and $z \in F(T)$. Then, since X is admissible, we obtain $c''_\kappa(d(Tx,z)) > 0$. Consequently, we have $0 \le c_\kappa(d(x,z)) - c_\kappa(d(Tx,z))$, which is the desired result. □

Lemma 4.11. *Let X be an admissible* CAT(κ) *space. Then every tightly quasinonexpansive mapping is asymptotically regular.*

Proof. Let $T\colon X \to X$ be tightly quasinonexpansive, and take $x \in X$ and $z \in F(T)$. Then we have $c_\kappa(d(T^n x, T^{n+1}x))c''_\kappa(d(T^{n+1}x,z)) \le c_\kappa(d(T^n x,z)) - c_\kappa(d(T^{n+1}x,z))$ for any $n \in \mathbb{N}$. We also have $\{d(T^n x,z)\}$ converges to some $\lambda \in [0, D_\kappa/2[$ by the previous lemma. This implies that $\inf_{k\in\mathbb{N}} c''_\kappa(d(T^{k+1}x,z)) > 0$ and hence

$$0 \le c_\kappa(d(T^n x, T^{n+1}x)) \le \frac{c_\kappa(d(T^n x,z)) - c_\kappa(d(T^{n+1}x,z))}{c''_\kappa(d(T^{n+1}x,z))} \le \frac{c_\kappa(d(T^n x,z)) - c_\kappa(d(T^{n+1}x,z))}{\inf_{k\in\mathbb{N}} c''_\kappa(d(T^{k+1}x,z))} \to 0$$

as $n \to \infty$, which is the desired result. □

The above three lemmas immediately prove the following.

Corollary 4.12. *Let X be an admissible* CAT(κ) *space. Then every firmly vicinal mapping with ψ such that $F(T) \neq \emptyset$ is quasinonexpansive and asymptotically regular.*

Now we consider an example of quasinonexpansive mappings and tightly quasinonexpansive mappings on the Euclidean space $\mathbb{R}^n$. Let $X = \mathbb{R}^n$ and d a metric on X such that $d(\cdot,\cdot) = \|\cdot - \cdot\|_{\mathbb{R}^n}$. Fix $c \in X$ and set $F = \{z \in X \mid d(z,c) \leq 1\}$. Let T be a mapping from X into itself such that $F(T) = F$. Then T is quasinonexpansive if and only if $Tx \in \bigcap_{z\in F}\{w \in X \mid d(w,z) \leq d(x,z)\}$ for any $x \in X$. Namely, the quasinonexpansiveness of T is equivalent to the fact that Tx always belongs to the closed ball that z and x are its center and its boundary, respectively. Since

$$\bigcap_{z\in F}\{w \in X \mid d(w,z) \leq d(x,z)\} = \{w \in X \mid d(x,c)^2 - d(w,c)^2 \geq 2d(w,x)\}$$

holds for any $x \in X$, we obtain that T is quasinonexpansive if and only if

$$d(x,c)^2 - d(Tx,c)^2 \geq 2d(Tx,x)$$

for any $x \in X$. Put $D_{\mathrm{qn}}(x) = \{w \in X \mid d(x,c)^2 - d(w,c)^2 \geq 2d(w,x)\}$ for each $x \in X$. It is the domain that Tx should be placed so that T is quasinonexpansive.

Next, we consider the tight quasinonexpansiveness of T. The mapping $T\colon X \to X$ is tightly quasinonexpansive for $\kappa = 0$ if and only if $d(x,Tx)^2 + d(Tx,z)^2 \leq d(x,z)^2$ for any $x \in X$ and $z \in F$. Then a inequality $d(x,Tx)^2 + d(Tx,z)^2 \leq d(x,z)^2$ is equivalent to $d(Tx,(x+z)/2) \leq d(x,z)/2$. Hence, the tight quasinonexpansiveness of T is equivalent to the fact that Tx always belongs to the closed ball whose diameter is the segment joining x and z, in other words, an angle between two vectors $\overrightarrow{(Tx)x}$ and $\overrightarrow{(Tx)z}$ is obtuse or right for any $x \in X$ and $z \in F$. Therefore, T is tightly quasinonexpansive if and only if

$$Tx \in \bigcap_{z\in F}\left\{w \in X \;\middle|\; d\left(w, \frac{x+z}{2}\right) \leq \frac{1}{2}d(x,z)\right\} = \left\{w \in X \;\middle|\; \frac{1}{4}d(x,c)^2 - d\left(w, \frac{x+c}{2}\right)^2 \geq d(w,x)\right\}$$

for all $x \in X$, that is,

$$\frac{1}{4}d(x,c)^2 - d\left(Tx, \frac{c+x}{2}\right)^2 \geq d(Tx,x)$$

for all $x \in X$. Put

$$D_{\mathrm{tqn}}(x) = \left\{w \in X \;\middle|\; \frac{1}{4}d(x,c)^2 - d\left(w, \frac{x+c}{2}\right)^2 \geq d(w,x)\right\}$$

for each $x \in X$, which is the domain that Tx should be placed so that T is tightly quasinonexpansive.

By basic calculations, we have

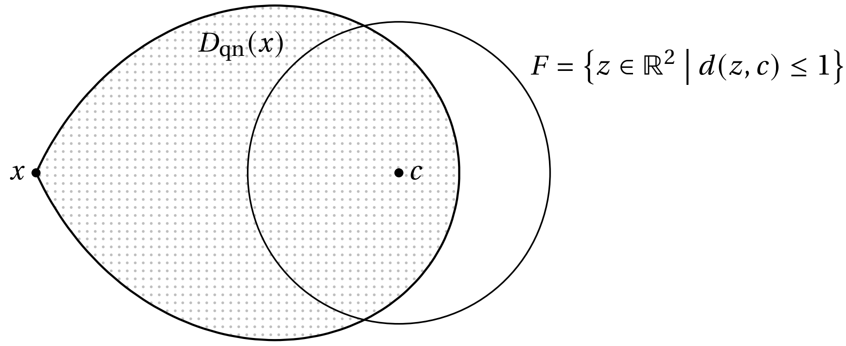
$$\begin{aligned} d(x,c)^2 - d(w,c)^2 \geq 2d(w,x) &\iff \|w-x\|^2 + 2\|w-x\| \leq 2\langle w-x, c-x\rangle \\ &\iff \left\|\frac{w-x}{2}\right\|^2 + \left\|\frac{w-x}{2}\right\| \leq \left\langle \frac{w-x}{2}, c-x\right\rangle \end{aligned}$$

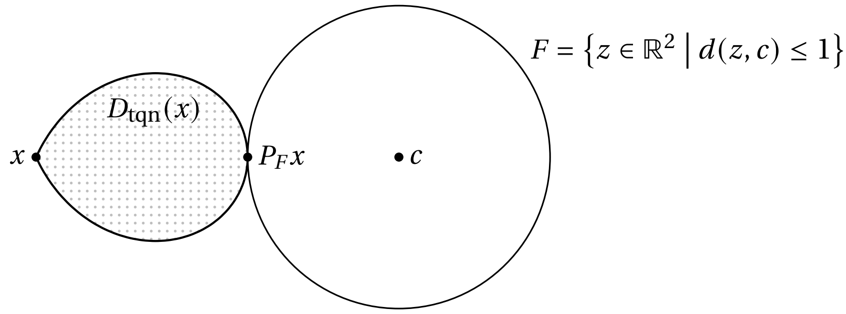
and

$$\frac{1}{4}d(x,c)^2 - d\left(w, \frac{x+c}{2}\right)^2 \geq d(w,x) \iff \|w-x\|^2 + \|w-x\| \leq \langle w-x, c-x\rangle.$$

Hence, $D_{\mathrm{tqn}}(x)$ and $D_{\mathrm{qn}}(x)$ are similar, and their homothetic ratio is $1:2$. We also get $D_{\mathrm{tqn}}(x) \subset D_{\mathrm{qn}}(x)$ and $D_{\mathrm{tqn}}(x) \cap F = \{P_F x\}$ for any $x \in X$.

For the sake of simplicity, let $X = \mathbb{R}^2$. Then the following figures show domains $D_{\mathrm{qn}}(x)$ and $D_{\mathrm{tqn}}(x)$ for some $x \in \mathbb{R}^2$.





These represent the case where $d(x,c) = 12/5$.

4.3 Mann type fixed point approximations

In this section, we show fixed point approximation theorems using Mann type iterative sequence for tightly quasinonexpansive mappings. For a uniquely D-geodesic space X and a mapping $T\colon X \to X$ with $F(T) \neq \emptyset$, Mann type iterative scheme generates a sequence $\{x_n\}$ on X by an iteration $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tx_n$ for $n \in \mathbb{N}$. Our aim is to investigate a convergence of such $\{x_n\}$ to a fixed point of T.

Lemma 4.13 (Kajimura and Kimura [8], Kimura and Kohsaka [14, 15])**.** *Let X be a complete* CAT(κ) *space and K an admissible closed convex subset of X. Let $\{z_n\}$ be a κ-bounded sequence on K. Take $\{\beta_n\} \subset [0,\infty[$ such that $\beta_1 > 0$. Define $g\colon K \to [0,\infty]$ by*

$$g(y) = \limsup_{n\to\infty} \frac{1}{\sum_{k=1}^n \beta_k} \sum_{k=1}^n \beta_k c_\kappa(d(y,z_k))$$

for $y \in K$. Then g has a unique minimizer on K.

Proof. Kajimura and Kimura [8] showed the case where $\kappa = -1$ with the assumption $\sum_{k=1}^\infty \beta_k = \infty$. In addition, Kimura and Kohsaka [14, 15] showed the case where $\kappa = 0$ and $\kappa = 1$ with $\sum_{k=1}^\infty \beta_k = \infty$. Now we give the proof for all $\kappa \in \mathbb{R}$ without the assumption $\sum_{k=1}^\infty \beta_k = \infty$.

Take $u \in K$ such that $\sup_{n\in\mathbb{N}} d(z_n, u) < D_\kappa/2$, and put $M = \sup_{n\in\mathbb{N}} d(z_n, u)$. Then we have $c_\kappa(d(z_k, u)) \le c_\kappa(M)$ for all $k \in \mathbb{N}$. It deduces that $0 \le \inf_{y\in K} g(y) \le g(u) \le c_\kappa(M) < \infty$.

Fix $x_1, x_2 \in K$ arbitrarily. Then we get

$$c_\kappa\Big(\frac{1}{2}d(x_1,x_2)\Big) + c''_\kappa\Big(\frac{1}{2}d(x_1,x_2)\Big)c_\kappa\Big(d\Big(\frac{1}{2}x_1 \oplus \frac{1}{2}x_2, z_k\Big)\Big) \le \frac{1}{2}c_\kappa(d(x_1,z_k)) + \frac{1}{2}c_\kappa(d(x_2,z_k))$$

for each $k \in \mathbb{N}$ by Lemma 2.9. It follows that

$$\begin{aligned} c_\kappa\Big(\frac{1}{2}d(x_1,x_2)\Big) + c''_\kappa\Big(\frac{1}{2}d(x_1,x_2)\Big) \cdot \frac{1}{\sum_{k=1}^n \beta_k}\sum_{k=1}^n \beta_k c_\kappa\Big(d\Big(\frac{1}{2}x_1 \oplus \frac{1}{2}x_2, z_k\Big)\Big) \\ \le \frac{1}{2\sum_{k=1}^n \beta_k}\sum_{k=1}^n \beta_k c_\kappa(d(x_1,z_k)) + \frac{1}{2\sum_{k=1}^n \beta_k}\sum_{k=1}^n \beta_k c_\kappa(d(x_2,z_k)) \end{aligned}$$

for any $n \in \mathbb{N}$. Hence we have

$$c_\kappa\Big(\frac{1}{2}d(x_1,x_2)\Big) + c''_\kappa\Big(\frac{1}{2}d(x_1,x_2)\Big)g\Big(\frac{1}{2}x_1 \oplus \frac{1}{2}x_2\Big) \le \frac{1}{2}g(x_1) + \frac{1}{2}g(x_2)$$

for every $x_1, x_2 \in K$.

Put $L = \inf_{y\in K} g(y)$, which satisfies $0 \le L \le c_\kappa(M) < \infty$. Then we can take $\{y_n\} \subset K$ such that $g(y_n) \ge g(y_{n+1})$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} g(y_n) = L$. Suppose that $n, m \in \mathbb{N}$ satisfies $n \le m$. From the inequality above, we get

$$\begin{aligned} g(y_n) &\ge \frac{1}{2}g(y_n) + \frac{1}{2}g(y_m) \\ &\ge c_\kappa\Big(\frac{1}{2}d(y_n,y_m)\Big) + c''_\kappa\Big(\frac{1}{2}d(y_n,y_m)\Big)L \\ &= L + (1 - L\kappa)c_\kappa\Big(\frac{1}{2}d(y_n,y_m)\Big). \end{aligned}$$

Therefore we obtain

$$(1 - L\kappa)c_\kappa\Big(\frac{1}{2}d(y_n,y_m)\Big) \le g(y_n) - L \to 0$$

as $n \to \infty$. Note that $1 - L\kappa > 0$ holds without regard to $\kappa \in \mathbb{R}$. Indeed, if $\kappa > 0$ then we have

$$1 - L\kappa = \Big(c_\kappa\Big(\frac{D_\kappa}{2}\Big) - L\Big)\kappa \ge (c_\kappa(M) - L)\kappa > 0.$$

Otherwise, if $\kappa \le 0$ then $1 - L\kappa \ge 1 > 0$. Hence we get $d(y_n, y_m) \to 0$ as $m, n \to \infty$. It follows that $\{y_n\}$ is a Cauchy sequence on the closed set K and thus it converges to some $p \in K$. Since g is continuous, we have $g(p) = \lim_{n\to\infty} g(y_n) = L$. Therefore, p is a minimizer of g.

Let q be another minimizer of g. Then we get

$$L + (1 - L\kappa)c_\kappa\Big(\frac{1}{2}d(p,q)\Big) \le \frac{1}{2}g(p) + \frac{1}{2}g(q) = L$$

and thus $p = q$ holds, which is the conclusion. □

Corollary 4.14. *Let X be a complete* CAT(κ) *space and K an admissible closed convex subset of X. Take $z_1, z_2, \dots, z_n \in K$ and $\gamma_1, \gamma_2, \dots, \gamma_n \in \,]0,\infty[$. Define $g\colon K \to [0,\infty[$ by*

$$g(y) = \sum_{k=1}^n \gamma_k c_\kappa(d(y,z_k))$$

for $y \in K$. Then g has a unique minimizer on K.

Proof. Using Lemma 4.13 for $\{\beta_n\} \subset]0,\infty[$ defined by $\beta_i = \gamma_i$ for $i = 1,2,\dots,n$ and $\beta_i = 0$ for $i > n$, we obtain the conclusion. □

Lemma 4.15. *Let X be a* CAT(κ) *space and T a quasinonexpansive mapping from X into itself such that $d(x,Tx) < D_\kappa$ for every $x \in X$. Let $\{\alpha_n\} \subset [0,1]$. Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by the following iteration:*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tx_n$$

for $n \in \mathbb{N}$. Then $\{d(x_n,p)\}$ is nonincreasing for any $p \in F(T)$.

Proof. Let $p \in F(T)$, then we get

$$\begin{aligned} c_\kappa(d(x_{n+1},p)) &\le \alpha_n c_\kappa(d(x_n,p)) + (1-\alpha_n)c_\kappa(d(Tx_n,p)) \\ &\le \alpha_n c_\kappa(d(x_n,p)) + (1-\alpha_n)c_\kappa(d(x_n,p)) \\ &= c_\kappa(d(x_n,p)) \end{aligned}$$

for any $n \in \mathbb{N}$ and hence we get the conclusion. □

Lemma 4.16. *Let X be an admissible* CAT(κ) *space and $T\colon X \to X$ a tightly quasinonexpansive mapping. Let $\{\alpha_n\} \subset [0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tx_n$$

for $n \in \mathbb{N}$. Then $d(x_n,Tx_n) \to 0$.

Proof. Let $p \in F(T)$. Then $\{d(x_n,p)\}$ is nonincreasing from Lemma 4.15, and hence there exists a limit $c \ge 0$ of $\{d(x_n,p)\}$. If $c = 0$, then $x_n \to p$, which implies the conclusion. In what follows, we assume that $c > 0$. Then we obtain $c < D_\kappa/2$ and

$$c_\kappa(d(Tx_n,p))c_\kappa''(d(x_n,Tx_n)) \le c_\kappa(d(x_n,p)) - c_\kappa(d(x_n,Tx_n))$$

for any $n \in \mathbb{N}$ by the definition of tightly quasinonexpansive mappings. Since X is admissible, we have $c_\kappa''(d(x_n,Tx_n)) > 0$ and hence

$$\begin{aligned} &c_\kappa(d(x_{n+1},p)) \\ &\quad\le \alpha_n c_\kappa(d(x_n,p)) + (1-\alpha_n)c_\kappa(d(Tx_n,p)) \\ &\quad\le \alpha_n c_\kappa(d(x_n,p)) + (1-\alpha_n)\cdot\frac{c_\kappa(d(Tx_n,p)) - c_\kappa(d(x_n,Tx_n))}{c_\kappa''(d(x_n,Tx_n))} \\ &\quad= c_\kappa(d(x_n,p)) + (1-\alpha_n)\cdot\frac{c_\kappa(d(Tx_n,p)) - c_\kappa(d(x_n,Tx_n)) - c_\kappa(d(x_n,p))c_\kappa''(d(x_n,Tx_n))}{c_\kappa''(d(x_n,Tx_n))} \end{aligned}$$

holds for any $n \in \mathbb{N}$. Thus we get

$$\begin{aligned} &c_\kappa(d(x_n,p)) - c_\kappa(d(x_{n+1},p)) \\ &\quad\ge (1-\alpha_n)\cdot\frac{-c_\kappa(d(Tx_n,p)) + c_\kappa(d(x_n,Tx_n)) + c_\kappa(d(x_n,p))c_\kappa''(d(x_n,Tx_n))}{c_\kappa''(d(x_n,Tx_n))} \\ &\quad\ge (1-\alpha_n)\cdot\frac{-c_\kappa(d(Tx_n,p)) + c_\kappa(d(x_n,p)) + c_\kappa(d(x_n,Tx_n))c_\kappa''(d(x_n,p))}{c_\kappa''(d(x_n,Tx_n))} \\ &\quad\ge (1-\alpha_n)\cdot\frac{c_\kappa(d(x_n,Tx_n))}{c_\kappa''(d(x_n,Tx_n))}\cdot c_\kappa''(d(x_n,p)) \\ &\quad\ge 0 \end{aligned}$$

for any $n \in \mathbb{N}$ by Lemma 2.8. Since $\lim_{n\to\infty} (c_\kappa(d(x_n, p)) - c_\kappa(d(x_{n+1}, p))) = 0$ and $c''_\kappa(d(x_n, p)) \to c''_\kappa(c) > 0$, we obtain

$$\lim_{n\to\infty} \left((1-\alpha_n) \cdot \frac{c_\kappa(d(x_n, Tx_n))}{c''_\kappa(d(x_n, Tx_n))} \right) = 0.$$

Put $\varepsilon = \liminf_{n\to\infty}(1-\alpha_n) > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $1-\alpha_n > \varepsilon/2$ for any $n \geq n_0$. Thus we have

$$\lim_{n\to\infty} \frac{c_\kappa(d(x_n, Tx_n))}{c''_\kappa(d(x_n, Tx_n))} = 0.$$

It means that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. In fact, if $\kappa > 0$, then

$$0 = \lim_{n\to\infty} \frac{c_\kappa(d(x_n, Tx_n))}{c''_\kappa(d(x_n, Tx_n))} = \lim_{n\to\infty} \frac{1}{\kappa} \left(\frac{1}{\cos\big(\sqrt{\kappa}\, d(x_n, Tx_n)\big)} - 1 \right)$$

and hence $\cos\big(\sqrt{\kappa}\, d(x_n, Tx_n)\big) \to 1$; if $\kappa \leq 0$ then

$$0 = \lim_{n\to\infty} \frac{c_\kappa(d(x_n, Tx_n))}{c''_\kappa(d(x_n, Tx_n))} = \lim_{n\to\infty} \frac{1}{-\kappa} \left(1 - \frac{1}{\cosh\big(\sqrt{-\kappa}\, d(x_n, Tx_n)\big)} \right),$$

which implies $\cosh\big(\sqrt{-\kappa}\, d(x_n, Tx_n)\big) \to 1$. Therefore we get the desired result. □

Lemma 4.17. *Let X be an admissible* CAT(κ) *space and $T\colon X \to X$ a quasinonexpansive mapping. Let $\{\beta_n\} \subset [0,1]$ such that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by*

$$x_{n+1} = \beta_n x_n \overset{\kappa}{\oplus} (1-\beta_n) T x_n$$

for $n \in \mathbb{N}$. Then $d(x_n, Tx_n) \to 0$.

Proof. Let $p \in F(T)$ and put

$$S_n = \sqrt{\beta_n^2 + 2\beta_n(1-\beta_n) c''_\kappa(d(x_n, Tx_n)) + (1-\beta_n)^2}$$

for each $n \in \mathbb{N}$. Then, from Theorem 3.15,

$$\begin{aligned} c_\kappa(d(x_{n+1}, p)) \leq \beta_n c_\kappa(d(x_n, p)) &+ (1-\beta_n) c_\kappa(d(Tx_n, p)) \\ &- \frac{2\beta_n(1-\beta_n) c_\kappa(d(x_n, Tx_n))}{1+S_n} \cdot \frac{\beta_n c''_\kappa(d(x_n, p)) + (1-\beta_n) c''_\kappa(d(Tx_n, p))}{S_n} \end{aligned}$$

holds for any $n \in \mathbb{N}$. It follows from quasinonexpansiveness of T that

$$\begin{aligned} 0 &\leq \frac{2\beta_n(1-\beta_n) c_\kappa(d(x_n, Tx_n))}{1+S_n} \cdot \frac{\beta_n c''_\kappa(d(x_n, p)) + (1-\beta_n) c''_\kappa(d(Tx_n, p))}{S_n} \\ &\leq \beta_n c_\kappa(d(x_n, p)) + (1-\beta_n) c_\kappa(d(Tx_n, p)) - c_\kappa(d(x_{n+1}, p)) \\ &\leq c_\kappa(d(x_n, p)) - c_\kappa(d(x_{n+1}, p)) \end{aligned}$$

for any $n \in \mathbb{N}$. We know that $\{d(x_n, p)\}$ is nonincreasing from Lemma 4.15, which implies $\lim_{n\to\infty} (c_\kappa(d(x_n, p)) - c_\kappa(d(x_{n+1}, p))) = 0$. Therefore we obtain

$$\lim_{n\to\infty} \beta_n(1-\beta_n) c_\kappa(d(x_n, Tx_n)) \big(\beta_n c''_\kappa(d(x_n, p)) + (1-\beta_n) c''_\kappa(d(Tx_n, p)) \big) = 0.$$

Note that $c''_\kappa(d(x_n,p))$ and $c''_\kappa(d(Tx_n,p))$ do not converge to 0. Indeed, if $\kappa > 0$ then we have $\sup_{n\in\mathbb{N}} c''_\kappa(d(Tx_n,p)) \geq \sup_{n\in\mathbb{N}} c''_\kappa(d(x_n,p)) \geq c''_\kappa(d(x_1,p)) > 0$. On the other hand, if $\kappa \leq 0$ then $c''_\kappa(d(x_n,p)) \geq 1$ for every $n \in \mathbb{N}$. This follows that

$$\lim_{n\to\infty} \beta_n(1-\beta_n)c_\kappa(d(x_n,Tx_n)) = 0.$$

Consequently we obtain $\lim_{n\to\infty} c_\kappa(d(x_n,Tx_n)) = 0$, which is the desired result. □

Corollary 4.18. *Let X and $T\colon X \to X$ be the same as Lemma 4.17. Let $\{\alpha_n\} \subset [0,1]$ such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tx_n$$

for $n \in \mathbb{N}$. Then $d(x_n,Tx_n) \to 0$.

Proof. Let $p \in F(T)$, and put $\beta_n = \zeta^\kappa_{d(x_n,Tx_n)}(\alpha_n)$ for each $n \in \mathbb{N}$. Then we have

$$\beta_n x_n \overset{\kappa}{\oplus} (1-\beta_n)Tx_n = \alpha_n x_n \oplus (1-\alpha_n)Tx_n$$

for any $n \in \mathbb{N}$. From Lemma 4.15, we get $\sup_{n\in\mathbb{N}} d(x_n,p) < D_\kappa/2$. It deduces that

$$\sup_{n\in\mathbb{N}} d(x_n,Tx_n) \leq \sup_{n\in\mathbb{N}}(d(x_n,p)+d(Tx_n,p)) < D_\kappa.$$

Therefore we obtain $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ by Corollary 3.26, which implies the conclusion from Lemma 4.17. □

Lemma 4.19. *Let X be an admissible complete* CAT(κ) *space and F a nonempty subset of X. Let $\{x_n\}$ be a κ-bounded sequence on X. Suppose that* (i) *and* (ii) *hold: For any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $w_0 = AC(\{x_{n_i}\})$,*

(i) *$w_0 \in F$;*
(ii) *$\{d(x_n,w_0)\}$ is convergent.*

Then $\{x_n\}$ Δ-converges to some element in F.

Proof. Let $x_0 = AC(\{x_n\})$ and take its subsequence $\{x_{n_i}\}$ arbitrarily. Put $w_0 = AC(\{x_{n_i}\})$. Then $w_0 \in F$, and $\{d(x_n,w_0)\}$ is convergent. It follows that

$$\lim_{n\to\infty} d(x_n,w_0) = \lim_{i\to\infty} d(x_{n_i},w_0) \leq \limsup_{i\to\infty} d(x_{n_i},x_0) \leq \limsup_{n\to\infty} d(x_n,x_0) \leq \lim_{n\to\infty} d(x_n,w_0).$$

Thus $x_0 = w_0 = AC(\{x_{n_i}\}) \in F$, which implies $x_n \overset{\Delta}{\rightharpoonup} x_0$. □

Lemma 4.20. *Let X be an admissible complete* CAT(κ) *space and T a Δ-demiclosed mapping from X into itself. Suppose that a κ-bounded sequence $\{x_n\} \subset X$ satisfies* (i) *and* (ii)*:*

(i) *$d(x_n,Tx_n) \to 0$;*
(ii) *$\{d(x_n,p)\}$ is convergent if p is a fixed point of T.*

Then $\{x_n\}$ Δ-converges to some fixed point of T.

Proof. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ and take $w_0 = AC(\{x_{n_i}\})$. We show w_0 is a fixed point of T. Take a Δ-convergent subsequence $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ and put $z_0 = \Delta\text{-}\lim_{j\to\infty} x_{n_{i_j}}$. Since T is Δ-demiclosed, we have $z_0 \in F(T)$. Then $\{d(x_n,z_0)\}$ is convergent and hence

$$\begin{aligned}\limsup_{i\to\infty} d(x_{n_i},w_0) &\leq \lim_{i\to\infty} d(x_{n_i},z_0)\\ &= \lim_{j\to\infty} d(x_{n_{i_j}},z_0) \leq \limsup_{j\to\infty} d(x_{n_{i_j}},w_0) \leq \limsup_{i\to\infty} d(x_{n_i},w_0).\end{aligned}$$

It implies $w_0 = z_0 \in F(T)$. Therefore, from Lemma 4.19, we get the conclusion. □

Now we show main results.

Theorem 4.21. *Let X be an admissible complete* $\mathrm{CAT}(\kappa)$ *space and $T\colon X \to X$ a vicinal mapping with ψ. Let $\{\alpha_n\} \subset [0,1[$ such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tx_n$$

for $n \in \mathbb{N}$. Let us denote (a) *and* (b) *by the following conditions:*

(a) *$\{Tx_n\}$ is κ-bounded;*
(b) $\sup_{n\in\mathbb{N}} d(x_n, Tx_n) < D_\kappa/2$.

Let $\psi = \varphi' \circ c_\kappa \colon [0, D_\kappa/2[\to]0,\infty[$ and define conditions (P1) *and* (P2) *as follows:*

(P1) *ψ is nondecreasing;*
(P2) $\sup_{n\in\mathbb{N}} \psi(d(x_n, Tx_n)) < \infty$.

Then the following hold:

(i) *Suppose that ψ satisfies* (P1). *Then $F(T) \neq \emptyset$ if* (a) *and* (b) *hold. Conversely, $F(T) \neq \emptyset$ only if* (a) *and* (b) *hold when T is tightly quasinonexpansive.*
(i)′ *Suppose that ψ satisfies* (P2). *Then $F(T) \neq \emptyset$ if and only if* (a) *holds.*

Theorem 4.22. *Let X be an admissible complete* $\mathrm{CAT}(\kappa)$ *space and T a quasinonexpansive and Δ-demiclosed mapping from X into itself. Suppose that $\{\alpha_n\}$ and $\{x_n\}$ are the same as the previous theorem. Then the following hold:*

(ii) *If T is tightly quasinonexpansive and $\limsup_{n\to\infty} \alpha_n < 1$, then $\{x_n\}$ Δ-converges to some fixed point of T.*
(iii) *If $\liminf_{n\in\mathbb{N}} \alpha_n(1-\alpha_n) > 0$, then $\{x_n\}$ Δ-converges to some fixed point of T.*

Proof of Theorem 4.21. Let T be a vicinal mapping with ψ from X into itself. Then, T is Δ-demiclosed from Lemma 4.2.

First we show the only if part of (i) and (i)′ simultaneously. Suppose that $F(T)$ is nonempty, and fix $p \in F(T)$. Then $\{d(x_n, p)\}$ is nonincreasing from Lemma 4.15. It implies $\sup_{n\in\mathbb{N}} d(Tx_n, p) \le \sup_{n\in\mathbb{N}} d(x_n, p) = d(x_1, p) < D_\kappa/2$ and thus (a) holds. Moreover, considering the case where (i) and assuming that T is tightly quasinonexpansive, we have

$$\begin{aligned} c_\kappa(d(x_n, Tx_n)) &\le c_\kappa(d(x_n, Tx_n)) + c_\kappa(d(Tx_n, p))c_\kappa''(d(x_n, Tx_n)) \\ &\le c_\kappa(d(x_n, p)) \\ &\le c_\kappa(d(x_1, p)) \end{aligned}$$

for any $n \in \mathbb{N}$, and then (b) holds.

Next, consider the if part of (i) and (i)′. Let $y \in X$ and $k \in \mathbb{N}$. Then we have

$$\begin{aligned} c_\kappa(d(x_{k+1}, Ty)) &\le \alpha_k c_\kappa(d(x_k, Ty)) + (1-\alpha_k)c_\kappa(d(Tx_k, Ty)) \\ &= c_\kappa(d(x_k, Ty)) + (1-\alpha_k)\big(c_\kappa(d(Tx_k, Ty)) - c_\kappa(d(x_k, Ty))\big). \end{aligned}$$

Therefore, since T is vicinal with ψ, we obtain

$$\begin{aligned} &(1-\alpha_k)\psi(d(y, Ty))\big(c_\kappa(d(Tx_k, y)) - c_\kappa(d(Tx_k, Ty))\big) \\ &\quad\ge (1-\alpha_k)\psi(d(x_k, Tx_k))\big(c_\kappa(d(Tx_k, Ty)) - c_\kappa(d(x_k, Ty))\big) \\ &\quad\ge \psi(d(x_k, Tx_k))\big(c_\kappa(d(x_{k+1}, Ty)) - c_\kappa(d(x_k, Ty))\big). \end{aligned}$$

Hence we get

$$
\begin{aligned}
&(1-\alpha_k)\psi(d(y,Ty))c_\kappa(d(Tx_k,Ty)) \\
&\qquad \leq (1-\alpha_k)\psi(d(y,Ty))c_\kappa(d(Tx_k,y)) - \psi(d(x_k,Tx_k))\bigl(c_\kappa(d(x_{k+1},Ty)) - c_\kappa(d(x_k,Ty))\bigr),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\psi(d(y,Ty))\cdot\frac{1-\alpha_k}{\psi(d(x_k,Tx_k))}\cdot c_\kappa(d(Tx_k,Ty)) \\
&\qquad \leq \psi(d(y,Ty))\cdot\frac{1-\alpha_k}{\psi(d(x_k,Tx_k))}\cdot c_\kappa(d(Tx_k,y)) + \bigl(c_\kappa(d(x_k,Ty)) - c_\kappa(d(x_{k+1},Ty))\bigr)
\end{aligned}
$$

since $\psi(d(x_k,Tx_k)) \neq 0$. It concludes

$$
\begin{aligned}
&\psi(d(y,Ty))\sum_{k=1}^{n}\frac{1-\alpha_k}{\psi(d(x_k,Tx_k))}c_\kappa(d(Tx_k,Ty)) \\
&\qquad \leq \psi(d(y,Ty))\sum_{k=1}^{n}\frac{1-\alpha_k}{\psi(d(x_k,Tx_k))}c_\kappa(d(Tx_k,y)) + \bigl(c_\kappa(d(x_1,Ty)) - c_\kappa(d(x_{n+1},Ty))\bigr) \quad (*)
\end{aligned}
$$

for any $y \in X$.

Put $W_n = \sum_{k=1}^{n}(1-\alpha_k)/\psi(d(x_k,Tx_k))$ for each $n \in \mathbb{N}$. Then we have $\lim_{n\to\infty} W_n = \infty$. Indeed, if (P1) and (b) is true, then

$$
\lim_{n\to\infty} W_n \geq \sum_{k=1}^{\infty}\frac{1-\alpha_k}{\psi\bigl(\sup_{k\in\mathbb{N}} d(x_k,Tx_k)\bigr)} = \infty;
$$

if (P2) is true, then

$$
\lim_{n\to\infty} W_n \geq \sum_{k=1}^{\infty}\frac{1-\alpha_k}{\sup_{k\in\mathbb{N}}\psi(d(x_k,Tx_k))} = \infty.
$$

Define $g\colon X \to \mathbb{R}$ by

$$
g(y) = \limsup_{n\to\infty}\frac{1}{W_n}\sum_{k=1}^{n}\frac{1-\alpha_k}{\psi(d(x_k,Tx_k))}c_\kappa(d(Tx_k,y))
$$

for $y \in X$. Then g has a unique minimizer $p \in X$ from Lemma 4.13 and (a). It satisfies $g(p) \leq g(Tp)$ obviously. We also have

$$
\begin{aligned}
\psi(d(y,Ty))g(Ty) &\leq \psi(d(y,Ty))g(y) + \limsup_{n\to\infty}\frac{c_\kappa(d(x_1,Ty)) - c_\kappa(d(x_{n+1},Ty))}{W_n} \\
&\leq \psi(d(y,Ty))g(y) + \lim_{n\to\infty}\frac{c_\kappa(d(x_1,Ty))}{W_n} \\
&= \psi(d(y,Ty))g(y)
\end{aligned}
$$

for all $y \in X$ by the inequality $(*)$. It implies $g(Tp) \leq g(p)$ and hence $p = Tp$, which implies $F(T) \neq \emptyset$. $\square$

Proof of Theorem 4.22. We first show (iii). Take $p \in F(T)$, and suppose that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then we obtain $d(x_n, Tx_n) \to 0$ from Corollary 4.18. We also have the convergence of $\{d(x_n, p)\}$ from Lemma 4.15. These imply the desired result from Lemma 4.20.

Next, we show (ii). Suppose that T is tightly quasinonexpansive and $\limsup_{n\to\infty} \alpha_n < 1$. Then we have $d(x_n, Tx_n) \to 0$ from Lemma 4.16. Hence $\{x_n\}$ Δ-converges to some $x_0 \in F(T)$ from Lemma 4.15 and Lemma 4.20. □

In Theorem 4.21, the condition (P2) always holds if ψ is bounded above. Related to Theorems 4.21 and 4.22, we can use κ-convex combinations instead of usual convex combinations as follows:

Theorem 4.23. *Let X be an admissible complete* CAT(κ) *space and $T: X \to X$ a vicinal mapping with ψ. Let $\{\beta_n\} \subset [0,1[$ such that $\sum_{n=1}^{\infty}(1-\beta_n) = \infty$. Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by*

$$x_{n+1} = \beta_n x_n \overset{\kappa}{\oplus} (1-\beta_n)Tx_n$$

for $n \in \mathbb{N}$. Let us denote (a) *and* (b) *by the following conditions:*

(a) *$\{Tx_n\}$ is κ-bounded;*
(b) $\sup_{n\in\mathbb{N}} d(x_n, Tx_n) < D_\kappa/2$.

Let $\psi = \varphi' \circ c_\kappa : [0, D_\kappa/2[\to]0,\infty[$ and define conditions (P1) *and* (P2) *as follows:*

(P1) *ψ is nondecreasing;*
(P2) $\sup_{n\in\mathbb{N}} \psi(d(x_n, Tx_n)) < \infty$.

Then the following hold:

(i) *Suppose that ψ satisfies* (P1). *Then $F(T) \neq \varnothing$ if* (a) *and* (b) *hold. Conversely, $F(T) \neq \varnothing$ only if* (a) *and* (b) *hold when T is tightly quasinonexpansive.*
(i)′ *Suppose that ψ satisfies* (P2). *Then $F(T) \neq \varnothing$ if and only if* (a) *holds.*

Theorem 4.24. *Let X be an admissible complete* CAT(κ) *space and T a quasinonexpansive and Δ-demiclosed mapping from X into itself. Let $\{\beta_n\} \subset [0,1[$ such that $\sum_{n=1}^{\infty}(1-\beta_n) = \infty$. Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by*

$$x_{n+1} = \beta_n x_n \overset{\kappa}{\oplus} (1-\beta_n)Tx_n$$

for $n \in \mathbb{N}$. Then the following hold:

(ii) *If T is tightly quasinonexpansive and* $\limsup_{n\to\infty} \beta_n < 1$, *then $\{x_n\}$ Δ-converges to some fixed point of T.*
(iii) *If* $\liminf_{n\in\mathbb{N}} \beta_n(1-\beta_n) > 0$, *then $\{x_n\}$ Δ-converges to some fixed point of T.*

Proof of Theorems 4.23 and 4.24. If $\kappa = 0$, then it is obvious from Theorems 4.21 and 4.22 since $\beta_n x_n \overset{0}{\oplus} (1-\beta_n)Tx_n = \beta_n x_n \overset{\kappa}{\oplus} (1-\beta_n)Tx_n$. Therefore, we hereinafter assume that $\kappa \neq 0$.

Let ζ_D^κ be a function defined in Section 2.5 for each $D \geq 0$. Put $D_n = d(x_n, Tx_n)$ for every $n \in \mathbb{N}$, and define a real sequence $\{\alpha_n\}$ on $[0,1[$ by $\alpha_n = \left(\zeta_{D_n}^\kappa\right)^{-1}(\beta_n)$ for every $n \in \mathbb{N}$. Then $\beta_n x_n \overset{\kappa}{\oplus} (1-\beta_n)Tx_n = \alpha_n x_n \oplus (1-\alpha_n)Tx_n$ for all $n \in \mathbb{N}$ by Lemma 3.6. Therefore, to prove desired theorems, we know that it is sufficient to show the following:

(a) If $\sum_{n=1}^{\infty}(1-\beta_n) = \infty$, then $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$;
(b) if $\limsup_{n\to\infty} \beta_n < 1$, then $\limsup_{n\to\infty} \alpha_n < 1$;
(c) if $\liminf_{n\in\mathbb{N}} \beta_n(1-\beta_n) > 0$, then $\liminf_{n\in\mathbb{N}} \alpha_n(1-\alpha_n) > 0$.

We obtain (b) and (c) by Lemma 3.25 and Corollary 3.26, respectively. Thus we only need to show (a). Suppose that $\sum_{n=1}^{\infty}(1-\beta_n)=\infty$.

If $\kappa < 0$, then we have

$$\frac{1}{2}(1-\beta_n) \le 1-\alpha_n$$

for all $n \in \mathbb{N}$ from Lemma 2.18, and hence

$$\sum_{n=1}^{\infty}(1-\alpha_n) \ge \frac{1}{2}\sum_{n=1}^{\infty}(1-\beta_n) = \infty.$$

We consider the case where $\kappa > 0$. Fix $n \in \mathbb{N}$ such that $x_n \neq Tx_n$. Then using Lemma 2.17, we obtain

$$1-\beta_n = 1-\zeta_{D_n}^{\kappa}(\alpha_n) < \frac{D_n}{c_\kappa'(D_n)}(1-\alpha_n).$$

Since X is admissible and a function $]0, D_\kappa[\ \ni t \mapsto t/c_\kappa'(t) \in\ ]1,\infty[$ is strictly increasing, we get

$$\frac{D_n}{c_\kappa'(D_n)} < \frac{D_\kappa/2}{c_\kappa'(D_\kappa/2)} = \frac{\pi}{2}$$

and hence

$$1-\beta_n \le \frac{\pi}{2}(1-\alpha_n).$$

It also holds even if $x_n = Tx_n$. Consequently, we have

$$\sum_{n=1}^{\infty}(1-\alpha_n) \ge \frac{2}{\pi}\sum_{n=1}^{\infty}(1-\beta_n) = \infty.$$

Therefore we get the conclusion. □

4.4 Halpern type fixed point approximations

For a uniquely D-geodesic space X and a mapping $T\colon X \to X$, Halpern type iterative scheme generates a sequence $\{x_n\}$ on X by an iteration $x_{n+1} = \alpha_n u \oplus (1-\alpha_n)Tx_n$ for $n \in \mathbb{N}$ and some fixed $u \in X$. We consider the convergence of such $\{x_n\}$ to a fixed point of T.

To prove approximation theorems of a fixed point using Halpern type approximation scheme, we use the following lemma.

Lemma 4.25 (Kimura and Saejung [16]; Saejung and Yotkaew [26])**.** *Let $\{\beta_n\} \subset\ ]0,1[$ such that $\sum_{n=1}^{\infty}\beta_n = \infty$. Take $\{a_n\} \subset [0,\infty[$ and $\{b_n\} \subset \mathbb{R}$ which satisfies $a_{n+1} \le (1-\beta_n)a_n + \beta_n b_n$ for all $n \in \mathbb{N}$. If $\liminf_{i\to\infty}\left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \ge 0$ implies $\limsup_{i\to\infty} t_{\varphi(i)} \le 0$ for any nondecreasing function $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that $\lim_{i\to\infty}\varphi(i) = \infty$, then $a_n \to 0$. Suppose that $\limsup_{i\to\infty} b_{\varphi(i)} \le 0$ for any nondecreasing function $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that $\liminf_{i\to\infty}\left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \ge 0$ and $\lim_{i\to\infty}\varphi(i) = \infty$. Then $\{a_n\}$ converges to 0.*

Lemma 4.26. *For $\kappa \neq 0$, let X be a* CAT(κ) *space. Take $x, y, z \in X$ such that $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$, and let $\alpha \in [0,1]$. Put*

$$\beta = 1 - \frac{1-\alpha}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)c_\kappa''(d(x,y)) + (1-\alpha)^2}}.$$

Then

$$
\begin{aligned}
&c_\kappa(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z)) \\
&\qquad \le (1-\beta)\, c_\kappa(d(y,z)) \\
&\qquad\qquad + \beta \cdot \frac{1}{\kappa}\left(1 - \frac{\left(1-\alpha+\sqrt{\alpha^2+2\alpha(1-\alpha)c_\kappa''(d(x,y))+(1-\alpha)^2}\right)c_\kappa''(d(x,z))}{\alpha+2(1-\alpha)c_\kappa''(d(x,y))}\right).
\end{aligned}
$$

Proof. Set $S=\sqrt{\alpha^2+2\alpha(1-\alpha)c_\kappa''(d(x,y))+(1-\alpha^2)}$. By using Theorem 3.14, we obtain

$$
\begin{aligned}
c_\kappa(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z)) &= \frac{1}{\kappa}\big(1-c_\kappa''(d(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z))\big) \\
&\le \frac{1}{\kappa} - \frac{1}{\kappa}\cdot\frac{\alpha c_\kappa''(d(x,z))+(1-\alpha)c_\kappa''(d(y,z))}{S} \\
&= \frac{1}{\kappa} - \frac{1-\alpha}{\kappa S}c_\kappa''(d(y,z)) - \frac{\alpha}{\kappa S}c_\kappa''(d(x,z)) \\
&= \frac{1}{\kappa} - \frac{1-\alpha}{\kappa S}\big(1-\kappa c_\kappa(d(y,z))\big) - \frac{\alpha}{\kappa S}c_\kappa''(d(x,z)) \\
&= \frac{1}{\kappa}\Big(1-\frac{1-\alpha}{S}\Big) + \frac{1-\alpha}{S}c_\kappa(d(y,z)) - \frac{\alpha}{\kappa S}c_\kappa''(d(x,z)) \\
&= \beta\cdot\frac{1}{\kappa} + (1-\beta)\, c_\kappa(d(y,z)) - \frac{\alpha}{\kappa S}c_\kappa''(d(x,z)) \\
&= (1-\beta)\, c_\kappa(d(y,z)) + \beta\cdot\frac{1}{\kappa}\Big(1-\frac{\alpha}{\beta S}c_\kappa''(d(x,z))\Big).
\end{aligned}
$$

We also have

$$
\frac{\alpha}{\beta S} = \frac{\alpha}{S-(1-\alpha)} = \frac{\alpha(S+(1-\alpha))}{S^2-(1-\alpha)^2} = \frac{1-\alpha+S}{\alpha+2(1-\alpha)c_\kappa''(d(x,y))}.
$$

Therefore we get the conclusion. □

Lemma 4.27. *Let* $\kappa \in \mathbb{R}$ *and* $\{\alpha_n\},\subset[0,1]$ *such that* $\lim_{n\to\infty}\alpha_n=0$. *Let* $\{d_n\}\subset[0,D_\kappa/2[$ *be a sequence and put*

$$
\beta_n = 1-\frac{1-\alpha_n}{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)c_\kappa''(d_n)+(1-\alpha_n)^2}}
$$

for all $n\in\mathbb{N}$. *Then* $\{\beta_n\}\subset[0,1]$.

Proof. The inequality $\beta_n\le 1$ is obvious for any $n\in\mathbb{N}$. Put $M=\sup_{n\in\mathbb{N}} d_n$. First, we assume that $\kappa>0$. Then we have $c_\kappa''(M)\ge 0$ since $\{d_n\}\subset[0,D_\kappa/2[$, and hence

$$
\begin{aligned}
\beta_n &\ge 1-\frac{1-\alpha_n}{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)c_\kappa''(M)+(1-\alpha_n)^2}} \\
&\ge \frac{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)c_\kappa''(M)+(1-\alpha_n)^2}-(1-\alpha_n)}{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)c_\kappa''(M)+(1-\alpha_n)^2}} \\
&= \frac{\alpha_n^2+2\alpha_n(1-\alpha_n)c_\kappa''(M)}{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)c_\kappa''(M)+(1-\alpha_n)^2}\left(\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)c_\kappa''(M)+(1-\alpha_n)^2}+1-\alpha_n\right)} \\
&\ge \frac{\alpha_n^2+2\alpha_n(1-\alpha_n)c_\kappa''(M)}{\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)+(1-\alpha_n)^2}\left(\sqrt{\alpha_n^2+2\alpha_n(1-\alpha_n)+(1-\alpha_n)^2}+1\right)}
\end{aligned}
$$

$$= \frac{\alpha_n^2}{2} + \alpha_n(1-\alpha_n)c_\kappa''(M)$$

for any $n \in \mathbb{N}$. It implies $\beta_n \geq 0$ for any $n \in \mathbb{N}$.

Next we assume that $\kappa \leq 0$. Then, since $c_\kappa''(d_n) \geq 1$ for any $n \in \mathbb{N}$, we have

$$\beta_n \geq 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cdot 1 + (1-\alpha_n)^2}} = \alpha_n \geq 0$$

for any $n \in \mathbb{N}$. □

Lemma 4.28. *Let $\kappa \in \mathbb{R}$ and $\{\alpha_n\} \subset [0,1]$ such that $\sum_{n=1}^\infty \alpha_n = \infty$. Let $\{d_n\} \subset [0, D_\kappa/2[$ and put*

$$\beta_n = 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)c_\kappa''(d_n) + (1-\alpha_n)^2}}$$

for all $n \in \mathbb{N}$. In the case where $\kappa > 0$, suppose that (i) *or* (ii) *holds:*

(i) $\sup_{n\in\mathbb{N}} d_n < D_\kappa/2$,

(ii) $\sum_{n=1}^\infty \alpha_n^2 = \infty$.

Then $\sum_{n=1}^\infty \beta_n = \infty$.

Proof. If $\kappa \leq 0$, then $c_\kappa''(d_n) \geq 1$ for any $n \in \mathbb{N}$. Hence we get

$$\sum_{n=1}^\infty \beta_n \geq \sum_{n=1}^\infty \left(1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)\cdot 1 + (1-\alpha_n)^2}}\right) = \sum_{n=1}^\infty \alpha_n = \infty.$$

Assume that $\kappa > 0$. Put $M = \sup_{n\in\mathbb{N}} d_n \leq D_\kappa/2$. Then we have $c_\kappa''(M) \geq 0$ and

$$\sum_{n=1}^\infty \beta_n \geq \sum_{n=1}^\infty \left(\frac{\alpha_n^2}{2} + \alpha_n(1-\alpha_n)c_\kappa''(M)\right)$$

by the same calculation as Lemma 4.27. If (i) is true, then $c_\kappa''(M) > 0$. Hence $\sum_{n=1}^\infty \beta_n = \infty$ holds if (i) or (ii) is true, which is the desired result. □

Theorem 4.29. *Let X be an admissible complete* CAT(κ) *space and $T\colon X \to X$ a tightly quasi-nonexpansive and Δ-demiclosed mapping. Let $\{\alpha_n\} \subset]0,1[$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Let $u, x_1 \in X$ arbitrarily and define $\{x_n\} \subset X$ by*

$$x_{n+1} = \alpha_n u \overset{\kappa}{\oplus} (1-\alpha_n)Tx_n$$

for any $n \in \mathbb{N}$. In the case where $\kappa > 0$, suppose that (i) *or* (ii) *holds:*

(i) $\sup_{n\in\mathbb{N}} d(u, Tx_n) < D_\kappa/2$,

(ii) $\sum_{n=1}^\infty \alpha_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_{F(T)}u$.

To prove the above theorem, we divide the two cases, $\kappa \neq 0$ and $\kappa = 0$.

Proof of Theorem 4.29 when $\kappa \neq 0$. Put $p = P_{F(T)}u$ and

$$
\begin{aligned}
a_n &= c_\kappa(d(x_n, p));\\
\beta_n &= 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)c_\kappa''(d(u,Tx_n)) + (1-\alpha_n)^2}};\\
b_n &= \frac{1}{\kappa}\left(1 - \frac{\left(1-\alpha_n + \sqrt{\alpha_n^2 + 2\alpha_n(1-\alpha_n)c_\kappa''(d(u,Tx_n)) + (1-\alpha_n)^2}\right)c_\kappa''(d(u,p))}{\alpha_n + 2(1-\alpha_n)c_\kappa''(d(u,Tx_n))}\right)
\end{aligned}
$$

for each $n \in \mathbb{N}$. Then we get $a_{n+1} \leq (1-\beta_n)a_n + \beta_n b_n$ for any $n \in \mathbb{N}$ by Lemmas 4.10, 4.26, and 4.27. Moreover, we obtain $\sum_{n=1}^{\infty} \beta_n = \infty$ by Lemma 4.28.

Note the quasinonexpansiveness of S and T. Since c_κ is nondecreasing, we obtain

$$
\begin{aligned}
c_\kappa(d(x_{n+1}, p)) &\leq \alpha_n c_\kappa(d(u,p)) + (1-\alpha_n)c_\kappa(d(Tx_n, p))\\
&\leq \alpha_n c_\kappa(d(u,p)) + (1-\alpha_n)c_\kappa(d(x_n, p))
\end{aligned}
$$

for any $n \in \mathbb{N}$, and it deduces that

$$
c_\kappa(d(x_n,p)) \leq \min\{c_\kappa(d(u,p)), c_\kappa(d(x_1,p))\} < c_\kappa(D_\kappa/2)
$$

for any $n \in \mathbb{N}$. Thus we get $\sup_{n\in\mathbb{N}} d(Tx_n, p) \leq \sup_{n\in\mathbb{N}} d(x_n,p) < D_\kappa/2$.

Let $\varphi\colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function such that $\lim_{i\to\infty}\varphi(i) = \infty$. Put $n_i = \varphi(i)$ for each $i \in \mathbb{N}$ and suppose that $\liminf_{i\to\infty}(a_{n_i+1} - a_{n_i}) \geq 0$. Then we get

$$
\begin{aligned}
0 &\leq \liminf_{i\to\infty}\left(a_{n_i+1} - a_{n_i}\right)\\
&= \liminf_{i\to\infty}\left(c_\kappa(d(x_{n_i+1},p)) - c_\kappa(d(x_{n_i},p))\right)\\
&\leq \liminf_{i\to\infty}\left(\alpha_{n_i}c_\kappa(d(u,p)) + (1-\alpha_{n_i})c_\kappa(d(Tx_{n_i},p)) - c_\kappa(d(x_{n_i},p))\right)\\
&= \liminf_{i\to\infty}\left(c_\kappa(d(Tx_{n_i},p)) - c_\kappa(d(x_{n_i},p))\right)\\
&\leq \limsup_{i\to\infty}\left(c_\kappa(d(Tx_{n_i},p)) - c_\kappa(d(x_{n_i},p))\right) \leq 0.
\end{aligned}
$$

Therefore, $\lim_{i\to\infty}\left(c_\kappa(d(Tx_{n_i},p)) - c_\kappa(d(x_{n_i},p))\right) = 0$ holds.

Put $L = \inf_{i\in\mathbb{N}} c_\kappa''(d(Tx_{n_i},p))$. Then we obtain $L > 0$. Indeed, if $\kappa < 0$, we have $L \geq 1$ obviously; if $\kappa > 0$, then

$$
L \geq \inf_{i\in\mathbb{N}} c_\kappa''(d(x_{n_i},p)) = c_\kappa''\left(\sup_{i\in\mathbb{N}} d(x_{n_i},p)\right) > c_\kappa''\left(\frac{D_\kappa}{2}\right) = 0.
$$

Since T is tightly quasinonexpansive, we have

$$
c_\kappa(d(x_{n_i}, Tx_{n_i}))c_\kappa''(d(Tx_{n_i},p)) \leq c_\kappa(d(x_{n_i},p)) - c_\kappa(d(Tx_{n_i},p))
$$

for any $i \in \mathbb{N}$, which yields

$$
\begin{aligned}
c_\kappa(d(x_{n_i}, Tx_{n_i})) &\leq \frac{c_\kappa(d(x_{n_i},p)) - c_\kappa(d(Tx_{n_i},p))}{c_\kappa''(d(Tx_{n_i},p))}\\
&\leq \frac{c_\kappa(d(x_{n_i},p)) - c_\kappa(d(Tx_{n_i},p))}{L} \to 0.
\end{aligned}
$$

as $i \to \infty$. Thus we get $d(x_{n_i}, Tx_{n_i}) \to 0$.

Take a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and its subsequence $\{x_{n_{i_{j_k}}}\}$ such that

$$\delta := \liminf_{i\to\infty} d(u, Tx_{n_i}) = \lim_{j\to\infty} d(u, Tx_{n_{i_j}})$$

and $x_{n_{i_{j_k}}} \xrightarrow{\Delta} z \in X$. Then we obtain $z \in F(T)$ since T is Δ-demiclosed.

Note that $\delta < D_\kappa/2$ always holds if $\kappa \le 0$. Indeed, we have $\sup_{i\in\mathbb{N}} d(u, Tx_{n_i}) \le d(u,p) + \sup_{i\in\mathbb{N}} d(Tx_{n_i}, p) < \infty = D_\kappa/2$ if $\kappa \le 0$.

In what follows, $\{k\}$ denotes $\{n_{i_{j_k}}\}$. Then we get from Corollary 2.7 that

$$\delta = \lim_{k\to\infty} d(u, Tx_k) = \lim_{k\to\infty} d(u, x_k) \ge d(u, z) \ge d(u, p).$$

Moreover, using $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$\begin{aligned}
\limsup_{i\to\infty} b_{n_i} &= \limsup_{i\to\infty} \frac{1}{\kappa}\left(1 - \frac{\left(1-\alpha_{n_i} + \sqrt{\alpha_{n_i}^2 + 2\alpha_{n_i}(1-\alpha_{n_i})c''_\kappa(d(u,Tx_{n_i})) + (1-\alpha_{n_i})^2}\right)c''_\kappa(d(u,p))}{\alpha_{n_i} + 2(1-\alpha_{n_i})c''_\kappa(d(u,Tx_{n_i}))}\right) \\
&= \limsup_{i\to\infty} \frac{1}{\kappa}\left(1 - \frac{c''_\kappa(d(u,p))}{c''_\kappa(d(u,Tx_{n_i}))}\right) \\
&= \frac{1}{\kappa}\left(1 - \lim_{k\to\infty} \frac{c''_\kappa(d(u,p))}{c''_\kappa(d(u,Tx_k))}\right).
\end{aligned}$$

If $\delta < D_\kappa/2$, then we have

$$\frac{1}{\kappa}\left(1 - \lim_{k\to\infty} \frac{c''_\kappa(d(u,p))}{c''_\kappa(d(u,Tx_k))}\right) \le \frac{1}{\kappa}\left(1 - \frac{c''_\kappa(d(u,p))}{c''_\kappa(d(u,p))}\right) = 0.$$

Otherwise, if $\delta = D_\kappa/2$, which occurs only if $\kappa > 0$, then we obtain $\lim_{k\to\infty} c''_\kappa(d(u,Tx_k)) = 0$ and $c''_\kappa(d(u,p)) > 0$; this follows that

$$\frac{1}{\kappa}\left(1 - \lim_{k\to\infty} \frac{c''_\kappa(d(u,p))}{c''_\kappa(d(u,Tx_k))}\right) = -\infty < 0.$$

Consequently, we get the conclusion from Lemma 4.25. □

Proof of Theorem 4.29 when $\kappa = 0$. Take a point $p = P_{F(T)}u$, and put $a_n = d(x_n, p)^2$ and $b_n = d(u,p)^2 - (1-\alpha_n)d(u, Tx_n)^2$ for $n \in \mathbb{N}$. Then we get $a_{n+1} \le \alpha_n d(u,p)^2 + (1-\alpha_n)d(Tx_n, p)^2 - \alpha_n(1-\alpha_n)d(u,Tx_n)^2 \le (1-\alpha_n)a_n + \alpha_n b_n$ for any $n \in \mathbb{N}$. We also have $d(x_{n+1}, p)^2 \le \alpha_n d(u,p)^2 + (1-\alpha_n)d(x_n,p)^2 \le \max\{d(u,p)^2, d(x_n,p)^2\}$ for any $n \in \mathbb{N}$, and thus $\sup_{n\in\mathbb{N}} d(x_n, p) \le \max\{d(u,p), d(x_1,p)\} < \infty$. It means that sequences $\{d(x_n,p)\}$, $\{d(Tx_n,p)\}$, $\{d(u,x_n)\}$ and $\{d(u,Tx_n)\}$ are bounded.

Let $\varphi\colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function such that $\lim_{i\to\infty} \varphi(i) = \infty$. Put $n_i = \varphi(i)$ for each $i \in \mathbb{N}$ and suppose that $\liminf_{i\to\infty}(a_{n_i+1} - a_{n_i}) \ge 0$. Then we get $\lim_{i\to\infty}\left(d(Tx_{n_i},p)^2 - d(x_{n_i},p)^2\right) = 0$ by the same calculation as the proof of Theorem 4.29. Therefore we obtain

$$d(x_{n_i}, Tx_{n_i})^2 \le d(x_{n_i}, p)^2 - d(Tx_{n_i}, p)^2 \to 0$$

as $i \to \infty$ by the tight quasinonexpansiveness of T, which implies $\lim_{i\to\infty} d(x_{n_i}, Tx_{n_i}) = 0$.

Take subsequences $\{x_{n_{i_{j_k}}}\} \subset \{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that $\liminf_{i\to\infty} d(u, Tx_{n_i}) = \lim_{j\to\infty} d(u, Tx_{n_{i_j}})$ and $x_{n_{i_{j_k}}} \xrightarrow{\Delta} z \in X$. Then $z \in F(T)$. Let us denote $\{n_{i_{j_k}}\}$ by $\{k\}$ simply. Then, since $\alpha_n \to 0$ as

$n \to \infty$, we have $\limsup_{i\to\infty} b_{n_i} = d(u,p)^2 - \liminf_{i\to\infty}(1-\alpha_{n_i})d(u,Tx_{n_i})^2 = d(u,p)^2 - \lim_{k\to\infty}(1-\alpha_k)d(u,Tx_k)^2 = d(u,p)^2 - \lim_{k\to\infty} d(u,Tx_k)^2 = d(u,p)^2 - \lim_{k\to\infty} d(u,x_k)^2 \le d(u,p)^2 - d(u,z)^2 \le 0$ by Lemma 2.6. Therefore we get the desired result from Lemma 4.25. □

By Theorem 4.29, we obtain an Halpern type approximation theorem for a firmly vicinal mapping with ψ an admissible complete CAT(κ) space as follows.

Corollary 4.30. *Let X be an admissible complete* CAT(κ) *space and $T\colon X \to X$ a firmly vicinal mapping with ψ such that $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$, u, and $\{x_n\}$ are the same as Theorem 4.29. In the case where $\kappa > 0$, suppose that* (i) *or* (ii) *holds:*

(i) $\sup_{n\in\mathbb{N}} d(u,Tx_n) < D_\kappa/2$,

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_{F(T)}u$.

Proof. By Lemmas 4.2 and 4.9, T is Δ-demiclosed and tightly quasinonexpansive. Therefore, from Theorem 4.29, we get the conclusion. □

Chapter 5

Equilibrium problems

Let X be a CAT(κ) space, K a nonempty closed convex subset of X, and $f\colon K^2 \to \mathbb{R}$. Then a mapping R_f is called a *resolvent* operator of the equilibrium problem for f if the set of all fixed points of R_f coincides with the set of all solutions to the equilibrium problem for f, that is, $F(R_f) = \operatorname{Equil} f$. Resolvents of the equilibrium problem play an important role in reducing the equilibrium problem to a fixed point problem.

In 2018, Kimura and Kishi [12] proposed the resolvent $Q_f\colon X \to 2^K$ of the equilibrium problem for $f\colon K^2 \to \mathbb{R}$ defined by

$$Q_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K} \left(f(z,y) + \frac{1}{2}d(x,y)^2 - \frac{1}{2}d(x,z)^2 \right) \geq 0 \right\}$$

on a complete CAT(0) space X and a nonempty closed convex subset K of X. They assumed that X has the convex hull finite property, and f satisfies conditions (E1)–(E4) when proving that Q_f is well-defined as a single-valued mapping. This mapping Q_f can be expressed by

$$Q_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) + \varphi(c_0(d(x,y))) - \varphi(c_0(d(x,z)))) \geq 0 \right\},$$

where $\varphi(t) = t$ for $t \in [0,\infty[$.

Later, in 2021, Kimura [11] showed the resolvent R_f defined by the following is a single-valued mapping under the appropriate conditions on an admissible complete CAT(1) space:

$$\begin{aligned} R_f x &= \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) - \log\cos d(x,y) + \log\cos d(x,z)) \geq 0 \right\} \\ &= \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) + \varphi(c_1(d(x,y))) - \varphi(c_1(d(x,z)))) \geq 0 \right\}, \end{aligned}$$

where $\varphi(t) = -\log(1-t)$ for $t \in [0,1[$.

Similarly, the resolvent S_f on a complete CAT(−1) space was proposed by Kimura and Ogihara [20]. It is defined by

$$\begin{aligned} S_f x &= \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) + \cosh d(x,y) - \cosh d(x,z)) \geq 0 \right\} \\ &= \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) + \varphi(c_{-1}(d(x,y))) - \varphi(c_{-1}(d(x,z)))) \geq 0 \right\}, \end{aligned}$$

where $\varphi(t) = t$ for $t \in [0,\infty[$.

In the same fashion, in general CAT(κ) spaces, we expect a resolvent defined by using a perturbation $\varphi(c_\kappa(d))$ to be a single-valued mapping with the appropriate conditions. In this chapter, we consider the sufficient conditions for the function φ to define the resolvent as a single-valued mapping on an admissible CAT(κ) space.

5.1 Resolvents of the equilibrium problem

In what follows, put $[n] = \{1, 2, \dots, n\}$ for each $n \in \mathbb{N}$.

Lemma 5.1. *Let X be a uniquely geodesic space and $E = \{y_1, y_2, \dots, y_n\}$ a subset of X. For a nonempty set A, let h be a bifunction from $A \times X$ into $\mathbb{R}$. Suppose that the function $h(z, \cdot) : X \to \mathbb{R}$ is convex for any $z \in A$. Then, for any $v \in \operatorname{co} E$, there exists $\{\mu_1, \mu_2, \dots, \mu_n\} \subset [0, 1]$ such that $\sum_{i=1}^n \mu_i = 1$ and $h(z, v) \le \sum_{i=1}^n \mu_i h(z, y_i)$ for all $z \in A$.*

Proof. Put $F_1 = E$ and $F_{j+1} = \{tu \oplus (1-t)u' \mid u, u' \in X,\ t \in [0, 1]\}$ for $j \in \mathbb{N}$. Then $\operatorname{co} E = \bigcup_{j \in \mathbb{N}} F_j$. Therefore, we need to show that the existence of such $\{\mu_1, \mu_2, \dots, \mu_n\} \subset [0, 1]$ for any $j \in \mathbb{N}$ and $v \in F_j$. We show it by induction for $j \in \mathbb{N}$.

Suppose $j = 1$ and let $v \in F_j = E$. Then there exists $i_n \in [n]$ such that $v = y_{i_0}$. Thus, putting $\mu_{i_0} = 1$ and $\mu_i = 0$ for $i \in [n] \setminus \{i_0\}$, we get $h(z, v) = \sum_{i=1}^n h(z, y_i)$ for any $z \in A$.

Next, assume that is true for some $j \in \mathbb{N}$. Let $v \in F_{j+1}$. Then there exists $t \in [0, 1]$ and $u, u' \in F_j$ such that $v = tu \oplus (1-t)u'$. Hence, from the assumption, there exists $\{\mu_1, \mu_2, \dots, \mu_n\} \subset [0, 1]$ such that $\sum_{i=1}^n \mu_i = 1$ and $h(z, u) \le \sum_{i=1}^n \mu_i h(z, y_i)$ for any $z \in A$. Similarly, we have the existence of $\{\mu'_1, \mu'_2, \dots, \mu'_n\} \subset [0, 1]$ such that $\sum_{i=1}^n \mu'_i = 1$ and $h(z, u') \le \sum_{i=1}^n \mu'_i h(z, y_i)$ for any $z \in A$. Take $z \in A$ arbitrarily. Then

$$h(z, v) \le t h(z, u) + (1-t) h(z, u') \le \sum_{i=1}^n \left(t\mu_i + (1-t)\mu'_i\right) h(z, y_i)$$

and $\sum_{i=1}^n \left(t\mu_i + (1-t)\mu'_i\right) = 1$ hold and thus we get the conclusion. □

Lemma 5.2 (Kimura [11])**.** *For $\kappa > 0$, let X be an admissible complete* CAT(κ) *space having the convex hull finite property and C a nonempty subset of X. Suppose that a mapping $M : C \to 2^X$ satisfies that $M(y)$ is closed for any $y \in X$. If* $\operatorname{cl}\operatorname{co} E \subset \bigcup_{y \in E} M(y)$ *holds for any finite subset E of X, then $\{M(y) \mid y \in C\}$ has the finite intersection property.*

Lemma 5.3 (Niculescu and Rovenţa [25])**.** *Let X be a complete* CAT(0) *space having the convex hull finite property and C a nonempty subset of X. Suppose that a mapping $M : C \to 2^X$ satisfies that $M(y)$ is nonempty closed convex for any $y \in X$. If* $\operatorname{cl}\operatorname{co} E \subset \bigcup_{y \in E} M(y)$ *holds for any finite subset E of X, then $\{M(y) \mid y \in C\}$ has the finite intersection property.*

Lemma 5.4 (Kimura [11])**.** *For $\kappa > 0$, let X be an admissible complete* CAT(κ) *space and C a nonempty closed convex subset of X satisfying* $\inf_{y \in C} \sup_{x \in C} d(x, y) < D_\kappa / 2$. *Let $\mathcal{M}$ be a family of closed convex subsets of X and suppose that $\mathcal{M}$ has the finite intersection property. Then, $\bigcap \mathcal{M} \neq \emptyset$.*

Lemma 5.5 (Kimura and Kishi [12])**.** *Let X be a complete* CAT(0) *space and C a Δ-compact subset of X. Let $\mathcal{M}$ be a family of Δ-closed subsets of X and suppose that $\mathcal{M}$ has the finite intersection property. Then, $\bigcap \mathcal{M} \neq \emptyset$.*

Lemma 5.6. *Let X be an admissible complete* CAT(κ) *space. Suppose that X has the convex hull finite property. Let K be a nonempty closed convex subset of X and suppose that a function $h : K^2 \to \mathbb{R}$ satisfies* (E1), (E2) *and* (E3). *Let C be a nonempty closed convex subset of K and* *define a set $M(y)$ by $M(y) = \{z \in C \mid h(y,z) \le 0\}$ for each $y \in C$. Then the following properties hold:*

(i) *$M(y)$ is nonempty closed convex for any $y \in C$,*
(ii) *a set $\{M(y) \mid y \in C\}$ has the finite intersection property,*
(iii) *if $\inf_{y\in C}\sup_{x\in C} d(x,y) < D_\kappa/2$, then $\bigcap_{y\in C} M(y) \neq \emptyset$.*

Proof. (i) Let $y \in C$. Since h satisfies (E1), we get $y \in M(y)$ and hence $M(y)$ is nonempty. Therefore, since h satisfies (E3), we obtain $M(y)$ is closed and convex.

(ii) Let $E = \{y_1, y_2, \dots, y_n\} \subset C$. We show $\operatorname{co} E \subset \bigcup_{i=1}^n M(y_i)$. Assume that it is false, and let $v \in \operatorname{co} E \setminus \bigcup_{i=1}^n M(y_i)$. Then we get $h(y_i, v) > 0$ for any $i \in [n]$. From $v \in \operatorname{co} E$ and Lemma 5.1, there exists $\{\mu_1, \mu_2, \dots, \mu_n\} \subset [0,1]$ such that $\sum_{i=1}^n \mu_i = 1$ and $h(y_k, v) \le \sum_{i=1}^n \mu_i h(y_k, y_i)$ for any $k \in [n]$. Thus we obtain

$$0 < \sum_{k=1}^n \mu_k h(y_k, v) \le \sum_{k=1}^n \sum_{i=1}^n \mu_k \mu_i h(y_k, y_i) = \frac{1}{2}\sum_{k=1}^n \sum_{i=1}^n \mu_k \mu_i (h(y_k, y_i) + h(y_i, y_k)) \le 0,$$

which is a contradiction. Hence we get $\operatorname{co} E \subset \bigcup_{i=1}^n M(y_i)$. It implies that

$$\operatorname{cl}\operatorname{co} E \subset \operatorname{cl}\bigcup_{i=1}^n M(y_i) = \bigcup_{i=1}^n M(y_i)$$

and thus $\{M(y) \mid y \in C\}$ has the finite intersection property by Lemma 5.2 or Lemma 5.3.

(iii) First we consider the case of $\kappa > 0$. By the result of (ii), $\{M(y) \mid y \in C\}$ has the finite intersection property. Hence we get $\bigcap_{y\in C} M(y) \neq \emptyset$ by using Lemma 5.4. We consider the case of $\kappa \le 0$. Suppose $\inf_{y\in C}\sup_{x\in C} d(x,y) < D_\kappa/2 = \infty$. It means that C is bounded and hence C is Δ-compact by Lemma 2.4. Furthermore, $M(y)$ is Δ-closed for any $y \in C$ since $M(y) \subset C$ from Lemma 2.5. Thus, from Lemma 5.5 and (ii), we have $\bigcap_{y\in C} M(y) \neq \emptyset$. □

Lemma 5.7. *Let X be an admissible complete* CAT(κ) *space. Suppose that X has the convex hull finite property. Let K be a nonempty closed convex subset of X and h a real function on K^2 with conditions* (E1)–(E4). *Suppose that there exist $u \in K$ and $R \in \,]0, D_\kappa/2[$ such that $h(z,u) \le 0$ for any $z \in K$ satisfying $d(u,z) = R$. Then* $\operatorname{Equil} h \neq \emptyset$.

Proof. Put $C = \{z \in K \mid d(u,z) \le R\}$, and let $M(y) = \{z \in C \mid h(y,z) \le 0\}$ for each $y \in C$. We know that C is a nonempty closed convex subset of K, and we have

$$\inf_{y\in C}\sup_{x\in C} d(x,y) \le \sup_{z\in C} d(u,z) \le R < \frac{D_\kappa}{2}.$$

Therefore, from Lemma 5.6 (iii), we obtain $\bigcap_{y\in C} M(y) \neq \emptyset$.

Let $z_0 \in \bigcap_{y\in C} M(y)$. Then we get $h(y, z_0) \le 0$ for any $y \in C$. We also have $d(u, z_0) \le R$. Let $w \in C$ and $t \in \,]0,1[$ arbitrarily. Then $tw \oplus (1-t)z_0 \in C$, and this implies

$$0 = h(tw \oplus (1-t)z_0, tw \oplus (1-t)z_0) \le t h(tw \oplus (1-t)z_0, w)$$

by using the condition (E3). It follows that $h(tw \oplus (1-t)z_0, w) \ge 0$. Since h satisfies the condition (E4), we obtain

$$h(z_0, w) \ge \limsup_{t\to 0} h(tw \oplus (1-t)z_0, w) \ge 0.$$

Therefore $h(z_0, w) \ge 0$ holds for any $w \in C$.

We show that $h(z_0, y) \geq 0$ holds for each $y \in K$. Let $y \in K$ and put

$$u_0 = \begin{cases} u & (\text{if } d(u, z_0) = R); \\ z_0 & (\text{if } d(u, z_0) < R). \end{cases}$$

Then we have $d(u, u_0) < R$. In fact, if $d(u, z_0) = R$, then we get $d(u, u_0) = d(u, u) = 0$. On the other hand, if $d(u, z_0) < R$, then we have $d(u, u_0) = d(u, z_0) < R$.

Since $d(u, u_0) < R$, we can take a sufficiently small $t_0 \in \,]0, 1[$ satisfying

$$t_0 c_\kappa(d(u, y)) + (1 - t_0) c_\kappa(d(u, u_0)) < c_\kappa(R).$$

Then we get

$$c_\kappa(d(u, t_0 y \oplus (1 - t_0) u_0)) \leq t_0 c_\kappa(d(u, y)) + (1 - t_0) c_\kappa(d(u, u_0)) < c_\kappa(R)$$

and thus $d(u, t_0 y \oplus (1 - t_0) u_0) < R$. Since K is convex, we get $t_0 y \oplus (1 - t_0) u_0 \in K$. Hence, by the definition of C, we obtain $t_0 y \oplus (1 - t_0) u_0 \in C$. Therefore

$$0 \leq h(z_0, t_0 y \oplus (1 - t_0) u_0) \leq t_0 h(z_0, y) + (1 - t_0) h(z_0, u_0).$$

Incidentally, we also have $h(z_0, u_0) \leq 0$ and hence $h(z_0, y) \geq 0$. Indeed, if $d(u, z_0) = R$, then $h(z_0, u_0) = h(z_0, u) \leq 0$ holds by the assumption, and if $d(u, z_0) < R$ then $h(z_0, u_0) = h(z_0, z_0) = 0$. Thus we get the conclusion. □

Remark 5.8. In the assumptions of Lemma 5.7, there need not exist $z \in K$ such that $d(u, z) = R$. This means that, we can adapt this lemma even when $d(u, z) < R$ for all $z \in K$.

In what follows for a function $\Phi\colon [0, D_\kappa/2[\to [0, \infty[$, let us write $\Phi(d)$ as Φd for every $d \in [0, D_\kappa/2[$.

Theorem 5.9. *Let X be an admissible complete* CAT(κ) *space. Suppose that X has the convex hull finite property. Let K be a nonempty closed convex subset of X and let $f\colon K^2 \to \mathbb{R}$ with conditions* (E1)–(E4). *Let $\Phi\colon [0, D_\kappa/2[\to [0, \infty[$ be a continuous convex function. Suppose the following:*

- *If $\kappa \leq 0$ and K is unbounded, then*

$$\liminf_{\substack{d(u,z)\to\infty \\ z \in K}} \frac{f(u, z)}{d(u, z)} + \lim_{d\to\infty} \frac{\Phi d}{d} > 0$$

  for some $u \in K$.
- *If $\kappa > 0$, then suppose that Φ is strictly increasing and* $\lim_{d\to D_\kappa/2} \Phi d = \infty$.

For $x \in X$, define a function $h_x\colon K^2 \to \mathbb{R}$ by

$$h_x(z, y) = f(z, y) + \Phi d(x, y) - \Phi d(x, z)$$

for any $(z, y) \in K^2$. Then $\operatorname{Equil} h_x \neq \emptyset$ *for any $x \in X$.*

Proof. Take $x \in X$ arbitrarily, and put $h := h_x$ for simplicity. Since Φ is continuous and convex, h satisfies the conditions (E1)–(E4).

First, we consider the case where $\kappa \leq 0$. Take a point $u \in K$ satisfying the assumption. We show the following condition $(*)$ holds:

$(*)$ there exists $R > 0$ such that for any $z \in K$ with $d(u, z) = R$, an inequality $h(u, z) \leq 0$ holds.

If K is bounded, then we can take $R > 0$ such that $d(u,z) < R$, thus $(*)$ holds. Suppose that K is unbounded. Then we have

$$\begin{aligned} h(z,u) &= f(z,u) + \Phi d(x,u) - \Phi d(x,z) \\ &\le -f(u,z) + \Phi d(x,u) - \Phi d(x,z) \\ &= \Phi d(x,u) - (f(u,z) + \Phi d(x,z)) \end{aligned}$$

for any $z \in K$. We also obtain

$$\liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{f(u,z) + \Phi d(x,z)}{d(u,z)} > 0.$$

In fact, in case (a), we have

$$\liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{f(u,z) + \Phi d(x,z)}{d(u,z)} \ge \liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{0 + \Phi d(x,z)}{d(u,z)} = \lim_{d\to\infty} \frac{\Phi d}{d} > 0$$

since $u \in \operatorname{Equil} f$; otherwise, in case (b),

$$\begin{aligned} \liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{f(u,z) + \Phi d(x,z)}{d(u,z)} &\ge \liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{f(u,z)}{d(u,z)} + \liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{\Phi d(x,z)}{d(u,z)} \\ &= \liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{f(u,z)}{d(u,z)} + \lim_{d\to\infty} \frac{\Phi d}{d} > 0. \end{aligned}$$

Hence we get $f(u,z) + \Phi d(x,z) \to \infty$ when $d(u,z) \to \infty$, and it means that $h(z,u) \to -\infty$ if $d(u,z) \to \infty$. Therefore we can take $R > 0$ such that $h(z,u) \le 0$ for any $z \in K$ with $d(u,z) = R$. This implies that $(*)$ holds regardless of whether the set K is bounded or not. Consequently, from Lemma 5.7, there exists $z_0 \in C$ such that $\inf_{y\in K} h(z_0,y) \ge 0$. Thus we get the conclusion if $\kappa \le 0$.

Next, we consider the case where $\kappa > 0$. By the assumptions for Φ, we can assume that Φ is bijective onto $[k,\infty[$ for $k := \Phi(0)$. Thus there exists the inverse $\Phi^{-1} \colon [k,\infty[\to [0, D_\kappa/2[$ of Φ.

Let $u = P_K x$ and put $L = \inf_{y\in K} f(u,y) - \Phi d(x,u)$. Since f satisfies the condition (E3), we obtain that $f(u,\cdot)$ is bounded below by Lemma 2.2. Hence we have

$$-\infty < L \le f(u,u) - \Phi d(x,u) = -\Phi d(x,u) \le -k.$$

If $L = -k$, then we obtain

$$h(u,y) = f(u,y) + \Phi d(x,y) - \Phi d(x,u) \ge -k + \Phi d(x,y) \ge 0$$

for all $y \in K$ and thus we get the conclusion.

Suppose $L < -k$. Using Corollary 2.12, we get $d(u,z) \le d(x,z)$ and it implies $\Phi d(u,z) \le \Phi d(x,z)$. Thus we have

$$\begin{aligned} h(z,u) &= f(z,u) + \Phi d(x,u) - \Phi d(x,z) \\ &\le -f(u,z) + \Phi d(x,u) - \Phi d(u,z) \\ &\le -L - \Phi d(u,z) \end{aligned}$$

for any $z \in K$. Put $R = \Phi^{-1}(-L) < D_\kappa/2$. Then we obtain

$$-L - \Phi d(u,z) = 0 \iff d(u,z) = R$$

for any $z \in K$. It implies that $h(z,u) \le 0$ for any $z \in K$ with $d(u,z) = R$. Therefore, from Lemma 5.7, we get the conclusion. □

Remark 5.10. If $\Phi\colon [0, D_\kappa/2[\to [0,\infty[$ is continuous and convex, then there exists a limit $\lim_{d\to\infty} \Phi d/d \in [0,\infty]$.

Remark 5.11. Consider the case where $\kappa \le 0$ in Theorem 5.9. Then we have

$$\liminf_{\substack{d(u,z)\to\infty\\ z\in K}} \frac{f(u,z)}{d(u,z)} \ge 0$$

for all $u \in \operatorname{Equil} f$ by the definition of $\operatorname{Equil} f$. Therefore, if $\operatorname{Equil} f \neq \emptyset$ and $\lim_{d\to\infty} \Phi d/d > 0$, then

$$\liminf_{\substack{d(u,z)\to\infty\\ z\in K}} \frac{f(u,z)}{d(u,z)} + \lim_{d\to\infty} \frac{\Phi d}{d} > 0$$

holds for all $u \in \operatorname{Equil} f \subset K$.

Remark 5.12. Consider the case where $\kappa \le 0$ in Theorem 5.9 again. If $\lim_{d\to\infty} \Phi d/d = \infty$, then we obtain

$$\liminf_{\substack{d(u,z)\to\infty\\ z\in K}} \frac{f(u,z)}{d(u,z)} + \lim_{d\to\infty} \frac{\Phi d}{d} > 0$$

for all $u \in K$. In fact, by Lemma 2.1, there exists $L \in]-\infty, 0]$ such that for any $u \in K$,

$$\liminf_{\substack{d(u,z)\to\infty\\ z\in K}} \frac{f(u,z)}{d(u,z)} \ge L.$$

This implies that

$$\liminf_{\substack{d(u,z)\to\infty\\ z\in K}} \frac{f(u,z)}{d(u,z)} + \lim_{d\to\infty} \frac{\Phi d}{d} \ge L + \lim_{d\to\infty} \frac{\Phi d}{d} = \infty$$

for all $u \in K$.

Remark 5.13. In this thesis, we say that a real function $f\colon [a,b[\to \mathbb{R}$ is differentiable if f is differentiable on $]a,b[$ and f is right differentiable at a. Then we write a right derivative of f at a simply by $f'(a)$. That is, $f'(a) = \lim_{t\searrow a}(f(t) - f(a))/(t-a)$. In addition, f is said to be continuous on $[a,b[$ if f is continuous at $]a,b[$ and f is right continuous at a.

Similarly, $g\colon]a,b] \to \mathbb{R}$ is said to be differentiable if g is differentiable on $]a,b[$ and g is left differentiable at b. Moreover, g is said to be continuous on $]a,b]$ if g is continuous at $]a,b[$ and g is left continuous at b.

Lemma 5.14. *Let X be an admissible* CAT(κ) *space and K a nonempty convex subset of X. Let $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$ be a differentiable function such that φ' is continuous on $[0, c_\kappa(D_\kappa/2)[$. Take $x \in X$ and $z, w \in K$ such that $z \neq w$, and put $D = d(z,w)$. Define a function Υ from $[0,1[$ into $[0, c_\kappa(D_\kappa/2)[$ by*

$$\Upsilon(t) = (t)_D^\kappa\big(c_\kappa(d(x,w)) - c_\kappa((1-t)D)\big) + (1-t)_D^\kappa\big(c_\kappa(d(x,z)) - c_\kappa(tD)\big)$$

for $t \in [0,1[$. Then

$$\lim_{t\to 0} \frac{\varphi(\Upsilon(t)) - \varphi(c_\kappa(d(x,z)))}{t} = \varphi'(c_\kappa(d(x,z))) \cdot \frac{D}{c_\kappa'(D)}\big(c_\kappa(d(x,w)) - c_\kappa(D) - c_\kappa''(D)c_\kappa(d(x,z))\big).$$

Proof. Define $L\colon [0,1[\ \to \mathbb{R}$ by $L(t) = \varphi(\Upsilon(t)) - \varphi(c_\kappa(d(x,z)))$ for every $t \in [0,1[$. Then L is differentiable on $]0,1[$, and $L(0) = 0$. In fact, since

$$\frac{d}{dt}(t)_D^\kappa = \frac{Dc_\kappa''(tD)}{c_\kappa'(D)} \quad \text{and} \quad \frac{d}{dt}(1-t)_D^\kappa = -\frac{Dc_\kappa''((1-t)D)}{c_\kappa'(D)}$$

for any $t \in\ ]0,1[$, we get

$$\begin{aligned}\Upsilon'(t) = \frac{Dc_\kappa''(tD)}{c_\kappa'(D)}\big(c_\kappa(d(x,w)) - c_\kappa((1-t)d(w,z))\big) + D(t)_D^\kappa c_\kappa((1-t)D) \\ - \frac{Dc_\kappa''((1-t)D)}{c_\kappa'(D)}\big(c_\kappa(d(x,z)) - c_\kappa(td(w,z))\big) - D(1-t)_D^\kappa c_\kappa(tD).\end{aligned}$$

This follows that

$$\lim_{t\to 0}\Upsilon'(t) = \frac{D}{c_\kappa'(D)}\big(c_\kappa(d(x,w)) - c_\kappa(D) - c_\kappa''(D)c_\kappa(d(x,z))\big).$$

since $t \mapsto (t)_\kappa^D$ is continuous at 0 and 1. Furthermore, since φ' is continuous,

$$\lim_{t\to 0}\varphi'(\Upsilon(t)) = \varphi'(\Upsilon(0)) = \varphi'(c_\kappa(d(x,z))).$$

Hence we obtain

$$\begin{aligned}\lim_{t\to 0}\frac{dL}{dt}(t) &= \lim_{t\to 0}\varphi'(\Upsilon(t))\Upsilon'(t) \\ &= \varphi'(c_\kappa(d(x,z)))\cdot\frac{D}{c_\kappa'(D)}\big(c_\kappa(d(x,w)) - c_\kappa(D) - c_\kappa''(D)c_\kappa(d(x,z))\big).\end{aligned}$$

Consequently, we get

$$\lim_{t\to 0}\frac{\varphi(\Upsilon(t)) - \varphi(c_\kappa(d(x,z)))}{t} = \lim_{t\to 0}\frac{L(t)}{t} = \lim_{t\to 0}\frac{dL}{dt}(t),$$

which is the desired result. □

Lemma 5.15. *Let X be an admissible complete* CAT(κ) *space. Suppose that X has the convex hull finite property. Let K be a nonempty closed convex subset of X and f a real function on K^2 with conditions* (E1)–(E4). *Let $\varphi\colon [0, c_\kappa(D_\kappa/2)[\ \to [0,\infty[$ be a nondecreasing and differentiable function such that φ' is continuous on $[0, c_\kappa(D_\kappa/2)[$. Define $\Phi\colon [0, D_\kappa/2[\ \to [0,\infty[$ by $\Phi d = \varphi(c_\kappa(d))$ for $d \in [0, D_\kappa/2[$. Fix $x \in X$ and define a subset $R_f x$ of K by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K}\big(f(z,y) + \Phi d(x,y) - \Phi d(x,z)\big) \ge 0 \right\}.$$

Suppose that $R_f x$ is nonempty. Then

$$0 \le f(z,w) + \varphi'(c_\kappa(d(x,z)))\cdot\frac{D}{c_\kappa'(D)}\big(c_\kappa(d(x,w)) - c_\kappa(D) - c_\kappa''(D)c_\kappa(d(x,z))\big)$$

holds for any $z \in R_f x$ and $w \in K\setminus\{z\}$, where $D = d(z,w)$.

Proof. Let $z \in R_f x$, $w \in K$, $t \in]0,1[$ and suppose that $w \neq z$. Since $tw \oplus (1-t)z \in K$, we have

$$\begin{aligned} 0 &\leq f(z, tw \oplus (1-t)z) + \Phi d(x, tw \oplus (1-t)z) - \Phi d(x,z) \\ &\leq tf(z,w) + (1-t)f(z,z) + \Phi d(x, tw \oplus (1-t)z) - \Phi d(x,z) \\ &= tf(z,w) + \Phi d(x, tw \oplus (1-t)z) - \Phi d(x,z), \end{aligned}$$

and thus

$$0 \leq f(z,w) + \frac{\Phi d(x, tw \oplus (1-t)z) - \Phi d(x,z)}{t}.$$

Put $D = d(z,w) > 0$ and

$$\Upsilon(t) = (t)_D^\kappa \big(c_\kappa(d(x,w)) - c_\kappa((1-t)D)\big) + (1-t)_D^\kappa \big(c_\kappa(d(x,z)) - c_\kappa(tD)\big).$$

Then, from Stewart's theorem on CAT(κ) spaces and the nondecreasingness of φ, we obtain

$$\begin{aligned} \Phi d(x, tw \oplus (1-t)z) - \Phi d(x,z) &= \varphi(c_\kappa(d(x, tw \oplus (1-t)z))) - \Phi d(x,z) \\ &\leq \varphi(\Upsilon(t)) - \Phi d(x,z). \end{aligned}$$

Therefore, using Lemma 5.14, we get

$$\begin{aligned} 0 &\leq f(z,w) + \frac{\Phi d(x, tw \oplus (1-t)z) - \Phi d(x,z)}{t} \\ &\leq f(z,w) + \frac{\varphi(\Upsilon(t)) - \Phi d(x,z)}{t} \\ &\to f(z,w) + \varphi'(c_\kappa(d(x,z))) \cdot \frac{D}{c_\kappa'(D)}\big(c_\kappa(d(x,w)) - c_\kappa(D) - c_\kappa''(D)c_\kappa(d(x,z))\big) \end{aligned}$$

as $t \searrow 0$ for any $z \in R_f x$ and $w \in K \setminus \{z\}$, where $D = d(z,w)$. □

Theorem 5.16. *Let X, K, f, φ, Φ, and R_f are same as Lemma 5.15. Suppose that $R_f x$ is nonempty for all $x \in X$. Then for any $x_1, x_2 \in X$, $z_1 \in R_f x_1$, and $z_2 \in R_f x_2$,*

$$\begin{aligned} &\big(\varphi'(c_\kappa(D_1))c_\kappa''(D_1) + \varphi'(c_\kappa(D_2))c_\kappa''(D_2)\big)c_\kappa(d(z_1,z_2)) \\ &\qquad \leq \varphi'(c_\kappa(D_1))\big(c_\kappa(d(x_1,z_2)) - c_\kappa(D_1)\big) + \varphi'(c_\kappa(D_2))\big(c_\kappa(d(x_2,z_1)) - c_\kappa(D_2)\big) \end{aligned} \tag{$*$}$$

holds, where $D_1 = d(x_1,z_1)$, and $D_2 = d(x_2,z_2)$. In addition, if R_f is well-defined as a single-valued mapping from X into K, then the following hold:

(i) *R_f is firmly vicinal with $\varphi' \circ c_\kappa \colon [0, D_\kappa/2[\to]0,\infty[$;*
(ii) *$F(R_f) = \operatorname{Equil} f$ holds, and $\operatorname{Equil} f$ is closed and convex.*

Proof. Let $x_1, x_2 \in X$, $z_1 \in R_f x_1$ and $z_2 \in R_f x_2$, and put $D = d(z_1,z_2)$. If $z_1 = z_2$, then $(*)$ holds obviously. Considering the case where $z_1 \neq z_2$, we have

$$0 \leq f(z_1,z_2) + \varphi'(c_\kappa(d(x_1,z_1))) \cdot \frac{D}{c_\kappa'(D)}\big(c_\kappa(d(x_1,z_2)) - c_\kappa(D) - c_\kappa''(D)c_\kappa(d(x_1,z_1))\big)$$

and

$$0 \leq f(z_2,z_1) + \varphi'(c_\kappa(d(x_2,z_2))) \cdot \frac{D}{c_\kappa'(D)}\big(c_\kappa(d(x_2,z_1)) - c_\kappa(D) - c_\kappa''(D)c_\kappa(d(x_2,z_2))\big)$$

from Lemma 5.15. Summing up these inequalities and dividing by $D/c'_\kappa(D)$, we obtain

$$\begin{aligned}0 \le\ & \varphi'(c_\kappa(d(x_1,z_1)))\big(c_\kappa(d(x_1,z_2)) - c_\kappa(D) - c''_\kappa(D)c_\kappa(d(x_1,z_1))\big)\\ &+ \varphi'(c_\kappa(d(x_2,z_2)))\big(c_\kappa(d(x_2,z_1)) - c_\kappa(D) - c''_\kappa(D)c_\kappa(d(x_2,z_2))\big).\end{aligned}$$

Let H be the right-hand side of the above inequality. Since $c''_\kappa(d) = 1 - \kappa c_\kappa(d)$ holds for any $d \in \mathbb{R}$, we have

$$\begin{aligned}H =\ & \varphi'(c_\kappa(d(x_1,z_1)))\big(c_\kappa(d(x_1,z_2)) - c_\kappa(D) - c_\kappa(d(x_1,z_1)) + \kappa c_\kappa(D)c_\kappa(d(x_1,z_1))\big)\\ &+ \varphi'(c_\kappa(d(x_2,z_2)))\big(c_\kappa(d(x_2,z_1)) - c_\kappa(D) - c_\kappa(d(x_2,z_2)) + \kappa c_\kappa(D)c_\kappa(d(x_2,z_2))\big)\\ =\ & \varphi'(c_\kappa(d(x_1,z_1)))\big(c_\kappa(d(x_1,z_2)) - c_\kappa(d(x_1,z_1))\big) + \varphi'(c_\kappa(d(x_2,z_2)))\big(c_\kappa(d(x_2,z_1)) - c_\kappa(d(x_2,z_2))\big)\\ &- \Big(\varphi'(c_\kappa(d(x_1,z_1)))\big(1-\kappa c_\kappa(d(x_1,z_1))\big) + \varphi'(c_\kappa(d(x_2,z_2)))\big(1-\kappa c_\kappa(d(x_2,z_2))\big)\Big)c_\kappa(D)\\ =\ & \varphi'(c_\kappa(d(x_1,z_1)))\big(c_\kappa(d(x_1,z_2)) - c_\kappa(d(x_1,z_1))\big) + \varphi'(c_\kappa(d(x_2,z_2)))\big(c_\kappa(d(x_2,z_1)) - c_\kappa(d(x_2,z_2))\big)\\ &- \big(\varphi'(c_\kappa(d(x_1,z_1)))\, c''_\kappa(d(x_1,z_1)) + \varphi'(c_\kappa(d(x_2,z_2)))\, c''_\kappa(d(x_2,z_2))\big)c_\kappa(D),\end{aligned}$$

which is the conclusion of $(*)$.

Henceforth, assume that R_f is single-valued. Take $x_1, x_2 \in X$ arbitrarily. Then, substituting $z_1 := R_f x_1$ and $z_2 := R_f x_2$ to $(*)$, we get (i) from Lemma 4.3.

We show $F(R_f) \subset \operatorname{Equil} f$. Suppose that $z \in F(R_f)$, namely, $z = R_f z$. Let $y \in K$ and put $D = d(z,y)$. Then from Lemma 5.15,

$$\begin{aligned}0 &\le f(z,y) + \varphi'(c_\kappa(d(z,z)))\cdot\frac{D}{c'_\kappa(D)}\big(c_\kappa(D) - c_\kappa(D) - c''_\kappa(D)c_\kappa(d(z,z))\big)\\ &= f(z,y) + \varphi'(0)\cdot\frac{D}{c'_\kappa(D)}\big(c_\kappa(D) - c_\kappa(D) - c''_\kappa(D)\cdot 0\big)\\ &= f(z,y)\end{aligned}$$

if $z \neq y$. We also have $f(z,y) \ge 0$ even if $z = y$ by (E2). Thus we get $f(z,y) \ge 0$ for all $y \in K$. This implies $z \in \operatorname{Equil} f$, and thus $F(R_f) \subset \operatorname{Equil} f$.

Finally, we show $F(R_f) \supset \operatorname{Equil} f$. Take $z \in \operatorname{Equil} f$. Since Φ is strictly increasing on $[0, D_\kappa/2[$, we obtain

$$\inf_{y\in K}\big(f(z,y) + \Phi d(z,y) - \Phi d(z,z)\big) \ge \inf_{y \in K} f(z,y) \ge 0$$

and hence $z = R_f z$, that is, $z \in F(R_f)$. It concludes that $F(R_f) = \operatorname{Equil} f$.

If $F(R_f)$ is nonempty, then R_f is quasinonexpansive by (iv). Thus the set $F(R_f)$ is closed and convex, and so is $\operatorname{Equil} f$. Consequently, we get the conclusion. □

We consider a condition for $R_f x \neq \emptyset$ to hold.

Lemma 5.17. *Let X be an admissible complete* CAT(κ) *space. Suppose that X has the convex hull finite property. Let K be a nonempty closed convex subset of X and f a real function on K^2 with conditions* (E1)–(E4). *Let $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$ be a nondecreasing and differentiable function such that φ' is continuous on $[0, c_\kappa(D_\kappa/2)[$. Define $\Phi\colon [0, D_\kappa/2[\to [0,\infty[$ by $\Phi d = \varphi(c_\kappa(d))$ for $d \in [0, D_\kappa/2[$. Suppose the following:*

- *Φ is convex.*
- *If $\kappa \le 0$ and K is unbounded, then*

$$\liminf_{\substack{d(u,z)\to\infty\\ z\in K}} \frac{f(u,z)}{d(u,z)} + \lim_{d\to\infty}\frac{\Phi d}{d} > 0$$

for some $u \in K$.

- *If* $\kappa > 0$*, then suppose that* φ *is strictly increasing and* $\lim_{d \to D_\kappa/2} \Phi d = \infty$.

Fix $x \in X$ *and define a subset* $R_f x$ *of* K *by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z,y) + \Phi d(x,y) - \Phi d(x,z)) \geq 0 \right\}.$$

Then $R_f x$ *is nonempty.*

Proof. By the assumption, Φ is continuous and convex. In addition, when $\kappa > 0$, then Φ is strictly increasing on $[0, D_\kappa/2[$ since so are φ and $c_\kappa|_{[0,D_\kappa]}$. Thus, from Theorem 5.9, we get $R_f x \neq \emptyset$ for every $x \in X$. □

Now we show two main results for the well-definedness of the resolvent for equilibrium problems with a perturbation function $\Phi = \varphi \circ c_\kappa$.

Theorem 5.18. *Let* X *be an admissible complete* CAT(κ) *space. Suppose that* X *has the convex hull finite property. Let* K *be a nonempty closed convex subset of* X *and* f *a real function on* K^2 *with conditions* (E1)–(E4). *Let* $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$ *be a strictly increasing and differentiable function such that* φ' *is continuous on* $[0, c_\kappa(D_\kappa/2)[$ *and nondecreasing. Define* $\Phi\colon [0, D_\kappa/2[\to [0,\infty[$ *by* $\Phi d = \varphi(c_\kappa(d))$ *for* $d \in [0, D_\kappa/2[$*. Suppose the following:*

- *If* $\kappa \leq 0$ *and* K *is unbounded, then*

$$\liminf_{\substack{d(u,z)\to\infty \\ z \in K}} \frac{f(u,z)}{d(u,z)} + \lim_{d\to\infty} \frac{\Phi d}{d} > 0$$

for some $u \in K$.

- *If* $\kappa > 0$*, then suppose that* $\lim_{d \to D_\kappa/2} \Phi d = \infty$.

Define a subset $R_f x$ *of* K *by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z,y) + \Phi d(x,y) - \Phi d(x,z)) \geq 0 \right\}$$

for each $x \in X$*. Then the following hold:*

(i) $R_f x$ *consists of exactly one point for every* $x \in X$*, and thus* $R_f\colon X \to K$ *is defined as a single-valued mapping;*
(ii) R_f *is firmly vicinal with* $\varphi' \circ c_\kappa\colon [0, D_\kappa/2[\to]0,\infty[$*;*
(iii) $F(R_f) = \operatorname{Equil} f$.

Proof. The nondecreasingness of φ' yields the convexity of φ. Furthermore, since c_κ is convex on $[0, D_\kappa/2[$, Φ is also convex. Therefore, from Lemma 5.17, $R_f x$ is nonempty for all $x \in X$. We also have $\varphi'(t) > 0$ for any $t \in]0, c_\kappa(D_\kappa/2)[$ since φ is strictly increasing.

Let $x \in X$ and $z_1, z_2 \in R_f x$. Then we obtain from Theorem 5.16 that

$$\begin{aligned} &c_\kappa(d(z_1,z_2)) \\ &\leq \frac{\varphi'(c_\kappa(d(x,z_1)))\big(c_\kappa(d(x,z_2)) - c_\kappa(d(x,z_1))\big) + \varphi'(c_\kappa(d(x,z_2)))\big(c_\kappa(d(x,z_1)) - c_\kappa(d(x,z_2))\big)}{\varphi'(c_\kappa(d(x,z_1)))c_\kappa''(d(x,z_1)) + \varphi'(c_\kappa(d(x,z_2)))c_\kappa''(d(x,z_2))} \\ &= \frac{\big(\varphi'(c_\kappa(d(x,z_1))) - \varphi'(c_\kappa(d(x,z_2)))\big)\big(c_\kappa(d(x,z_2)) - c_\kappa(d(x,z_1))\big)}{\varphi'(c_\kappa(d(x,z_1)))c_\kappa''(d(x,z_1)) + \varphi'(c_\kappa(d(x,z_2)))c_\kappa''(d(x,z_2))}. \end{aligned}$$

Since we are now assuming that φ' is nondecreasing, we obtain $c_\kappa(d(z_1, z_2)) \le 0$ and thus $z_1 = z_2$. Therefore we can consider R_f to be a single-valued mapping from X into K. Conditions (ii) and (iii) are obtained from Theorem 5.16. □

Theorem 5.19. *For $\kappa \le 0$, let X be an admissible complete* CAT(κ) *space and suppose that X has the convex hull finite property. Let K be a nonempty closed convex subset of X and f a real function on K^2 with conditions* (E1)–(E4). *Let $\varphi\colon [0,\infty[\to [0,\infty[$ be a nondecreasing and differentiable function such that φ' is continuous on $[0,\infty[$. Define $\Phi\colon [0,\infty[\to [0,\infty[$ by $\Phi d = \varphi(c_\kappa(d))$ for $d \in [0,\infty[$. If K is unbounded, then assume that*

$$\liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{f(u,z)}{d(u,z)} + \lim_{d\to\infty} \frac{\Phi d}{d} > 0$$

for some $u \in K$. Furthermore, suppose that the following two conditions hold:

- *$\Phi = \varphi \circ c_\kappa$ is convex on $[0,\infty[$;*
- *$\Phi d(x,\cdot)$ is strictly midpoint convex on K for any $x \in X$, namely,*

$$\Phi d\Big(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\Big) < \frac{1}{2}\Phi d(x,y_1) + \frac{1}{2}\Phi d(x,y_2)$$

  holds for any $x \in X$ and $y_1, y_2 \in K$ with $y_1 \ne y_2$.

Define a subset $R_f x$ of K by

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) + \Phi d(x,y) - \Phi d(x,z)) \ge 0 \right\}$$

for each $x \in X$. Then the following hold:

(i) *$R_f x$ consists of exactly one point for every $x \in X$, and thus $R_f\colon X \to K$ is defined as a single-valued mapping;*
(ii) *R_f is firmly vicinal with $\varphi' \circ c_\kappa\colon [0,\infty[\to]0,\infty[$;*
(iii) *$F(R_f) = \operatorname{Equil} f$.*

Proof. (i) Take $x \in X$ and $z \in K$ arbitrarily, and put $g_z(\cdot) = f(z,\cdot) + \Phi d(x,\cdot)$. Then $g_z\colon K \to \mathbb{R}$ is lower semicontinuous and convex. By the assumptions for f and Φ, we get

$$\begin{aligned} g_z\Big(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\Big) &= f\Big(z, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\Big) + \Phi d\Big(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\Big) \\ &< \frac{1}{2}g_z(y_1) + \frac{1}{2}g_z(y_2) \end{aligned}$$

for any $y_1, y_2 \in K$ with $y_1 \ne y_2$. Moreover, we obtain

$$\begin{aligned} \liminf_{\substack{d(z,y)\to\infty \\ y\in K}} \frac{g_z(y)}{d(z,y)} &\ge \liminf_{\substack{d(z,y)\to\infty \\ y\in K}} \frac{f(z,y)}{d(z,y)} + \liminf_{\substack{d(z,y)\to\infty \\ y\in K}} \frac{\Phi d(x,y)}{d(z,y)} \\ &= \liminf_{\substack{d(z,y)\to\infty \\ y\in K}} \frac{f(z,y)}{d(z,y)} + \lim_{d\to\infty} \frac{\Phi d}{d} > 0 \end{aligned}$$

and hence $g_z(y) \to \infty$ if $d(z,y) \to \infty$, that is, g_z is coercive. It concludes that g_z has the unique minimizer by Lemma 6.7 (in Chapter 6).

Put $y_z = \operatorname{argmin}_{y\in K} g_z(y)$ for each $z \in K$. Let $z_1, z_2 \in R_f x$. Then we have $f(z_1, y_{z_1}) + \Phi d(x, y_{z_1}) - \Phi d(x, z_1) \geq 0$ and $f(z_2, y_{z_2}) + \Phi d(x, y_{z_2}) - \Phi d(x, z_2) \geq 0$. Thus

$$f(z_1, y_{z_1}) + f(z_2, y_{z_2}) \geq \Phi d(x, z_1) + \Phi d(x, z_2) - \Phi d(x, y_{z_1}) - \Phi d(x, y_{z_2})$$

holds. Assume that $z_1 \neq z_2$. Then we obtain

$$\begin{aligned} f(z_1, y_{z_1}) + \Phi d(x, y_{z_1}) &\leq f\Big(z_1, \frac{1}{2}z_1 \oplus \frac{1}{2}z_2\Big) + \Phi d\Big(x, \frac{1}{2}z_1 \oplus \frac{1}{2}z_2\Big) \\ &< \frac{1}{2}f(z_1, z_2) + \frac{1}{2}\Phi d(x, z_1) + \frac{1}{2}\Phi d(x, z_2) \end{aligned}$$

and similarly we get

$$f(z_2, y_{z_2}) + \Phi d(x, y_{z_2}) < \frac{1}{2}f(z_2, z_1) + \frac{1}{2}\Phi d(x, z_2) + \frac{1}{2}\Phi d(x, z_1).$$

Summing up these inequalities, we obtain

$$f(z_1, y_{z_1}) + f(z_2, y_{z_2}) < \Phi d(x, z_1) + \Phi d(x, z_2) - \Phi d(x, y_{z_1}) - \Phi d(x, y_{z_2}),$$

which is a contradiction. Thus $R_f x$ is a singleton for every $x \in X$. (ii) and (iii) are directly obtained by Theorem 5.16. □

In Theorem 5.19, the assumption that φ is nondecreasing is not much different from assuming that φ is strictly increasing. Indeed, we get the following.

Fact 5.20. *Let X be an admissible* CAT(κ) *space and $D \in \,]0, D_\kappa/2[$. Let K be a convex subset of X such that there exist $p, q \in K$ such that $d(p, q) \geq D$. Suppose that a function $\varphi\colon [0, c_\kappa(D_\kappa/2)[$ is nondecreasing. Put $\Phi d = \varphi(c_\kappa(d))$ for $d \in [0, D_\kappa/2[$, and suppose that $\Phi d(x, \cdot)$ is strictly midpoint convex on K for all $x \in X$. Then, φ is strictly increasing on $[0, c_\kappa(D)[$. In particular, if $\kappa \leq 0$ and K is unbounded, then φ is strictly increasing on $[0, \infty[$.*

Proof. Assume that φ is not strictly increasing on $[0, c_\kappa(D)[$. Then there exist $t_1, t_2 \in [0, c_\kappa(D)[$ such that $t_1 < t_2$ and $\varphi(t) = \varphi(t_1) = \varphi(t_2)$ for all $t \in [t_1, t_2]$. Take $d_1, d_2 \in [0, D[$ such that $t_1 = c_\kappa(d_1)$ and $t_2 = c_\kappa(d_2)$. Then we obtain $\Phi d = \Phi d_1 = \Phi d_2$ for all $d \in [d_1, d_2]$.

Let $p, q \in K$ such that $d(p, q) \geq D$. Then there exist $u, w \in [p, q]$ which satisfy $d(u, w) = d_2$. Take $v \in [u, w]$ such that $d(u, v) = d_1$. Then we get

$$\Phi d\Big(u, \frac{1}{2}v \oplus \frac{1}{2}w\Big) < \frac{1}{2}\Phi d(u, v) + \frac{1}{2}\Phi d(u, w) = \frac{1}{2}\Phi d_1 + \frac{1}{2}\Phi d_2 = \Phi d_1.$$

We also obtain

$$d\Big(u, \frac{1}{2}v \oplus \frac{1}{2}w\Big) = \frac{d_1 + d_2}{2} \in [d_1, d_2]$$

and thus

$$\Phi d\Big(u, \frac{1}{2}v \oplus \frac{1}{2}w\Big) = \Phi d_1,$$

which is a contradiction. Therefore φ is strictly increasing on $[0, c_\kappa(D)[$.

If $\kappa \leq 0$ and K is unbounded, then we can take any large $D \in [0, \infty[$ satisfying the assumption. This is the conclusion. □

As a consequences of previous results, we obtain sufficient conditions that the perturbation $\Phi = \varphi \circ c_\kappa$ makes R_f a single-valued mapping as follows. Recall that the differentiability and the continuity of the mapping is specified in Remark 5.13.

In what follows, suppose that X is an admissible complete CAT(κ) space which has the convex hull finite property. Let K be a nonempty closed convex subset of X and f a real function on K^2 with conditions (E1)–(E4). Let $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$ and put $\Phi = \varphi \circ c_\kappa\colon [0, D_\kappa/2[\to [0,\infty[$. Define a set-valued mapping $R_f\colon X \to 2^K$ by

$$R_f x = \left\{ z \in K \,\middle|\, \inf_{y\in K} \left(f(z,y) + \Phi(d(x,y)) - \Phi(d(x,z))\right) \geq 0 \right\} \tag{$\star$}$$

for $x \in X$.

First, we consider case where $\kappa \leq 0$. If $\varphi\colon [0,\infty[\to [0,\infty[$ has the following conditions **(a)**, **(b)**, **(c)**, and **(d$_1$)**, then a resolvent R_f of the equilibrium problem for f is well-defined by Theorem 5.18 or Theorem 5.19:

- **(a)** φ is strictly increasing and differentiable;
- **(b)** φ' is continuous on $[0,\infty[$;
- **(c)** at least one of the following hold:
  - **(c$_1$)** φ' is nondecreasing;
  - **(c$_2$)** $\Phi = \varphi \circ c_\kappa$ is convex on $[0,\infty[$, and $\Phi d(x,\cdot)$ is strictly midpoint convex on K for any $x \in X$;
- **(d$_1$)** K is bounded; otherwise, an inequality

$$\liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{f(u,z)}{d(u,z)} + \lim_{d\to\infty} \frac{\Phi d}{d} > 0$$

  holds for some $u \in K$.

Note the following remarks.

- If the condition **(c$_2$)** holds, then we can change the condition of the strict increasingness of φ in **(a)** to the nondecreasingness of φ. These two conditions for φ are equivalent under **(c$_2$)** if K is unbounded, see Fact 5.20.
- If $\lim_{d\to\infty} \Phi d/d > 0$ and $\operatorname{Equil} f \neq \emptyset$, then **(d$_1$)** is always true, see Remark 5.11.
- If $\lim_{d\to\infty} \Phi d/d = \infty$, then **(d$_1$)** is always true, see Remark 5.12. Therefore **(d$_1$)** is true if $\lim_{t\to\infty} \varphi'(t) = \infty$. Indeed, if $\lim_{t\to\infty} \varphi'(t) = \infty$, then we have $\lim_{t\to\infty} \varphi(t)/t = \infty$ and hence $\lim_{d\to\infty} \Phi d/d = \infty$.
- If K is unbounded, then the condition **(d$_1$)** is true if $f(v,\cdot)$ is bounded below for any $v \in K$ and $\lim_{d\to\infty} \Phi d/d > 0$.

Next, consider the case where $\kappa > 0$. If $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$ has the following conditions **(a)**, **(b)**, **(c$_1$)**, and **(d$_2$)**, then a resolvent R_f of the equilibrium problem for f is well-defined by Theorem 5.18.

- **(a)** φ is strictly increasing and differentiable;
- **(b)** φ' is continuous on $[0, c_\kappa(D_\kappa/2)[$;
- **(c$_1$)** φ' is nondecreasing;
- **(d$_2$)** Φ satisfies $\lim_{d\to D_\kappa/2} \Phi d = \infty$.

Note the following remarks.

- $\lim_{d\to D_\kappa/2} \Phi d = \infty$ if and only if $\lim_{d\to D_\kappa/2} \Phi d/d = \infty$ since $D_\kappa = \pi/\sqrt{\kappa}$ for $\kappa > 0$.

- Since $\kappa > 0$, we have $c_\kappa(D_\kappa/2) = 1/\kappa$. Therefore, **(d₂)** is equivalent to $\lim_{\lambda \nearrow 1} \varphi(\lambda/\kappa) = \infty$.

We now consider some specific cases. First, we confirm that the same results as in previous studies [11, 12, 20] in several perturbation functions follow from Theorems 5.18 and 5.19. In the following three cases, a resolvent R_f is well-defined as a single-valued mapping from Theorem 5.18.

Corollary 5.21 (Kimura [11])**.** *Let $\kappa = 1$ and $\varphi(t) = -\log(1-t)$ for $t \in [0,1[$. Then, $\Phi d = -\log\cos d$ for $d \in [0,\pi/2[$, and a resolvent $R_f\colon X \to K$ defined by the equation $(\star)$ is well-defined. Moreover, the following hold:*

(i) *R_f is firmly vicinal with $\psi\colon [0,\pi/2[\ni d \mapsto 1/\cos d$, that is,*

$$2\cos d(R_f x_1, R_f x_2) \geq \frac{\cos d(x_1, R_f x_2)}{\cos d(x_1, R_f x_1)} + \frac{\cos d(x_2, R_f x_1)}{\cos d(x_2, R_f x_2)}$$

for any $x_1, x_2 \in X$;

(ii) *R_f is vicinal with the same ψ as* (i)*;*

(iii) *R_f is vicinal with a constant function $\mathbb{1}\colon [0,\pi/2[\ni d \mapsto 1$.*

Proof. A derivative of φ is expressed by $\varphi'(t) = 1/(1-t)$ for $t \in [0,1[$. Thus φ satisfies conditions **(a)**, **(b)**, **(c₁)**, and **(d₂)**. Therefore, R_f is well-defined as a single-valued mapping. Put $\psi = \varphi' \circ c_\kappa$, then $\psi(d) = 1/\cos d$ for $d \in [0,\pi/2[$, and R_f is firmly vicinal with ψ. It follows that (i) and (ii) hold. Moreover, (i) implies an inequality $2\cos d(R_f x_1, R_f x_2) \geq \cos d(x_1, R_f x_2) + \cos d(x_2, R_f x_1)$ for every $x_1, x_2 \in X$, which means the vicinity of R_f with a constant function $\mathbb{1}$. □

Corollary 5.21 (iii) means that such a mapping R_f is spherically nonspreading of sum type, see Section 4.1.

Corollary 5.22 (Kimura and Kishi [12])**.** *Let $\kappa = 0$ and $\varphi(t) = t$ for $t \in [0,\infty[$. Then, $\Phi d = d^2/2$ for $d \in [0,\infty[$, and a resolvent $R_f\colon X \to K$ defined by the equation $(\star)$ is well-defined. Moreover, R_f is firmly metrically nonspreading.*

Proof. We easily obtain that conditions **(a)**, **(b)**, **(c₁)**, and **(d₁)** are true, hence R_f is well-defined. Put $\psi = \varphi' \circ c_\kappa$ for any $d \in [0,\infty[$. Then $\psi(d) = 1$ for any $d \in [0,\infty[$. Thus R_f is firmly metrically nonspreading, see Section 4.1. □

Corollary 5.23 (Kimura and Ogihara [20])**.** *Let $\kappa = -1$ and $\varphi(t) = t+1$ for $t \in [0,\infty[$. Then, $\Phi d = \cosh d$ for $d \in [0,\infty[$, and a resolvent $R_f\colon X \to K$ defined by the equation $(\star)$ is well-defined. Moreover, it satisfies*

$$\big(\cosh d(x_1, R_f x_1) + \cosh d(x_2, R_f x_2)\big)\cosh d(R_f x_1, R_f x_2) \leq \cosh d(x_1, R_f x_2) + \cos d(x_2, R_f x_1)$$

for any $x_1, x_2 \in X$.

Proof. Now conditions **(a)**, **(b)**, **(c₁)**, and **(d₁)** are true. Put $\psi = \varphi' \circ c_\kappa$, that is, $\psi(d) = 1$ for any $d \in [0,\infty[$. Then R_f is firmly vicinal with ψ, which follows the desired inequality. □

Now we introduce three corollaries, which are our new result. The following Corollaries 5.24 and 5.26 are obtained from Theorem 5.18, and Corollary 5.27 from Theorem 5.19.

Corollary 5.24. *We consider the case where $\kappa = 1$ and $\varphi(t) = 1/(1-t) - (1-t)$ for $t \in [0,1[$. Then, $\Phi d = \tan d \sin d$ for $d \in [0,\pi/2[$. Define a resolvent $R_f\colon X \to K$ by the equation $(\star)$. Then*

R_f is well-defined, and it satisfies

$$\left(\left(\frac{1}{\cos^2 d(x_1,R_fx_1)}+1\right)\cos d(x_1,R_fx_1)+\left(\frac{1}{\cos^2 d(x_2,R_fx_2)}+1\right)\cos d(x_2,R_fx_2)\right)\cos d(R_fx_1,R_fx_2)$$
$$\geq\left(\frac{1}{\cos^2 d(x_1,R_fx_1)}+1\right)\cos d(x_1,R_fx_2)+\left(\frac{1}{\cos^2 d(x_2,R_fx_2)}+1\right)\cos d(x_2,R_fx_1)$$

for any $x_1,x_2\in X$. Furthermore, R_f is firmly spherically nonspreading in the sense of Kimura and Kohsaka [13]*. Namely,*

$$\cos^2 d(R_fx_1,R_fx_2)\geq\frac{2}{\cos d(x_1,R_fx_1)+\cos d(x_1,R_fx_1)}\cos d(x_1,R_fx_2)\cos d(x_2,R_fx_1)$$

for any $x_1,x_2\in X$.

Proof. We get $\varphi'(t)=1+1/(1-t)^2$ for any $t\in[0,1[$. Thus φ satisfies **(a)**, **(b)**, **(c$_1$)**, and **(d$_2$)**, therefore R_f is well-defined as a single-valued mapping. Put $\psi=\varphi'\circ c_\kappa$, that is, $\psi(d)=1/\cos^2 d+1$ for $d\in[0,\pi/2[$, then R_f is firmly vicinal with ψ. Therefore, we get the desired first inequality.

Let $x_1,x_2\in X$ and put $\varphi_1=\cos d(x_1,R_fx_2)$, $\varphi_2=\cos d(x_2,R_fx_1)$ and put $C_1=\cos d(x_1,R_fx_1)$ and $C_2=\cos d(x_2,R_fx_2)$. Then we get

$$\cos d(R_fx_1,R_fx_2)\geq\frac{\left(\frac{1}{C_1^2}+1\right)\varphi_1+\left(\frac{1}{C_2^2}+1\right)\varphi_2}{\left(\frac{1}{C_1^2}+1\right)C_1+\left(\frac{1}{C_2^2}+1\right)C_2}$$
$$=\frac{\frac{C_2}{C_1}\varphi_1+\frac{C_1}{C_2}\varphi_2+C_1C_2(\varphi_1+\varphi_2)}{(C_1+C_2)(1+C_1C_2)}\geq\frac{2\sqrt{\varphi_1\varphi_2}+2C_1C_2\sqrt{\varphi_1\varphi_2}}{(C_1+C_2)(1+C_1C_2)}=\frac{2\sqrt{\varphi_1\varphi_2}}{C_1+C_2}$$

and thus

$$\cos^2 d(R_fx_1,R_fx_2)\geq\left(\frac{2}{C_1+C_2}\right)^2\varphi_1\varphi_2\geq\frac{2}{C_1+C_2}\cos d(x_1,R_fx_2)\cos d(x_2,R_fx_1)$$

for any $x_1,x_2\in X$. This is the desired result. □

In 2016, the well-definedness of the resolvent of the convex function defined by using the perturbation $\tan d\sin d$ was proved by Kimura and Kohsaka [13]. The result above implies that we can use the same perturbation $\tan d\sin d$ to define the resolvent of the equilibrium problem as the single-valued mapping.

Corollary 5.25. *We consider the case where $\kappa=1$ and $\varphi(t)=1/(1-t)$ for $t\in[0,1[$. Then, $\Phi d=1/\cos d$ for $d\in[0,\pi/2[$. Define a resolvent $R_f\colon X\to K$ by the equation ($\star$). Then R_f is well-defined, and it satisfies*

$$\left(\frac{1}{\cos d(x_1,R_fx_1)}+\frac{1}{\cos d(x_2,R_fx_2)}\right)\cos d(R_fx_1,R_fx_2)$$
$$\geq\frac{1}{\cos^2 d(x_1,R_fx_1)}\cdot\cos d(x_1,R_fx_2)+\frac{1}{\cos^2 d(x_2,R_fx_2)}\cdot\cos d(x_2,R_fx_1)$$

for any $x_1,x_2\in X$.

Proof. Since $\varphi'(t) = 1/(1-t)^2$ for any $t \in [0,1[$, φ satisfies **(a)**, **(b)**, **(c$_1$)**, and **(d$_2$)**. Hence R_f is well-defined as a single-valued mapping. Moreover, putting $\psi = \varphi' \circ c_\kappa$, we have $\psi(d) = 1/\cos^2 d$ for $d \in [0, \pi/2[$ and R_f is firmly vicinal with ψ. This is the conclusion. □

Corollary 5.26. *We consider the case where $\kappa = -1$ and $\varphi(t) = \log(t+1)$ for $t \in [0,\infty[$. Then, $\Phi d = \log\cosh d$ for $d \in [0,\infty[$. Suppose that $f\colon K^2 \to \mathbb{R}$ satisfies* (E1)–(E4) *and*

$$\liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{f(u,z)}{d(u,z)} + 1 > 0$$

for some $u \in K$. Then a resolvent $R_f\colon X \to K$ defined by the equation $(\star)$ is well-defined, and

$$2\cosh d(R_f x_1, R_f x_2) \le \frac{\cosh d(x_1, R_f x_2)}{\cosh d(x_1, R_f x_1)} + \frac{\cosh d(x_2, R_f x_1)}{\cosh d(x_2, R_f x_2)}$$

holds for any $x_1, x_2 \in X$.

Proof. We get $\varphi'(t) = 1/(t+1)$, and thus φ satisfies **(a)**, **(b)**, and **(c$_1$)**. Furthermore, since $\lim_{d\to\infty} \Phi d/d = \lim_{d\to\infty}(\log\cosh d)/d = 1$, we obtain **(d$_1$)**. Thus R_f is well-defined as a single-valued mapping. Put $\psi = \varphi' \circ c_\kappa$, that is, $\psi(d) = 1/\cosh d$ for $d \in [0,\infty[$. Then R_f is firmly vicinal with ψ and hence we get the conclusion. □

Corollary 5.27. *We consider the case where $\kappa = -1$ and $\varphi(t) = t + 1 - 1/(t+1)$ for $t \in [0,\infty[$. Then, $\Phi d = \tanh d \sinh d$ for $d \in [0,\infty[$. Define a resolvent $R_f\colon X \to K$ by the equation $(\star)$. Then R_f is well-defined, and it satisfies*

$$\begin{aligned}\left(\left(\frac{1}{\cosh^2 d(x_1,R_f x_1)}+1\right)\cosh d(x_1,R_f x_1)+\left(\frac{1}{\cosh^2 d(x_2,R_f x_2)}+1\right)\cosh d(x_2,R_f x_2)\right)\cosh d(R_f x_1,R_f x_2)\\ \le \left(\frac{1}{\cosh^2 d(x_1,R_f x_1)}+1\right)\cosh d(x_1,R_f x_2)+\left(\frac{1}{\cosh^2 d(x_2,R_f x_2)}+1\right)\cosh d(x_2,R_f x_1)\end{aligned}$$

for any $x_1, x_2 \in X$.

To show it, we use the next lemma.

Lemma 5.28 (Kajimura and Kimura [7])**.** *Let X be a complete* CAT(-1) *space and $x \in X$. Define a real function g by*

$$g(\cdot) = \tanh d(x,\cdot)\sinh d(x,\cdot).$$

Then g is strictly midpoint convex.

Proof of Corollary 5.27. A function Φ is convex, and $\Phi d(x,\cdot) = \tanh d(x,\cdot)\sinh d(x,\cdot)$ is strictly midpoint convex by the above lemma. Moreover, we get $\lim_{d\to\infty}\Phi d/d = \infty$. It means that φ satisfies **(a)**, **(b)**, **(c$_2$)** and **(d$_1$)**. Therefore R_f is well-defined from Theorem 5.19. Put $\psi = \varphi' \circ c_\kappa$. Then $\psi(d) = 1 + 1/\cosh^2 d$ for any $d \in [0,\infty[$, and then R_f is firmly vicinal with ψ. It implies the desired result. □

In Theorems 5.16, 5.18, and 5.19, we used a perturbation $\varphi \circ c_\kappa$ to define a resolvent R_f. Henceforth, we consider that we use a perturbation $\overline{\varphi} \circ c''_\kappa$ instead of $\varphi \circ c_\kappa$, and consider conditions to well-define a resolvent. Let X be a complete CAT(κ) space with the convex hull finite property, and K a nonempty closed convex subset of X.

Consider the case where $\kappa < 0$. Let $\varphi\colon [0,\infty[\to [0,\infty[$ and $\overline{\varphi}\colon [1,\infty[\to [0,\infty[$. Suppose that $\varphi \circ c_\kappa = \overline{\varphi} \circ c''_\kappa$, and let $R_f\colon X \to 2^K$ be a set-valued mapping defined by

$$\begin{aligned} R_f x &= \left\{ z \in K \;\middle|\; \inf_{y\in K} \left(f(z,y) + \varphi(c_\kappa(d(x,y))) - \varphi(c_\kappa(d(x,z))) \right) \geq 0 \right\} \\ &= \left\{ z \in K \;\middle|\; \inf_{y\in K} \left(f(z,y) + \overline{\varphi}(c''_\kappa(d(x,y))) - \overline{\varphi}(c''_\kappa(d(x,z))) \right) \geq 0 \right\} \end{aligned}$$

for $x \in X$. Then we know that R_f is well-defined if $\varphi\colon [0,\infty[\to [0,\infty[$ satisfies conditions **(a)**, **(b)**, **(c)**, and **(d₁)**. Moreover since $c''_\kappa(d) = 1 - \kappa c_\kappa(d)$ for $d \in \mathbb{R}$, we get $\varphi(t) = \overline{\varphi}(1-\kappa t)$ for any $t \geq 0$. Indeed, we obtain $\varphi(c_\kappa(d)) = \overline{\varphi}(c''_\kappa(d)) = \overline{\varphi}(1 - \kappa c_\kappa(d))$ for all $d \geq 0$. Under this setting, we have the following.

Theorem 5.29. *Let $\kappa < 0$, $\varphi\colon [0,\infty[\to [0,\infty[$, and $\overline{\varphi}\colon [1,\infty[\to [0,\infty[$. Suppose that $\varphi(c_\kappa(d)) = \overline{\varphi}(c''_\kappa(d))$ for every $d \in [0,\infty[$. Define conditions* **(a)**, **(b)** *and* **(c₁)** *for φ by*

- **(a)** *φ is strictly increasing and differentiable;*
- **(b)** *φ' is continuous on $[0,\infty[$;*
- **(c₁)** *φ' is nondecreasing.*

Similarly, define conditions **(a′)**, **(b′)** *and* **(c₁′)** *for $\overline{\varphi}$ by*

- **(a′)** *$\overline{\varphi}$ is strictly increasing and differentiable;*
- **(b′)** *$\overline{\varphi}'$ is continuous on $[1,\infty[$;*
- **(c₁′)** *$\overline{\varphi}'$ is nondecreasing.*

Then conditions **(a)** *and* **(a′)** *are equivalent. Moreover, under the conditions* **(a)** *and* **(a′)**, *the following hold:*

- **(b)** *and* **(b′)** *are equivalent;*
- **(c₁)** *and* **(c₁′)** *are equivalent.*

Proof. Since $-\kappa > 0$ and $\varphi(t) = \overline{\varphi}(1-\kappa t)$ for any $t \geq 0$, we easily get that φ is strictly increasing if and only if so is $\overline{\varphi}$. Moreover, from $\varphi((s-1)/(-\kappa)) = \overline{\varphi}(s)$ for any $s \geq 1$, it is clear that φ is differentiable if and only if $\overline{\varphi}$ is differentiable. Assuming that **(a)** is true, we get $\varphi'(t) = -\kappa\overline{\varphi}'(1-\kappa t)$ for all $t \in [0,\infty[$, which is the conclusion. □

In the same way, we also get the following for $\kappa > 0$.

Theorem 5.30. *Let $\kappa > 0$, $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$, and $\overline{\varphi}\colon]0,1] \to [0,\infty[$. Suppose that $\varphi(c_\kappa(d)) = \overline{\varphi}(c''_\kappa(d))$ for every $d \in [0, D_\kappa/2[$. Define conditions* **(a)**, **(b)** *and* **(c₁)** *for φ by*

- **(a)** *φ is strictly increasing and differentiable;*
- **(b)** *φ' is continuous on $[0,\infty[$;*
- **(c₁)** *φ' is nondecreasing.*

Similarly, define conditions **(a′)**, **(b′)** *and* **(c₁′)** *for $\overline{\varphi}$ by*

- **(a″)** *$\overline{\varphi}$ is strictly decreasing and differentiable;*
- **(b′)** *$\overline{\varphi}'$ is continuous on $]0,1]$;*
- **(c₁′)** *$\overline{\varphi}'$ is nondecreasing.*

Then conditions **(a)** *and* **(a″)** *are equivalent. Moreover, under the conditions* **(a)** *and* **(a″)**, *the following hold:*

- **(b)** *and* **(b′)** *are equivalent;*
- **(c₁)** *and* **(c₁′)** *are equivalent.*

Proof. By assumptions, we get $c_\kappa(D_\kappa/2) = 1/\kappa$ and $\varphi(t) = \overline{\varphi}(1-\kappa t)$ for any $t \in [0, 1/\kappa[$. In other words, $\varphi((1-s)/\kappa) = \overline{\varphi}(s)$ for any $s \in]0,1]$. These imply the conclusion. □

Theorems 5.29 and 5.30 imply that using c_κ and c''_κ to define the perturbation function are essentially equivalent for any $\kappa \neq 0$. Note that since $\varphi'(t) = -\kappa\overline{\varphi}'(1-\kappa t)$ for all $t \in [0,\infty[$, we get $\varphi' \circ c_\kappa = -\kappa(\overline{\varphi}' \circ c''_\kappa)$. This means that a mapping $T\colon X \to X$ is (firmly) vicinal with $\varphi' \circ c_\kappa$ if and only if T is (firmly) vicinal with $-\kappa(\overline{\varphi}' \circ c''_\kappa)$.

The following table describes natures of a resolvent R_f of the equilibrium problem defined by using a perturbation $\Phi = \varphi \circ c_\kappa = \overline{\varphi} \circ c''_\kappa$, where 'FV' and 'V' mean 'firmly vicinal' and 'vicinal', respectively.

	Perturbation $\Phi(d)$	$\varphi(t)$	$\overline{\varphi}(s)$	Nature of φ' and $\overline{\varphi}'$	Natures of R_f
$\kappa = 1$	$-\log\cos d$	$-\log(1-t)$	$-\log s$	nondecreasing	FV with $d \mapsto \dfrac{1}{\cos d}$; V with $d \mapsto 1$.
	$\tan d \sin d$	$\dfrac{1}{1-t} - (1-t)$	$\dfrac{1}{s} - s$	nondecreasing	FV with $d \mapsto \dfrac{1}{\cos^2 d} + 1$.
	$\dfrac{1}{\cos d}$	$\dfrac{1}{1-t}$	$\dfrac{1}{s}$	nondecreasing	FV with $d \mapsto \dfrac{1}{\cos^2 d}$.
$\kappa = 0$	$\dfrac{1}{2}d^2$	t	(null)	nondecreasing	FV with $d \mapsto 1$.
$\kappa = -1$	$\log\cosh d$	$\log(t-1)$	$\log s$	nondecreasing	FV with $d \mapsto \dfrac{1}{\cosh d}$.
	$\tanh d \sinh d$	$t + 1 - \dfrac{1}{t+1}$	$s - \dfrac{1}{s}$	nonincreasing	FV with $d \mapsto \dfrac{1}{\cosh^2 d} + 1$.
	$\cosh d$	$t+1$	s	nondecreasing	FV with $d \mapsto 1$.

Note that perturbations defining R_f and functions defining firm vicinity of R_f are related to an integral with the formula

$$\int_0^d f(c''_\kappa(t))c'_\kappa(t)\,dt.$$

For instance, if $\kappa = 1$, which implies $c''_\kappa(t) = \cos t$ and $c'_\kappa(t) = \sin t$, then we get

$$-\log\cos d = \int_0^d \frac{1}{\cos t} \cdot \sin t\,dt, \quad \tan d \sin d = \int_0^d \left(\frac{1}{\cos^2 t} + 1\right) \sin t\,dt,$$

and

$$\frac{1}{\cos d} = 1 + \int_0^d \frac{1}{\cos^2 t} \cdot \sin t\,dt.$$

Similarly, if $\kappa = -1$, which implies $c''_\kappa(t) = \cosh t$, $c'_\kappa(t) = \sinh t$, then we obtain

$$\log\cosh d = \int_0^d \frac{1}{\cosh t} \cdot \sinh t\,dt, \quad \tanh d \sinh d = \int_0^d \left(\frac{1}{\cosh^2 t} + 1\right) \sinh t\,dt,$$

and

$$\cosh d = 1 + \int_0^d 1 \cdot \sinh t\,dt.$$

This fact is obtained by the following result.

Theorem 5.31. *For $\kappa \neq 0$, let X be a complete* CAT(κ) *space with the convex hull finite property, and K a nonempty closed convex subset of X. Let $\varphi\colon [0,\infty[\to [0,\infty[$ be a function which satisfies the same conditions as Theorem 5.16, and $R_f\colon X \to K$ a resolvent for f well defined by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) + \varphi(c_\kappa(d(x,y))) - \varphi(c_\kappa(d(x,z)))) \geq 0 \right\}$$

for $x \in X$ as a single-valued mapping. Put $A_\kappa = [1,\infty[$ if $\kappa < 0$, and $A_\kappa =]0,1]$ if $\kappa > 0$. Let $c \in \mathbb{R}$ and $g\colon A_\kappa \to \mathbb{R}$ such that $\varphi'(c_\kappa(0)) = g(c_\kappa''(0))$ and

$$\varphi(c_\kappa(d)) = c + \int_0^d g(c_\kappa''(t))c_\kappa'(t)\,dt$$

for every $d \in [0, D_\kappa/2[$. Then R_f is firmly vicinal with $g \circ c_\kappa''$.

Proof. By assumptions, we obtain

$$\varphi'(c_\kappa(d))c_\kappa'(d) = g(c_\kappa''(d))c_\kappa'(d)$$

for every $d \in [0, D_\kappa/2[$. This means that $\varphi'(c_\kappa(d)) = g(c_\kappa''(d))$ for all $d \in [0, D_\kappa/2[$. Therefore, from Theorem 5.16, we get the conclusion. □

5.2 Applications

By applying results of Lemma 4.9 and Theorem 5.16 to Theorems 4.21, 4.22, 4.23, and 4.24, we obtain the following convergence theorem with Mann type approximation scheme.

Corollary 5.32. *Let X, K, f, φ and Φ be the same as Theorem 5.16, and $R_f\colon X \to K$ resolvent well defined by an equation ($\star$), see p. 67. Let $\{\alpha_n\} \subset [0,1[$ such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by either*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)R_f x_n$$

for $n \in \mathbb{N}$ or

$$x_{n+1} = \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n)R_f x_n$$

for $n \in \mathbb{N}$. Let us denote (a) *and* (b) *by the following conditions:*

- (a) *$\{R_f x_n\}$ is κ-bounded;*
- (b) $\sup_{n\in\mathbb{N}} d(x_n, R_f x_n) < D_\kappa/2$.

Suppose that R_f is vicinal with $\psi\colon [0, D_\kappa/2[\to]0,\infty[$, and define conditions (P1) *and* (P2) *as follows:*

- (P1) *ψ is nondecreasing;*
- (P2) $\sup_{n\in\mathbb{N}} \psi(d(x_n, R_f x_n)) < \infty$.

Then the following hold:

- (i) *Suppose that ψ satisfies* (P1). *Then* Equil $f \neq \emptyset$ *if and only if* (a) *and* (b) *hold.*
- (i)′ *Suppose that ψ satisfies* (P2). *Then* Equil $f \neq \emptyset$ *if and only if* (a) *holds.*

Corollary 5.33. *Let X, K, f, φ and Φ be the same as Theorem 5.16, and $R_f\colon X \to K$ resolvent well defined by an equation* ($\star$). *Suppose that* $\operatorname{Equil} f \neq \emptyset$. *Let $\{\alpha_n\} \subset [0,1[$ such that* $\limsup_{n\to\infty} \alpha_n < 1$ *and* $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. *Take $x_1 \in X$ and generate $\{x_n\} \subset X$ by either*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) R_f x_n$$

for $n \in \mathbb{N}$ or

$$x_{n+1} = \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) R_f x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to some element in $\operatorname{Equil} f$.

In Corollary 5.32, the function ψ need not be given by $\varphi' \circ c_\kappa$. Therefore for instance, if R_f is the resolvent defined by Corollary 5.21, which uses the perturbation function $\Phi d = \varphi(c_1(d)) = -\log\cos d$, then we can use not only $\varphi' \circ c_1$ but also the constant function $\mathbb{1}$ as the function ψ. In this case, $\varphi' \circ c_1$ satisfies (P1) since $\varphi'(c_1(d)) = 1/\cos d$, and $\mathbb{1}$ satisfies both (P1) and (P2). Note that the condition (P2) always holds if ψ is bounded above.

We also get the following convergence theorem with Halpern type approximation scheme from Theorem 4.29.

Corollary 5.34. *Let X, K, f, φ and Φ be the same as Theorem 5.16, and $R_f\colon X \to K$ resolvent well defined by an equation* ($\star$). *Suppose that* $\operatorname{Equil} f \neq \emptyset$. *Let $\{\alpha_n\} \subset]0,1[$ such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Let $u, x_1 \in X$ arbitrarily and define $\{x_n\} \subset X$ by*

$$x_{n+1} = \alpha_n u \overset{\kappa}{\oplus} (1-\alpha_n) R_f x_n$$

for any $n \in \mathbb{N}$. In the case where $\kappa > 0$, suppose that (i) *or* (ii) *holds:*

(i) $\sup_{n\in\mathbb{N}} d(u, R_f x_n) < D_\kappa/2$,

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then $\{x_n\}$ converges to some element in $\operatorname{Equil} f$.

Finally, we show a property for two resolvents $R_{\lambda f}$ and $R_{\mu f}$ for $\lambda, \mu > 0$.

Theorem 5.35. *Let X be an admissible complete* $\mathrm{CAT}(\kappa)$ *space. Let K be a nonempty closed convex subset of X and f a real function on K^2 with conditions* (E2). *Let $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$ be a differentiable function. For $\lambda, \mu > 0$, let $R_{\lambda f}, R_{\mu f}$ be mappings from X into K. Assume that for each $\nu \in \{\lambda, \mu\}$, an inequality*

$$0 \le \nu f(R_{\nu f} x, w) + \varphi'(c_\kappa(d(x, R_{\nu f} x))) \cdot \frac{E}{c_\kappa'(E)} \big(c_\kappa(d(x,w)) - c_\kappa(E) - c_\kappa''(E) c_\kappa(d(x, R_{\nu f} x)) \big)$$

holds for any $x \in X$ and $w \in K \setminus \{R_{\nu f} x\}$, where $E = d(R_{\nu f} x, w)$. Then for any $x, y \in X$, the following inequalities hold:

$$\begin{aligned} 0 \le \lambda \varphi'(c_\kappa(D_{yy}))\big(c_\kappa(D_{yx}) - c_\kappa(D) - c_\kappa''(D) c_\kappa(D_{yy})\big) \\ + \mu \varphi'(c_\kappa(D_{xx}))\big(c_\kappa(D_{xy}) - c_\kappa(D) - c_\kappa''(D) c_\kappa(D_{xx})\big); \end{aligned} \tag{i}$$

$$\begin{aligned} \big(\lambda \varphi'(c_\kappa(D_{yy})) c_\kappa''(D_{yy}) + \mu \varphi'(c_\kappa(D_{xx})) c_\kappa''(D_{xx})\big) c_\kappa(D) \\ \le \lambda \varphi'(c_\kappa(D_{yy}))\big(c_\kappa(D_{yx}) - c_\kappa(D_{yy})\big) \\ + \mu \varphi'(c_\kappa(D_{xx}))\big(c_\kappa(D_{xy}) - c_\kappa(D_{xx})\big); \end{aligned} \tag{ii}$$

$$\frac{1}{\kappa}\big(\lambda\varphi'(c_\kappa(D_{yy}))c''_\kappa(D_{yy}) + \mu\varphi'(c_\kappa(D_{xx}))c''_\kappa(D_{xx})\big)c''_\kappa(D)$$
$$\geq \frac{1}{\kappa}\big(\lambda\varphi'(c_\kappa(D_{yy}))c''_\kappa(D_{yx}) + \mu\varphi'(c_\kappa(D_{xx}))c''_\kappa(D_{xy})\big), \quad \text{(iii)}$$

where $D_{xx} = d(x, R_{\lambda f}x)$, $D_{xy} = d(x, R_{\mu f}y)$, $D_{yx} = d(y, R_{\lambda f}x)$, $D_{yy} = d(y, R_{\mu f}y)$, *and* $D = d(R_{\lambda f}x, R_{\mu f}y)$. *The inequality* (iii) *is considered only when* $\kappa \neq 0$.

Proof. Let $\lambda, \mu > 0$ and $x, y \in X$. If $R_{\lambda f}x = R_{\mu f}y$, then we get the conclusion obviously. Suppose that $R_{\lambda f}x \neq R_{\mu f}y$. Then, for a function $\mu f \colon K^2 \to \mathbb{R}$, we have

$$0 \leq \mu f(R_{\mu f}y, R_{\lambda f}x) + \varphi'(c_\kappa(D_{yy})) \cdot \frac{D}{c'_\kappa(D)}\big(c_\kappa(D_{yx}) - c_\kappa(D) - c''_\kappa(D)c_\kappa(D_{yy})\big)$$

and hence

$$0 \leq \frac{c'_\kappa(D)}{D}\lambda\mu f(R_{\mu f}y, R_{\lambda f}x) + \lambda\varphi'(c_\kappa(D_{yy}))\big(c_\kappa(D_{yx}) - c_\kappa(D) - c''_\kappa(D)c_\kappa(D_{yy})\big).$$

Similarly, for a function λf, we get

$$0 \leq \frac{c'_\kappa(D)}{D}\mu\lambda f(R_{\lambda f}x, R_{\mu f}y) + \mu\varphi'(c_\kappa(D_{xx}))\big(c_\kappa(D_{xy}) - c_\kappa(D) - c''_\kappa(D)c_\kappa(D_{xx})\big).$$

Summing up these two inequalities, we obtain (i) since f satisfies (E2).

Using Lemma 2.8, we obtain that the inequality (i) is equivalent to

$$0 \leq \lambda\varphi'(c_\kappa(D_{yy}))\big(c_\kappa(D_{yx}) - c_\kappa(D_{yy}) - c''_\kappa(D_{yy})c_\kappa(D)\big)$$
$$+ \mu\varphi'(c_\kappa(D_{xx}))\big(c_\kappa(D_{xy}) - c_\kappa(D_{xx}) - c''_\kappa(D_{xx})c_\kappa(D)\big),$$

and so is (ii).

Assume that $\kappa \neq 0$. Then, using Lemma 2.8 again, we obtain that (i) is equivalent to

$$0 \leq \lambda\varphi'(c_\kappa(D_{yy}))\left(\frac{1 - c''_\kappa(D_{yx})}{\kappa} - \frac{1 - c''_\kappa(D_{yy})c''_\kappa(D)}{\kappa}\right)$$
$$+ \mu\varphi'(c_\kappa(D_{xx}))\left(\frac{1 - c''_\kappa(D_{xy})}{\kappa} - \frac{1 - c''_\kappa(D_{xx})c''_\kappa(D)}{\kappa}\right),$$

and so is (iii). □

Corollary 5.36. *For* $\lambda, \mu > 0$, *let* $R_{\lambda f}$ *and* $R_{\mu f}$ *be well-defined resolvents of the equilibrium problem under the assumption of Theorem 5.16 for* λf *and* μf, *respectively. Then these satisfy inequalities* (i), (ii) *and* (iii) *of Theorem 5.35 for any* $x, y \in X$.

Proof. Suppose that $R_{\lambda f}$ and $R_{\mu f}$ are well-defined as a single-valued mapping. From Lemma 5.15, we obtain

$$0 \leq \lambda f(R_{\lambda f}x, w) + \varphi'(c_\kappa(d(x, R_{\lambda f}x))) \cdot \frac{D_\lambda}{c'_\kappa(D_\lambda)}\big(c_\kappa(d(x,w)) - c_\kappa(D_\lambda) - c''_\kappa(D_\lambda)c_\kappa(d(x, R_{\lambda f}x))\big)$$

for any $x \in X$ and $w \in K \setminus \{R_{\lambda f}x\}$, where $D_\lambda = d(R_{\lambda f}x, w)$. We also have

$$0 \leq \mu f(R_{\mu f}x, w) + \varphi'(c_\kappa(d(x, R_{\mu f}x))) \cdot \frac{D_\mu}{c'_\kappa(D_\mu)}\big(c_\kappa(d(x,w)) - c_\kappa(D_\mu) - c''_\kappa(D_\mu)c_\kappa(d(x, R_{\mu f}x))\big)$$

for any $x \in X$ and $w \in K \setminus \{R_{\mu f}x\}$, where $D_\mu = d(R_{\mu f}x, w)$. From Theorem 5.35, we obtain the conclusion. □

Chapter 6

Convex functions

In this chapter, we consider a convex minimization problem and a new type of convex functions.

Definition 6.1. Let X be a uniquely D-geodesic space and f a proper function from X into $]-\infty, \infty]$. Then f is said to be *midpoint convex* if

$$f\left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) \le \frac{1}{2}f(y_1) + \frac{1}{2}f(y_2)$$

for any $y_1, y_2 \in \mathrm{dom}(f)$ with $d(y_1, y_2) < D$. In addition, f is said to be *strictly midpoint convex* if

$$f\left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) < \frac{1}{2}f(y_1) + \frac{1}{2}f(y_2)$$

for any $y_1, y_2 \in \mathrm{dom}(f)$ with $0 < d(y_1, y_2) < D$.

Definition 6.2. Let X be a uniquely D-geodesic space and f a proper function from X into $]-\infty, \infty]$. Then f is said to be *quasiconvex* if

$$f(ty_1 \oplus (1-t)y_2) \le \max\{f(y_1), f(y_2)\}$$

for any $y_1, y_2 \in \mathrm{dom}(f)$ and $t \in]0, 1[$. In addition, f is said to be *strictly midpoint quasiconvex* if

$$f\left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) < \max\{f(y_1), f(y_2)\}$$

for any $y_1, y_2 \in \mathrm{dom}(f)$ with $0 < d(y_1, y_2) < D$.

It is clear that if f is convex then f is quasiconvex. Similarly, if f is strictly midpoint convex, then f is strictly midpoint quasiconvex.

6.1 Convex minimization problems

In this section, we consider applying the result of fixed point approximation theorems shown in the previous chapter to solve convex minimization problems on admissible complete CAT(κ) spaces.

Let f be a proper convex function from X into $]-\infty, \infty]$. Then we call a mapping S_f from X into itself a *resolvent* operator of f if the set of all fixed points of S_f coincides with the set of all minimizers of f, that is, $F(S_f) = \operatorname{argmin} f$.

Regarding resolvents of convex functions, Kajimura and Kimura proved the following result.

Lemma 6.3 (Kajimura and Kimura [9])**.** *Let X be an admissible complete* $\mathrm{CAT}(\kappa)$ *space and f a proper convex function from X into* $]-\infty,\infty]$. *Suppose that a function* $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$ *is nondecreasing and differentiable, and φ' is continuous at* $[0, c_\kappa(D_\kappa/2)[$. *Define a set-valued mapping S_f from X into $2^{\mathrm{dom}(f)}$ by*

$$S_f x = \operatorname*{argmin}_{y\in X} \big(f(y) + \varphi(c_\kappa(d(x,y)))\big)$$

for $x \in X$. Assume that S_f is well-defined as a single-valued mapping from X into $\mathrm{dom}(f)$. *Then S_f is firmly vicinal with* $\varphi' \circ c_\kappa\colon [0, D_\kappa/2[\to]0,\infty[$.

Lemma 6.3 is obtained by the next lemma.

Lemma 6.4. *Let X, f and φ are the same as Lemma 6.3. Define a set-valued mapping S_f from X into $2^{\mathrm{dom}(f)}$ by*

$$S_f x = \operatorname*{argmin}_{y\in X} \big(f(y) + \varphi(c_\kappa(d(x,y)))\big)$$

for $x \in X$. Then the following hold:

(i) *For any $x \in X$, $z \in S_f x$ and $w \in X \setminus \{z\}$, an inequality*

$$0 \le f(w) - f(z) + \varphi'(c_\kappa(d(x,z))) \cdot \frac{D}{c'_\kappa(D)}\big(c_\kappa(d(x,w)) - c_\kappa(D) - c''_\kappa(D)c_\kappa(d(x,z))\big)$$

holds, where $D = d(z,w)$;

(ii) *for any $x_1, x_2 \in X$, $z_1 \in S_f x_1$ and $z_2 \in S_f x_2$, an inequality*

$$\begin{aligned}&\big(\varphi'(c_\kappa(D_1))c''_\kappa(D_1) + \varphi'(c_\kappa(D_2))c''_\kappa(D_2)\big)c_\kappa(d(z_1,z_2))\\ &\qquad \le \varphi'(c_\kappa(D_1))\big(c_\kappa(d(x_1,z_2)) - c_\kappa(D_1)\big) + \varphi'(c_\kappa(D_2))\big(c_\kappa(d(x_2,z_1)) - c_\kappa(D_2)\big)\end{aligned}$$

holds, where $D_1 = d(x_1,z_1)$, and $D_2 = d(x_2,z_2)$.

Proof. (i) Since f is proper convex, $\mathrm{dom}(f)$ is a nonempty convex set. Define $\overline{f}\colon \mathrm{dom}(f)^2 \to \mathbb{R}$ by $\overline{f}(z,y) = f(y) - f(z)$ for $z, y \in \mathrm{dom}(f)$. Put $\Phi d := \varphi(c_\kappa(d))$ for every $d \in [0, D_\kappa/2[$. Then we obtain

$$\begin{aligned}S_f x &= \operatorname*{argmin}_{y\in X} \big(f(y) + \varphi(c_\kappa(d(x,y)))\big)\\ &= \operatorname*{argmin}_{y\in \mathrm{dom}(f)} \big(f(y) + \Phi d(x,y)\big)\\ &= \Big\{z \in \mathrm{dom}(f) \;\Big|\; \inf_{y\in\mathrm{dom}(f)} \big(f(y) + \Phi d(x,y)\big) \ge f(z) + \Phi d(x,z)\Big\}\\ &= \Big\{z \in \mathrm{dom}(f) \;\Big|\; \inf_{y\in\mathrm{dom}(f)} \big(\overline{f}(z,y) + \Phi d(x,y) - \Phi d(x,z)\big) \ge 0\Big\}\end{aligned}$$

for every $x \in X$. Therefore, from Lemma 5.15 we obtain

$$0 \le f(w) - f(z) + \varphi'(c_\kappa(d(x,z))) \cdot \frac{D}{c'_\kappa(D)}\big(c_\kappa(d(x,w)) - c_\kappa(D) - c''_\kappa(D)c_\kappa(d(x,z))\big)$$

for any $x \in X$, $z \in S_f x$ and $w \in \mathrm{dom}(f) \setminus \{z\}$, where $D = d(z,w)$. This inequality is true obviously if $w \in X \setminus \mathrm{dom}(f)$.

(ii) By Theorem 5.16, we get the conclusion. □

From Lemma 4.3 and the inequality (ii) in Lemma 6.4, we obtain that S_f is firmly vicinal with $\varphi' \circ c_\kappa$ if S_f is well-defined as a single-valued mapping from X into $\operatorname{dom}(f)$. This completes the proof of Lemma 6.3.

Now we show the following crucial fact.

Lemma 6.5. *The single-valued mapping $S_f\colon X \to \operatorname{dom}(f)$ well defined in Lemma 6.3 satisfies $F(S_f) = \operatorname{argmin} f$.*

Proof. Let $z \in \operatorname{argmin} f$. Then since φ is nondecreasing, we have

$$f(z) + \varphi(c_\kappa(d(z,z))) = f(z) + \varphi(0) \le f(y) + \varphi(0) \le f(y) + \varphi(c_\kappa(d(z,y)))$$

for any $y \in X$. It implies that $z = S_f z$, that is, $z \in F(S_f)$.

Conversely, take $z \in F(S_f)$. Then we get

$$\begin{aligned} f(z) &\le f(w) + \varphi'(c_\kappa(d(z,z))) \cdot \frac{d(z,w)}{c'_\kappa(d(z,w))}\big(c_\kappa(d(z,w)) - c_\kappa(d(z,w)) - c''_\kappa(d(z,w))c_\kappa(d(z,z))\big) \\ &= f(w) \end{aligned}$$

for any $w \in X \setminus \{z\}$ from Lemma 6.4 (i). It means that $z \in \operatorname{argmin} f$. □

We consider a sufficient condition of φ such that such a resolvent S_f is well-defined as a single-valued mapping. In 2016, Kimura and Kohsaka gave a sufficient condition on f so that f has the unique minimizer.

Lemma 6.6 (Kimura and Kohsaka [13])**.** *For $\kappa > 0$, let X be an admissible complete* CAT(κ) *space and f a proper lower semicontinuous quasiconvex $(D_\kappa/2)$-coercive function from X into $]-\infty,\infty]$. Then f has at least one minimizer on X. Moreover, if f is also strictly midpoint convex, then f has the unique minimizer on X.*

Later, in 2019, Kajimura and Kimura showed the following result.

Lemma 6.7 (Kajimura and Kimura [7])**.** *Let X be a complete* CAT(0) *space and f a proper lower semicontinuous convex coercive function from X into $]-\infty,\infty]$. Suppose that f is strictly midpoint convex. Then f has the unique minimizer on X.*

From the three lemmas above, we get the following result.

Theorem 6.8. *Let X be an admissible complete* CAT(κ) *space and f a proper lower semicontinuous convex function from X into $]-\infty,\infty]$. Suppose that a function $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$ is strictly increasing, differentiable, and φ' is nondecreasing and continuous on $[0, c_\kappa(D_\kappa/2)[$. Define $\Phi\colon [0, D_\kappa/2[\to [0,\infty[$ by $\Phi = \varphi \circ c_\kappa$. Suppose the following:*

- *If $\kappa \le 0$, then*
$$\liminf_{d(u,z)\to\infty} \frac{f(z)}{d(u,z)} + \lim_{d\to\infty} \frac{\Phi(d)}{d} > 0$$
for some $u \in X$.
- *If $\kappa > 0$, then suppose that $\lim_{d\to D_\kappa/2} \Phi(d) = \infty$.*

Define a set-valued mapping $S_f\colon X \to 2^{\operatorname{dom}(f)}$ by

$$S_f x = \operatorname*{argmin}_{y\in X} \big(f(y) + \Phi(d(x,y))\big)$$

for $x \in X$. Then the following hold.

(i) $S_f\colon X \to \operatorname{dom}(f)$ *is well-defined as a single-valued mapping;*
(ii) S_f *is firmly vicinal with* $\varphi' \circ c_\kappa\colon [0, D_\kappa/2[\to]0, \infty[$*;*
(iii) $F(S_f) = \operatorname{argmin} f$.

Proof. This is obtained by similar proof of Theorem 5.18. Since φ' is nondecreasing, we obtain that φ is convex. Moreover, since c_κ is convex on $[0, D_\kappa/2[$, Φ is also convex. Therefore, from Lemmas 6.6 and 6.7, we get $S_f x \neq \emptyset$ for every $x \in X$. Since φ is strictly increasing and φ' is nondecreasing, we also have $\varphi'(t) > 0$ for all $t \in]0, c_\kappa(D_\kappa/2)[$.

Let $x \in X$ and $z_1, z_2 \in S_f x$. Then from Lemma 6.4 (ii), we obtain

$$\begin{aligned}
&c_\kappa(d(z_1, z_2))\\
&\quad\le \frac{\varphi'(c_\kappa(d(x,z_1)))\big(c_\kappa(d(x,z_2)) - c_\kappa(d(x,z_1))\big) + \varphi'(c_\kappa(d(x,z_2)))\big(c_\kappa(d(x,z_1)) - c_\kappa(d(x,z_2))\big)}{\varphi'(c_\kappa(d(x,z_1)))c''_\kappa(d(x,z_1)) + \varphi'(c_\kappa(d(x,z_2)))c''_\kappa(d(x,z_2))}\\
&\quad= \frac{\big(\varphi'(c_\kappa(d(x,z_1))) - \varphi'(c_\kappa(d(x,z_2)))\big)\big(c_\kappa(d(x,z_2)) - c_\kappa(d(x,z_1))\big)}{\varphi'(c_\kappa(d(x,z_1)))c''_\kappa(d(x,z_1)) + \varphi'(c_\kappa(d(x,z_2)))c''_\kappa(d(x,z_2))}.
\end{aligned}$$

Since φ' is nondecreasing, we get $c_\kappa(d(z_1, z_2)) \le 0$, and hence $z_1 = z_2$. This is the conclusion of (i). Note that (ii) and (iii) are obtained from Lemmas 6.3 and 6.5. □

Theorem 6.9. *Let X be an admissible complete* $\mathrm{CAT}(\kappa)$ *space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Suppose that a function $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0, \infty[$ is nondecreasing, differentiable, and φ' is continuous on $[0, c_\kappa(D_\kappa/2)[$. Define $\Phi\colon [0, D_\kappa/2[\to [0, \infty[$ by $\Phi = \varphi \circ c_\kappa$. Furthermore, suppose that $\Phi(d(x, \cdot))$ is strictly midpoint convex for any $x \in X$. Suppose the following:*

- *If $\kappa \le 0$, then*
$$\liminf_{d(u,z)\to\infty} \frac{f(z)}{d(u,z)} + \lim_{d\to\infty} \frac{\Phi(d)}{d} > 0$$
for some $u \in X$.
- *If $\kappa > 0$, then suppose that $\lim_{d\to D_\kappa/2} \Phi(d) = \infty$.*

Define a set-valued mapping $S_f\colon X \to 2^{\operatorname{dom}(f)}$ by

$$S_f x = \operatorname*{argmin}_{y\in X} \big(f(y) + \Phi(d(x,y))\big)$$

for $x \in X$. Then the following hold.

(i) $S_f\colon X \to \operatorname{dom}(f)$ *is well-defined as a single-valued mapping;*
(ii) S_f *is firmly vicinal with* $\varphi' \circ c_\kappa\colon [0, D_\kappa/2[\to]0, \infty[$*;*
(iii) $F(S_f) = \operatorname{argmin} f$.

Proof. Fix $x \in X$ and put $g(\cdot) = f(\cdot) + \Phi(d(x, \cdot))$. Then $g\colon X \to]-\infty, \infty]$ is proper and lower semicontinuous. Since $\Phi(d(x, \cdot))$ is continuous and midpoint convex, we obtain that $\Phi(d(x, \cdot))$ is convex on $\operatorname{dom}(f)$. Hence, for any $y_1, y_2 \in \operatorname{dom}(f)$ and $t \in]0, 1[$,

$$\begin{aligned}
g(ty_1 \oplus (1-t)y_2) &= f(ty_1 \oplus (1-t)y_2) + \Phi(d(x, ty_1 \oplus (1-t)y_2))\\
&\le tf(y_1) + (1-t)f(y_2) + t\Phi(d(x,y_1)) + (1-t)\Phi(d(x,y_2)) = tg(y_1) + (1-t)g(y_2).
\end{aligned}$$

Thus g is convex. We also have

$$g\Big(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\Big) = f\Big(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\Big) + \Phi\Big(d\Big(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\Big)\Big) < \frac{1}{2}g(y_1) + \frac{1}{2}g(y_2)$$

for any $y_1, y_2 \in \mathrm{dom}(f)$ with $y_1 \neq y_2$. Hence g is strictly midpoint convex.

We show g is $(D_\kappa/2)$-coercive. If $\kappa > 0$, then we have $\lim_{d\to D_\kappa/2} \Phi(d) = \infty$ by the assumption. It follows that $\lim_{d(x,z)\nearrow D_\kappa/2} g(z) = \infty$, which implies the coercivity of g. We consider the case where $\kappa \le 0$. Take a point $u \in X$ satisfying the assumption. Then we obtain

$$\liminf_{d(x,z)\to\infty} \frac{g(z)}{d(x,z)} \ge \liminf_{d(u,z)\to\infty} \left(\frac{f(z)}{d(u,z)} \cdot \frac{d(u,z)}{d(x,z)} \right) + \lim_{d\to\infty} \frac{\Phi(d)}{d} > 0$$

and hence g is coercive.

Consequently, by Lemmas 6.6 and 6.7, $S_f x$ is a singleton for any $x \in X$. Moreover, from Lemmas 6.3 and 6.5, we get (ii) and (iii). □

In the previous theorem, we consider the case where $\kappa \le 0$. Suppose that f is bounded below and $\liminf_{d\to\infty}(\Phi(d)/d) > 0$. Then the condition $\liminf_{d(x,z)\to\infty}(f(z)/d(x,z)) + \liminf_{d\to\infty}(\Phi(d)/d) > 0$ is true. Indeed, we get $\liminf_{d(x,z)\to\infty}(f(z)/d(x,z)) = 0$ if f is bounded below.

Note that we obtain from Lemmas 4.2 and 4.9 that, if $T\colon X \to X$ is firmly vicinal with ψ and $F(T) \neq \emptyset$, then T is tightly quasinonexpansive and Δ-demiclosed. From this fact, we get the following results from Theorems 4.21, 4.22, 4.23, and 4.24.

Corollary 6.10. *Let X, f, and φ be the same as Lemma 6.3, and $S_f\colon X \to \mathrm{dom}(f)$ resolvent well defined by an equation*

$$S_f x = \operatorname*{argmin}_{y\in X} \big(f(y) + \varphi(c_\kappa(d(x,y)))\big). \tag{$\star\star$}$$

Let $\{\alpha_n\} \subset [0,1[$ such that $\sum_{n=1}^\infty (1-\alpha_n) = \infty$. Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by either

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) S_f x_n$$

for $n \in \mathbb{N}$ or

$$x_{n+1} = \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) S_f x_n$$

for $n \in \mathbb{N}$. Let us denote (a) *and* (b) *by the following conditions:*

(a) *$\{S_f x_n\}$ is κ-bounded;*
(b) $\sup_{n\in\mathbb{N}} d(x_n, S_f x_n) < D_\kappa/2$.

Suppose that S_f is vicinal with $\psi\colon [0, D_\kappa/2[\to]0,\infty[$, and define conditions (P1) *and* (P2) *as follows:*

(P1) *ψ is nondecreasing;*
(P2) $\sup_{n\in\mathbb{N}} \psi(d(x_n, S_f x_n)) < \infty$.

Then the following hold:

(i) *Suppose that ψ satisfies* (P1). *Then* $\operatorname{argmin} f \neq \emptyset$ *if and only if* (a) *and* (b) *hold.*
(i)′ *Suppose that ψ satisfies* (P2). *Then* $\operatorname{argmin} f \neq \emptyset$ *if and only if* (a) *holds.*

Corollary 6.11. *Let X, f, and φ be the same as Lemma 6.3, and suppose that* $\operatorname{argmin} f \neq \emptyset$. *Let $S_f\colon X \to \mathrm{dom}(f)$ a resolvent well defined by an equation* ($\star\star$). *Let $\{\alpha_n\} \subset [0,1[$ such that* $\limsup_{n\to\infty} \alpha_n < 1$ *and $\sum_{n=1}^\infty (1-\alpha_n) = \infty$. Take $x_1 \in X$ and generate $\{x_n\} \subset X$ by either*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) S_f x_n$$

for $n \in \mathbb{N}$ or

$$x_{n+1} = \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) S_f x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to some minimizer of f.

We also get the following result from Theorem 4.29.

Corollary 6.12. *Let X, f, and φ be the same as Lemma 6.3, and suppose that* $\operatorname{argmin} f \neq \emptyset$. *Let $S_f\colon X \to \operatorname{dom}(f)$ a resolvent well defined by an equation* ($\star\star$). *Let $\{\alpha_n\} \subset \,]0,1[$ such that* $\lim_{n\to\infty} \alpha_n = 0$ *and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $u, x_1 \in X$ arbitrarily and define $\{x_n\} \subset X$ by*

$$x_{n+1} = \alpha_n u \overset{\kappa}{\oplus} (1-\alpha_n) S_f x_n$$

for any $n \in \mathbb{N}$. In the case where $\kappa > 0$, suppose that (i) *or* (ii) *holds:*

(i) $\sup_{n\in\mathbb{N}} d(u, S_f x_n) < D_\kappa/2$,

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then $\{x_n\}$ converges to some minimizer of f.

6.2 (-1)-convex functions

Using (-1)-convex combination, we can define another type of convex functions named '(-1)-convex function.' In this section, we investigate its natures and perform some numerical experiments.

6.2.1 Natures of (-1)-convex functions on geodesic spaces

Let X be a uniquely geodesic space and f a function from X into $]-\infty,\infty]$. f is said to be (-1)*-convex* [27] if

$$f(\alpha x \overset{-1}{\oplus} (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y)$$

for any $x, y \in X$ and $\alpha \in [0,1]$.

Then we easily get the following:

- If $f, g\colon X \to \,]-\infty,\infty]$ is (-1)-convex, then so is $f+g$.
- If $f\colon X \to \,]-\infty,\infty]$ is (-1)-convex, then so is kf for any $k \ge 0$.
- If $f\colon \mathbb{R} \to \,]-\infty,\infty]$ is (-1)-convex, then so is $g\colon \mathbb{R} \ni t \longmapsto f(t+c)$ for any $c \in \mathbb{R}$.
- If $f\colon \mathbb{R} \to \,]-\infty,\infty]$ is (-1)-convex, then so is $g\colon \mathbb{R} \ni t \longmapsto f(-t)$.

We can get several examples as follows.

- Let X be a CAT(-1) space and $z \in X$. Then a function $f\colon X \to \,]-\infty,\infty]$ defined by $f(\cdot) = \cosh d(\cdot, z)$ is (-1)-convex.
- A function $f\colon \mathbb{R} \ni t \longmapsto \cosh t$ is (-1)-convex.
- A function $f\colon \mathbb{R} \ni t \longmapsto \exp t$ is (-1)-convex.
- For $b \in \mathbb{R}$, a function $f\colon \mathbb{R} \ni t \longmapsto b$ is (-1)-convex.
- For $a, b \in \mathbb{R}$ such that $a \neq 0$, a function $f\colon \mathbb{R} \ni t \longmapsto at+b$ is not (-1)-convex.
- A function $f\colon \mathbb{R} \ni t \longmapsto t^2$ is not (-1)-convex.

Theorem 6.13. *Let X be a uniquely geodesic space. Then a function $f\colon X \to \,]-\infty,\infty]$ is (-1)-convex if and only if for any $x, y \in X$ with $x \neq y$ and $t \in \,]0,1[$,*

$$\begin{aligned} f(tx \oplus (1-t)y) \le{}& \frac{\sinh(td(x,y))}{\sinh(td(x,y)) + \sinh((1-t)d(x,y))} f(x) \\ &+ \frac{\sinh((1-t)d(x,y))}{\sinh(td(x,y)) + \sinh((1-t)d(x,y))} f(y). \end{aligned}$$

Proof. $f(\alpha x \overset{-1}{\oplus} (1-\alpha)y) = \alpha f(x) + (1-\alpha)f(y)$ always holds if $x = y$ or $\alpha \in \{0,1\}$. Therefore, f is (-1)-convex if and only if $f(\alpha x \overset{-1}{\oplus} (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for any $x, y \in X$ with $x \neq y$ and $\alpha \in \,]0,1[$. It is equivalent from Lemma 3.6 to $f(tx \oplus (1-t)y) \leq \zeta^{\kappa}_{d(x,y)}(t)f(x) + \zeta^{\kappa}_{d(x,y)}(1-t)f(y)$ for any $x, y \in X$ with $x \neq y$ and $t \in \,]0,1[$. This implies the conclusion. □

Corollary 6.14. *Let X be a uniquely geodesic space. Then a function $f\colon X \to \,]-\infty,\infty]$ is (-1)-convex if and only if for any $x, y \in X$ with $x \neq y$ and $z \in \,]x,y[$,*

$$f(z) \leq \frac{(\sinh d(y,z))f(x) + (\sinh d(x,z))f(y)}{\sinh d(y,z) + \sinh d(x,z)}.$$

Corollary 6.15. *Let X be a uniquely geodesic space. Then a function $f\colon X \to \,]-\infty,\infty]$ is (-1)-convex if and only if for any $x, y \in X$ with $x \neq y$ and $z \in \,]x,y[$,*

$$(\sinh d(y,z))(f(x) - f(z)) + (\sinh d(x,z))(f(y) - f(z)) \geq 0.$$

Corollary 6.16. *Let X be a uniquely geodesic space. Then a function $f\colon X \to \mathbb{R}$ is (-1)-convex if and only if for any $x, y \in X$ with $x \neq y$ and $z \in \,]x,y[$,*

$$f(z) \leq \frac{f(x)+f(y)}{2} + \frac{\tanh \dfrac{d(y,z) - d(x,z)}{2}}{\tanh \dfrac{d(y,z) + d(x,z)}{2}} \cdot \frac{f(x) - f(y)}{2}.$$

A (-1)-convex function has the following relationship with convex functions.

Lemma 6.17. *Let X be a uniquely geodesic space. Then every continuous (-1)-convex function $f\colon X \to \,]-\infty,\infty]$ is convex.*

Proof. Assume that f is continuous and (-1)-convex. Then we obtain from Corollary 3.7 that

$$f\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) = f\left(\frac{1}{2}x \overset{-1}{\oplus} \frac{1}{2}y\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

for any $x, y \in X$, and hence f is midpoint convex. Therefore, since f is continuous, we get the desired result. □

Theorem 6.18. *Let X be a uniquely geodesic space and $f\colon X \to \,]-\infty,\infty]$ a proper (-1)-convex function. Let $u, v \in \mathrm{dom}(f)$. Then $f|_{]u,v[}$ is continuous.*

Proof. Take an arbitrary point $x \in \,]u,v[$. Let $\{x_n\}$ be a sequence on $]u,v[$ which converges to $x \in \,]u,v[$. Put $A = \{n \in \mathbb{N} \mid x_n \in \,]u,x[\}$ and $B = \{n \in \mathbb{N} \mid x_n \in [x,v[\}$. Then there exists $\{\alpha_n\}_{n\in A}$ and $\{\beta_n\}_{n\in B}$ such that

$$x_n = \alpha_n u \overset{-1}{\oplus} (1-\alpha_n)x \quad \text{and} \quad x_n = \beta_n v \overset{-1}{\oplus} (1-\beta_n)x$$

for any $n \in A$ and $n \in B$, respectively. Since $x_n \to x$ as $n \to \infty$, we obtain from Lemma 3.22 that $\{\alpha_n\}$ and $\{\beta_n\}$ both converge to 0. By the (-1)-convexity of f, we get

$$f(x_n) \leq \alpha_n f(u) + (1-\alpha_n)f(x) \quad \text{and} \quad f(x_n) \leq \beta_n f(v) + (1-\beta_n)f(x)$$

for any $n \in A$ and $n \in B$, respectively. It follows that $\limsup_{n\to\infty} f(x_n) \leq f(x)$.

Take $\{\gamma_n\}_{n\in A}$ and $\{\delta_n\}_{n\in B}$ satisfying

$$x = \gamma_n v \overset{-1}{\oplus} (1-\gamma_n)x_n \quad \text{and} \quad x = \delta_n u \overset{-1}{\oplus} (1-\delta_n)x_n$$

for any $n \in A$, and $n \in B$, respectively. Then $\{\gamma_n\}$ and $\{\delta_n\}$ both converge to 0 from Lemma 3.23. We also have

$$f(x) \le \gamma_n f(u) + (1-\gamma_n) f(x_n) \quad \text{and} \quad f(x) \le \delta_n f(v) + (1-\delta_n) f(x_n)$$

for any $n \in A$, and $n \in B$, respectively. Therefore we obtain $f(x) \le \liminf_{n\to\infty} f(x_n)$.

Consequently, we get $f(x) = \lim_{n\to\infty} f(x_n)$, which is the desired result. □

Using Lemma 6.17 and Theorem 6.18, we obtain the following fact.

Theorem 6.19. *Let X be a uniquely geodesic space and $f\colon X \to]-\infty,\infty]$ a proper (-1)-convex function. Then $f|_{\operatorname{dom}(f)}$ is convex.*

Proof. Let $u, v \in \operatorname{dom}(f)$. It is obvious if $u = v$, therefore we assume that $u \neq v$. We immediately obtain that $f|_{]u,v[}$ is convex from Lemma 6.17 and Theorem 6.18.

Take $x \in [u,v[\subset \operatorname{dom}(f)$ arbitrarily. Since f is (-1)-convex, we have $f(tx \overset{-1}{\oplus} (1-t)v) \le tf(x) + (1-t)f(v)$ for any $t \in]0,1[$. Note that there exists a limit $L = \lim_{\alpha\to 0} f(\alpha u \oplus (1-\alpha)v)$. It follows that

$$f(v) \ge \limsup_{t\to 0} \frac{f(tx \overset{-1}{\oplus} (1-t)v) - tf(x)}{1-t} = \lim_{t\to 0} f(tx \overset{-1}{\oplus} (1-t)v) = L.$$

Since $v \in \operatorname{dom}(f)$, we have $L < \infty$. Then we get

$$\begin{aligned} tf(x) + (1-t)f(v) - f(tx \oplus (1-t)v) &= tf(x) + (1-t)f(v) - \lim_{s\to 0} f(tx \oplus (1-t)(sx \oplus (1-s)v)) \\ &\ge tf(x) + (1-t)f(v) - tf(x) - (1-t)\lim_{s\to 0} f(sx \oplus (1-s)v) \\ &= (1-t)(f(v) - L) \\ &\ge 0 \end{aligned}$$

for any $t \in]0,1[$. Thus f is convex on $]u,v]$.

Similarly, we also obtain $tf(u) + (1-t)f(y) - f(tu \oplus (1-t)y) \ge 0$ for any $y \in]u,v]$ and $t \in]0,1[$, and hence f is convex on $[u,v[$. Consequently, we get the conclusion. □

Lemma 6.20. *Let X be a uniquely geodesic space and $f\colon X \to]-\infty,\infty]$ a proper (-1)-convex function. Then for any $u, v \in \operatorname{dom}(f)$ such that $f(u) \neq f(v)$,*

$$f\Big(\frac{1}{2}u \oplus \frac{1}{2}v\Big) < \frac{1}{2}f(u) + \frac{1}{2}f(v).$$

Proof. Take $u, v \in \operatorname{dom}(f)$ such that $f(u) < f(v)$. Put $D = d(u,v)/4 > 0$. Then we have

$$f\Big(\frac{3}{4}u \oplus \frac{1}{4}v\Big) \le \frac{(\sinh(3D))f(u) + (\sinh D)f(v)}{\sinh(3D) + \sinh D}$$

by Theorem 6.13. Hence we obtain

$$\begin{aligned}
f\Big(\frac{1}{2}u \oplus \frac{1}{2}v\Big) &= f\Big(\frac{2}{3}\Big(\frac{3}{4}u \oplus \frac{1}{4}v\Big)u \oplus \frac{1}{3}v\Big) \\
&\le \frac{\sinh(2D)}{\sinh(2D) + \sinh D} f\Big(\frac{3}{4}u \oplus \frac{1}{4}v\Big) + \frac{\sinh D}{\sinh(2D) + \sinh D} f(v) \\
&\le \frac{\sinh(2D)\sinh(3D)}{(\sinh(2D) + \sinh D)(\sinh(3D) + \sinh D)} f(u) \\
&\quad + \frac{(\sinh(2D) + \sinh(3D) + \sinh D)\sinh D}{(\sinh(2D) + \sinh D)(\sinh(3D) + \sinh D)} f(v) \\
&= \frac{C(3+4S^2)}{2(2C+1)(1+S^2)} f(u) + \frac{C+2+2S^2}{2(2C+1)(1+S^2)} f(v),
\end{aligned}$$

where $S = \sinh D$ and $C = \cosh D$. Therefore, we have

$$\begin{aligned}
&\frac{1}{2}f(u) + \frac{1}{2}f(v) - \left(\frac{C(3+4S^2)}{2(2C+1)(1+S^2)} f(u) + \frac{C+2+2S^2}{2(2C+1)(1+S^2)} f(v)\right) \\
&= \frac{(2S^2+1)C^2 - S^2 - 1}{(2C+1)(1+S^2)}(f(v) - f(u)) \\
&\ge \frac{(2S^2+1)\cdot 1 - S^2 - 1}{(2C+1)(1+S^2)}(f(v) - f(u)) \\
&= \frac{S^2}{(2C+1)(1+S^2)}(f(v) - f(u)) > 0.
\end{aligned}$$

It follows that

$$f\Big(\frac{1}{2}u \oplus \frac{1}{2}v\Big) < \frac{1}{2}f(u) + \frac{1}{2}f(v),$$

which is the desired result. □

Corollary 6.21. *Let X be a uniquely geodesic space and $f\colon X \to]-\infty,\infty]$ a proper (-1)-convex function. Suppose that*

$$f\Big(\frac{1}{2}u \oplus \frac{1}{2}v\Big) = \frac{1}{2}f(u) + \frac{1}{2}f(v).$$

for any $u, v \in \operatorname{dom}(f)$. Then $f|_{\operatorname{dom}(f)}$ is a constant function.

Theorem 6.22. *Let X be a uniquely geodesic space and $f\colon X \to]-\infty,\infty]$ a proper (-1)-convex function. Suppose that there exists $u, v \in \operatorname{dom}(f)$ such that $f(u) \neq f(v)$. Then*

$$f(tu \oplus (1-t)v) < tf(u) + (1-t)f(y)$$

for any $t \in]0,1[$.

Proof. From Theorem 6.19, $f|_{\operatorname{dom}(f)}$ is convex. Let $t \in]0,1[$. If $t = 1/2$, then we get the conclusion by Lemma 6.20.

Suppose that $t < 1/2$. Then putting $m = (1/2)u \oplus (1/2)v$ and $s = 2t$, we have

$$\begin{aligned}
f(tu \oplus (1-t)v) = f(sm \oplus (1-s)v) &\le sf(m) + (1-s)f(v) \\
&< \frac{1}{2}sf(u) + \Big(1 - \frac{1}{2}s\Big)f(v) \\
&= tf(u) + (1-t)f(v).
\end{aligned}$$

Next, suppose that $t > 1/2$. Put $r = 2t - 1$. Then we obtain

$$\begin{aligned} f(tu \oplus (1-t)v) = f(ru \oplus (1-r)m) &\leq rf(u) + (1-r)f(m) \\ &< \left(\frac{1}{2} + \frac{1}{2}r\right)f(u) + \frac{1}{2}(1-r)f(v) \\ &= tf(u) + (1-t)f(v). \end{aligned}$$

Thus we get the conclusion. □

Corollary 6.23. *Let X be a uniquely geodesic space and $f\colon X \to \,]-\infty, \infty]$ a proper (-1)-convex function. Let $u, v \in \operatorname{dom}(f)$ such that $u \neq v$. Assume that $f(x) \neq f(y)$ for any $x, y \in [u, v]$. Then f is strictly midpoint convex on $[u, v]$.*

Proof. Take two points $x, y \in [u, v]$ with $x \neq y$ and $d(u, x) < d(u, y)$, and let $t \in \,]0, 1[$. Then we get $f(x) \neq f(y)$ by the assumption of f. Therefore, we get $f(tx \oplus (1-t)y) < tf(x) + (1-t)f(y)$ from Theorem 6.22, which is the desired result. □

In Corollary 6.23, if there exist $x, y \in [u, v]$ satisfying $f(x) = f(y)$, then f is not always strictly midpoint convex on $[u, v]$. For instance, a function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0 & (\text{if } t \leq 0); \\ \cosh t - 1 & (\text{if } t \geq 0) \end{cases}$$

is (-1)-convex, and not strictly midpoint convex on $[-1, 1]$.

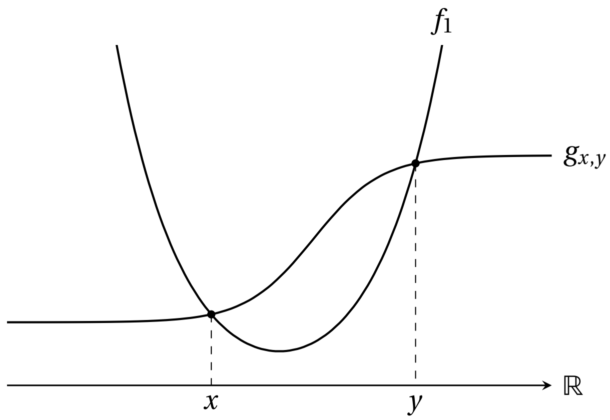
6.2.2 Natures of (-1)-convex functions on the real line

We consider (-1)-convex functions on $\mathbb{R}$. Henceforth, we use a function $g_{x,y}\colon \mathbb{R} \to \mathbb{R}$ defined by

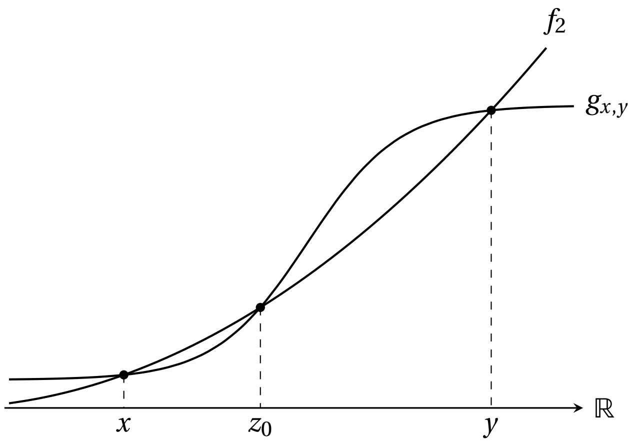
$$\begin{aligned} g_{x,y}(z) &= \frac{\sinh(y-z)}{\sinh(y-z) + \sinh(z-x)} f(x) + \frac{\sinh(z-x)}{\sinh(y-z) + \sinh(z-x)} f(y) \\ &= \frac{f(x) + f(y)}{2} - \frac{\tanh\left(z - \frac{y+x}{2}\right)}{\tanh \frac{y-x}{2}} \cdot \frac{f(x) - f(y)}{2} \end{aligned}$$

for $x, y, z \in \mathbb{R}$ such that $x < y$. By Theorem 6.13 and Corollary 6.16, a function $f\colon \mathbb{R} \to \mathbb{R}$ is (-1)-convex if and only if for any $x, y, z \in \mathbb{R}$ such that $x < z < y$, an inequality $f(z) \leq g_{x,y}(z)$ holds. In other words, we can explain that the (-1)-convexity of f means that $\operatorname{epi} f \supset \operatorname{epi} g_{x,y}$ for any $x < y$.

The following figure represents a graph of $f_1\colon \mathbb{R} \ni t \mapsto \cosh t \in \mathbb{R}$ and $g_{x,y}$ defined by the above formula, where $x = -1$ and $y = 2$. This function f_1 is (-1)-convex, and hence $g_{x,y}$ is always above the graph of f_1 on any bounded interval $]x, y[$.



Define a function $f_2 \colon \mathbb{R} \ni t \longmapsto t^2$. Then f_2 is not (-1)-convex, since $f_2(z) > g_{x,y}(z)$ holds for $x = 2$, $y = 6$, and $2 < z < z_0$, where $z_0 \approx 3.48659558$.



Similarly, we can obtain that a function $f \colon \mathbb{R} \ni t \longmapsto at$ is not (-1)-convex for any $a \neq 0$.

Theorem 6.24. *Every* (-1)*-convex function* $f \colon \mathbb{R} \to \mathbb{R}$ *is convex.*

Proof. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a (-1)-convex function. Then, since $\mathrm{dom}(f) = \mathbb{R}$, we get the conclusion from Theorem 6.19. ◻

Let $I \subset \mathbb{R}$ be a closed interval and f a function from I into $\mathbb{R}$. Then, f is (-1)-convex if and only if

$$f(t) \leq \inf_{\substack{s \leq t \leq u \\ s<u \\ s,u \in I}} \left(\frac{\sinh(u-t)}{\sinh(u-t)+\sinh(t-s)} f(s) + \frac{\sinh(t-s)}{\sinh(u-t)+\sinh(t-s)} f(u) \right) \tag{$*$}$$

for any $t \in I$ by Theorem 6.13. It means that, for a function f to be (-1)-convex, the value $f(t)$ must satisfy the inequality $(*)$.

Let I be a bounded closed interval on $\mathbb{R}$, and f a function from I into $\mathbb{R}$. Define a function $g \colon I \to \mathbb{R}$ by

$$g(t) = \min_{\substack{s \leq t \leq u \\ s<u \\ s,u \in I}} \left(\frac{\sinh(u-t)}{\sinh(u-t)+\sinh(t-s)} f(s) + \frac{\sinh(t-s)}{\sinh(u-t)+\sinh(t-s)} f(u) \right)$$

for $t \in I$. Then f is (-1)-convex if and only if $f = g$.

Lemma 6.25. *Define a function* $h\colon\,]-\infty,0] \to \mathbb{R}$ *by*

$$h(t) = \tanh\frac{t}{2}$$

for $t \in\,]-\infty,0]$*. Then* h *is* (-1)*-convex.*

Proof. Let $s,t,u \in\,]-\infty,0]$ such that $s<t<u$. Then we have

$$\begin{aligned}
&(\sinh(u-t))(h(s)-h(t)) + (\sinh(t-s))(h(u)-h(t))\\
&= \sinh(u-t)\Big(\tanh\frac{s}{2} - \tanh\frac{t}{2}\Big) + \sinh(t-s)\Big(\tanh\frac{u}{2} - \tanh\frac{t}{2}\Big)\\
&= 2\sinh\frac{u-t}{2}\cosh\frac{u-t}{2}\cdot\frac{-\sinh\frac{t-s}{2}}{\cosh\frac{s}{2}\cosh\frac{t}{2}} + 2\sinh\frac{t-s}{2}\cosh\frac{t-s}{2}\cdot\frac{\sinh\frac{u-t}{2}}{\cosh\frac{u}{2}\cosh\frac{t}{2}}\\
&= \frac{2\sinh\frac{u-t}{2}\sinh\frac{t-s}{2}}{\cosh\frac{s}{2}\cosh\frac{t}{2}\cosh\frac{u}{2}}\Big(-\cosh\frac{u-t}{2}\cosh\frac{u}{2} + \cosh\frac{t-s}{2}\cosh\frac{s}{2}\Big)\\
&= \frac{2\sinh\frac{u-t}{2}\sinh\frac{t-s}{2}\sinh\frac{u-s}{2}\sinh\frac{t-s-u}{2}}{\cosh\frac{s}{2}\cosh\frac{t}{2}\cosh\frac{u}{2}}.
\end{aligned}$$

Since $t-s-u \le -s \le 0$, we obtain

$$(\sinh(u-t))(h(s)-h(t)) + (\sinh(t-s))(h(u)-h(t)) \ge 0.$$

It means that h is (-1)-convex from Corollary 6.15. □

Now we prove the following crucial result.

Theorem 6.26. *Let* $f\colon \mathbb{R}\to\mathbb{R}$ *be a* (-1)*-convex function. Then for any* $v\in\mathbb{R}$,

$$\liminf_{|u-v|\to\infty} \frac{f(u)}{|u-v|} \ge 0.$$

Proof. By Theorem 6.24, we have f is continuous and convex, Thus, from Lemma 2.1, there exists $L \in\,]-\infty,0]$ such that for any $v\in\mathbb{R}$,

$$\liminf_{|u-v|\to\infty} \frac{f(u)}{|u-v|} \ge L.$$

Suppose $L<0$ and assume that there exists $v\in\mathbb{R}$ such that the inequality above holds as an equation. Then

$$\lim_{u\to\infty}\frac{f(u)}{u-v} = L \quad\text{or}\quad \lim_{u\to-\infty}\frac{f(u)}{-(u-v)} = L$$

holds. Without loss of generality, we may assume the first equation holds, which implies

$$\lim_{u\to\infty}\frac{f(u)}{u} = \lim_{u\to\infty}\Big(\frac{f(u)}{u-v}\cdot\frac{u-v}{u}\Big) = L.$$

Take a real number ε such that $0 < \varepsilon < -L/7$. Then there exists $u_0 > 0$ such that for any $u \geq u_0$, an inequality $(L-\varepsilon)u < f(u) < (L+\varepsilon)u$ holds. Thus, for any $\lambda > 0$, we get

$$\begin{aligned}(L-\varepsilon)(u_0+3\lambda) < f(u_0+3\lambda) &\leq \frac{(\sinh\lambda)f(u_0)+(\sinh 3\lambda)f(u_0+4\lambda)}{\sinh\lambda+\sinh 3\lambda}\\ &\leq (L+\varepsilon)\cdot\frac{(\sinh\lambda)u_0+(\sinh 3\lambda)(u_0+4\lambda)}{\sinh\lambda+\sinh 3\lambda}\\ &\leq (L+\varepsilon)u_0+(4L+4\varepsilon)\cdot\frac{(\sinh 3\lambda)\lambda}{\sinh\lambda+\sinh 3\lambda}.\end{aligned}$$

It deduces that

$$\begin{aligned}0 &< \frac{1}{\lambda}\Big((-L+\varepsilon)(u_0+3\lambda)+(L+\varepsilon)u_0+(4L+4\varepsilon)\cdot\frac{(\sinh 3\lambda)\lambda}{\sinh\lambda+\sinh 3\lambda}\Big)\\ &< \frac{2\varepsilon}{\lambda}u_0-3L+3\varepsilon+(4L+4\varepsilon)\cdot\frac{\sinh 3\lambda}{\sinh\lambda+\sinh 3\lambda}\\ &\to L+7\varepsilon<0\end{aligned}$$

as $\lambda \to \infty$, which is a contradiction. Hence we get the conclusion. □

6.2.3 Numerical experiments for (−1)-convex functions on the real line

In what follows, we consider numerical experiments for (−1)-convex functions on $\mathbb{R}$. First, we generate a "maximum" (−1)-convex function on $\mathbb{R}$ joining two points. Let x_1, x_2, y_1, y_2 be real numbers such that $x_1 < x_2$ and $y_1 < y_2$. Let $f_1 \colon [x_1, x_2] \to [y_1, y_2]$ be an affine function such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Namely,

$$f_1(t) = y_1 \cdot \frac{t-x_2}{x_1-x_2} + y_2 \cdot \frac{t-x_1}{x_2-x_1}$$

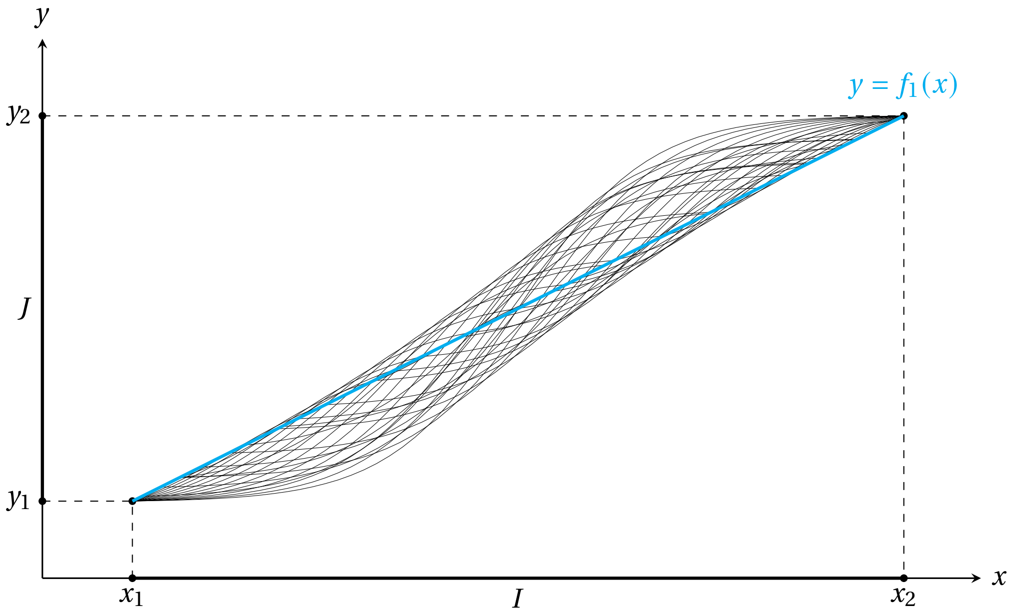
for $t \in [x_1, x_2]$. Then f_1 is not (−1)-convex.

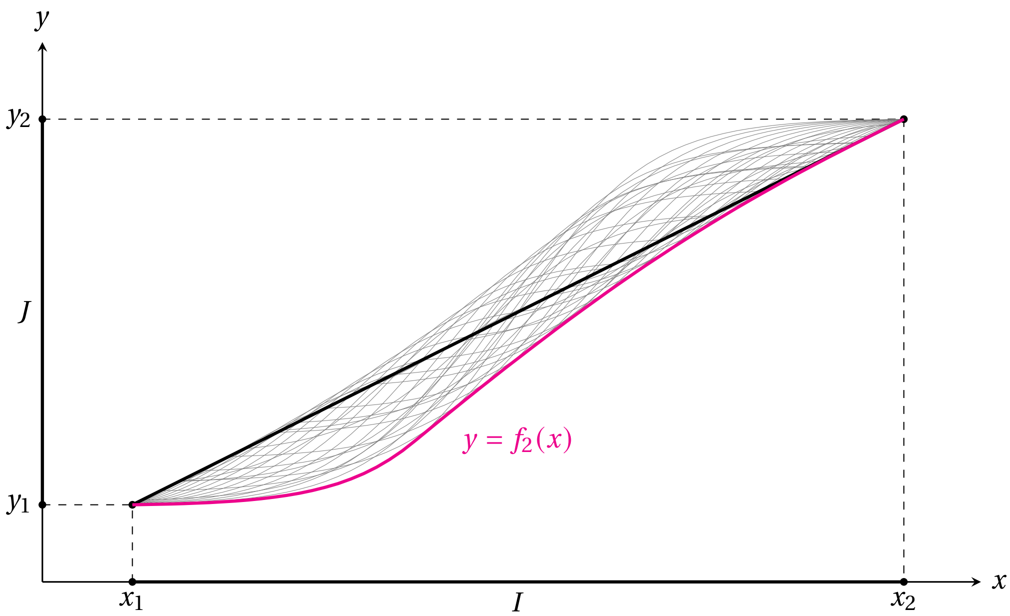
Starting with f_1, we attempt to create a sequence of mappings $\{f_n\}$ whose limit $\lim_{n\to\infty} f_n$ being (−1)-convex. Put $I = [x_1, x_2]$ and $J = [y_1, y_2]$. Define a function $f_2 \colon I \to J$ by

$$f_2(t) = \min_{\substack{s\leq t\leq u\\ s<u\\ s,u\in I}}\left(\frac{\sinh(u-t)}{\sinh(u-t)+\sinh(t-s)}f_1(s)+\frac{\sinh(t-s)}{\sinh(u-t)+\sinh(t-s)}f_1(u)\right)$$

for $t \in I$. Then f_2 is not (−1)-convex.

The following graph describes the construction of f_2 from f_1. epi f_2 is given by the lower envelope created by the family of epigraphs $\{\text{epi}\, g_{s,u} \mid s, u \in I,\ s < u\}$.





In the same fashion, for each $n \in \mathbb{N}$, define a function $f_{n+1}: I \to J$ by

$$f_{n+1}(t) = \min_{\substack{s \le t \le u \\ s<u \\ s,u \in I}} \left(\frac{\sinh(u-t)}{\sinh(u-t)+\sinh(t-s)} f_n(s) + \frac{\sinh(t-s)}{\sinh(u-t)+\sinh(t-s)} f_n(u) \right)$$

inductively. Then the following hold.

Lemma 6.27. *$f_n(x_1) = y_1$ and $f_n(x_2) = y_2$ for any $n \in \mathbb{N}$.*

Proof. It is obvious if $n = 1$. Take $n \in \mathbb{N}$. Then we have

$$\begin{aligned} f_{n+1}(x_1) &= \min_{\substack{s \le x_1 \le u \\ s<u \\ s,u \in I}} \left(\frac{\sinh(u - x_1)}{\sinh(u - x_1) + \sinh(x_1 - s)} f_n(s) + \frac{\sinh(x_1 - s)}{\sinh(u - x_1) + \sinh(x_1 - s)} f_n(u) \right) \\ &= \min_{\substack{x_1 < u \\ u \in I}} \left(\frac{\sinh(u - x_1)}{\sinh(u - x_1) + 0} f_n(x_1) + 0 \right) = f_n(x_1). \end{aligned}$$

It implies that $f_n(x_1) = y_1$ for any $n \in \mathbb{N}$. Similarly, we get $f_{n+1}(x_2) = f_n(x_2) = y_2$ for all $n \in \mathbb{N}$. □

Lemma 6.28. $f_{n+1}(t) \le f_n(t)$ *for any* $n \in \mathbb{N}$ *and* $t \in I$.

Proof. We get

$$\begin{aligned} f_{n+1}(t) &= \min_{\substack{s \le t \le u \\ s<u \\ s,u \in I}} \left(\frac{\sinh(u - t)}{\sinh(u - t) + \sinh(t - s)} f_n(s) + \frac{\sinh(t - s)}{\sinh(u - t) + \sinh(t - s)} f_n(u) \right) \\ &\le \min_{\substack{s \le t \le u \\ s<u \\ s,u \in I}} \max\{f_n(s), f_n(u)\} \le f_n(t) \end{aligned}$$

for all $n \in \mathbb{N}$ and $t \in I$. □

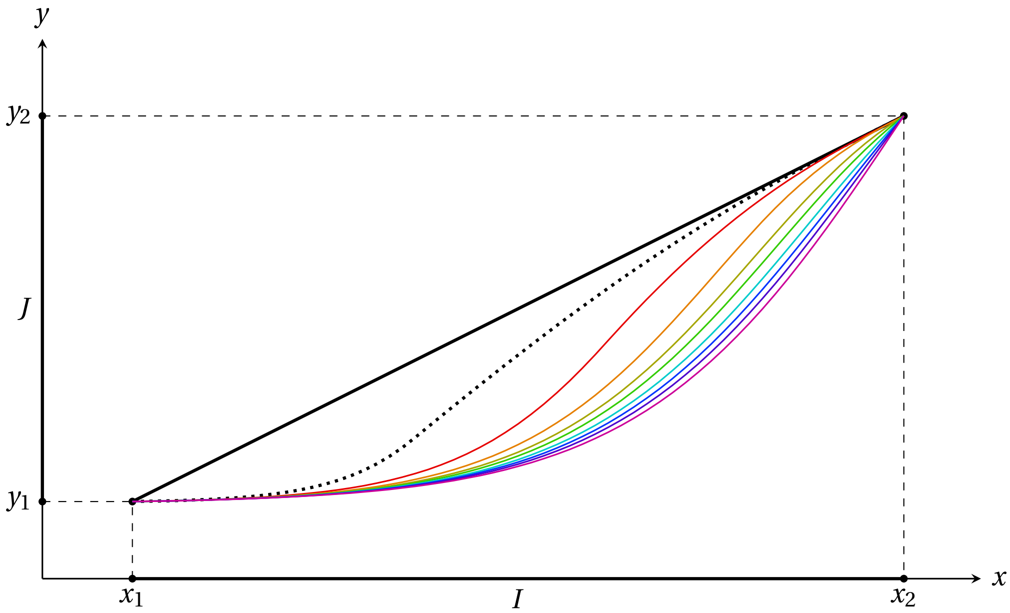
Lemma 6.29. $y_1 \le f_n(t)$ *for any* $n \in \mathbb{N}$ *and* $t \in I$.

Proof. It is obvious when $n = 1$. Suppose that some $n \in \mathbb{N}$ satisfies for all $t \in I$, $y_1 \le f_n(t)$. Then we get

$$f_{n+1}(t) \ge \min_{\substack{s \le t \le u \\ s<u \\ s,u \in I}} \left(\frac{\sinh(u - t)}{\sinh(u - t) + \sinh(t - s)} y_1 + \frac{\sinh(t - s)}{\sinh(u - t) + \sinh(t - s)} y_1 \right) = y_1$$

for any $t \in I$. Hence we obtain the conclusion. □

By above lemmas, we obtain that there exists a limit $\lim_{n\to\infty} f_n(t)$ for each $t \in I$. Define a function $f_\infty \colon I \to J$ by $f_\infty(t) = \lim_{n\to\infty} f_n(t)$ for $t \in I$.



Black solid line: $y = f_1(x)$, dotted line: $y = f_2(x)$,
$y = f_3(x)$, $y = f_4(x)$, $y = f_5(x)$, $y = f_6(x)$, $y = f_8(x)$, $y = f_{11}(x)$, $y = f_{18}(x)$, $y = f_\infty(x)$.

Then we expect the following holds.

Conjecture 6.30. *For any $t \in I$,*

$$f_\infty(t) = \frac{\sinh\left(\frac{2x_2 - x_1 - t}{2}\right)}{\sinh\left(\frac{2x_2 - x_1 - t}{2}\right) + \sinh\left(\frac{t - x_1}{2}\right)} y_1 + \frac{\sinh\left(\frac{t - x_1}{2}\right)}{\sinh\left(\frac{2x_2 - x_1 - t}{2}\right) + \sinh\left(\frac{t - x_1}{2}\right)} (2y_2 - y_1)$$
$$= y_2 + (y_2 - y_1) \cdot \frac{\tanh \frac{t - x_2}{2}}{\tanh \frac{x_2 - x_1}{2}}.$$

We can show that the following hold.

Theorem 6.31. *Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ such that $x_1 < x_2$ and $y_1 < y_2$. Define $g\colon]-\infty, x_2] \to \mathbb{R}$ by*

$$g(t) = y_2 + (y_2 - y_1) \cdot \frac{\tanh \frac{t - x_2}{2}}{\tanh \frac{x_2 - x_1}{2}}$$

for $t \in]-\infty, x_2]$. Then g is (-1)-convex.

Proof. Define $h\colon]-\infty, 0] \to \mathbb{R}$ by

$$h(t) = \frac{g(t + x_2) - y_2}{y_2 - y_1} \tanh \frac{x_2 - x_1}{2} = \tanh \frac{t}{2}$$

for $t \in]-\infty, 0]$. Then g is (-1)-convex if and only if h is (-1)-convex. Therefore we get the conclusion from Lemma 6.25. □

This implies that if Conjecture 6.30 is true, then the function f_∞ is (-1)-convex.
Next, we propose the following conjectures:

Conjecture 6.32. *Let f be a (-1)-convex function from $\mathbb{R}$ into itself. Then f is bounded below.*

Conjecture 6.33. *Let X be a* CAT(-1) *space and f a proper (-1)-convex function from X into $]-\infty,\infty]$. Then for any $v \in X$,*

$$\liminf_{d(u,v)\to\infty} \frac{f(u)}{d(u,v)} \geq 0.$$

Note that Conjecture 6.33 is true if $X = \mathbb{R}$, which is obtained by Theorem 6.26. Now we present the experiment that led us to the Conjecture 6.32. Specifically, we attempt to simulate the extension of a (-1)-convex function. In preparation for the numerical experiment, we notice the following fact. Let $s, t, u \in \mathbb{R}$ such that $s < t < u$. Suppose that a function $f\colon [s,u] \to \mathbb{R}$ is (-1)-convex on $[s,t]$. Then, for f to be (-1)-convex on $[s,u]$, f must satisfy

$$f(u) \geq \frac{(\sinh(u-t') + \sinh(t'-s'))f(t') - (\sinh(u-t'))f(s')}{\sinh(t'-s')}$$

for all $s', t' \in \mathbb{R}$ such that $s \leq s' < t' \leq t$. Note that the inequality above is equivalent to $f(t') \leq g_{s',u}(t')$. Naturally, satisfying this condition is not sufficient for f to be (-1)-convex.

Let $\varepsilon > 0$ such that $|\varepsilon| \ll 1$ and take $\delta < 0$ arbitrarily. Put $A = \{k\varepsilon \mid k \in \mathbb{N} \cup \{0\}\}$. We will create a function $f\colon A \to \mathbb{R}$. Define $f(0) = 0$ and $f(\varepsilon) = \delta$. In addition, for any $k \in \mathbb{N}$, define $f((k+1)\varepsilon)$ by

$$\begin{aligned} f((k+1)\varepsilon) &= \max_{\substack{0 \leq s' < t' \leq \varepsilon k \\ s', t' \in A}} \frac{(\sinh((k+1)\varepsilon - t') + \sinh(t'-s'))f(t') - \sinh((k+1)\varepsilon - t')f(s')}{\sinh(t'-s')} \\ &= \max_{\substack{0 \leq m < n \leq k \\ m,n \in \mathbb{N}\cup\{0\}}} \frac{(\sinh((k+1-n)\varepsilon) + \sinh((n-m)\varepsilon))f(n\varepsilon) - \sinh((k+1-n)\varepsilon)f(m\varepsilon)}{\sinh((n-m)\varepsilon)}. \end{aligned}$$

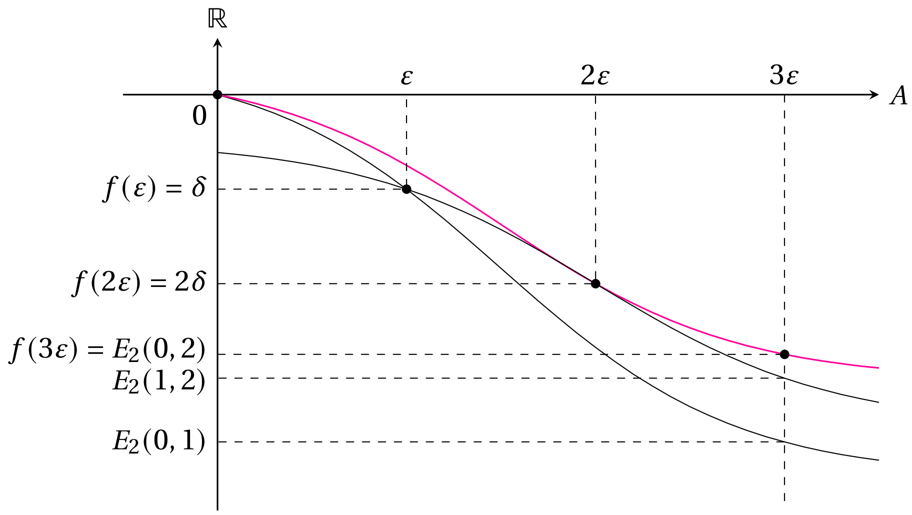
Put $P_k = \{(m,n) \in \mathbb{Z}^2 \mid 0 \leq m < n \leq k\}$ and

$$E_k(m,n) = \frac{(\sinh((k+1-n)\varepsilon) + \sinh((n-m)\varepsilon))f(n\varepsilon) - \sinh((k+1-n)\varepsilon)f(m\varepsilon)}{\sinh((n-m)\varepsilon)}$$

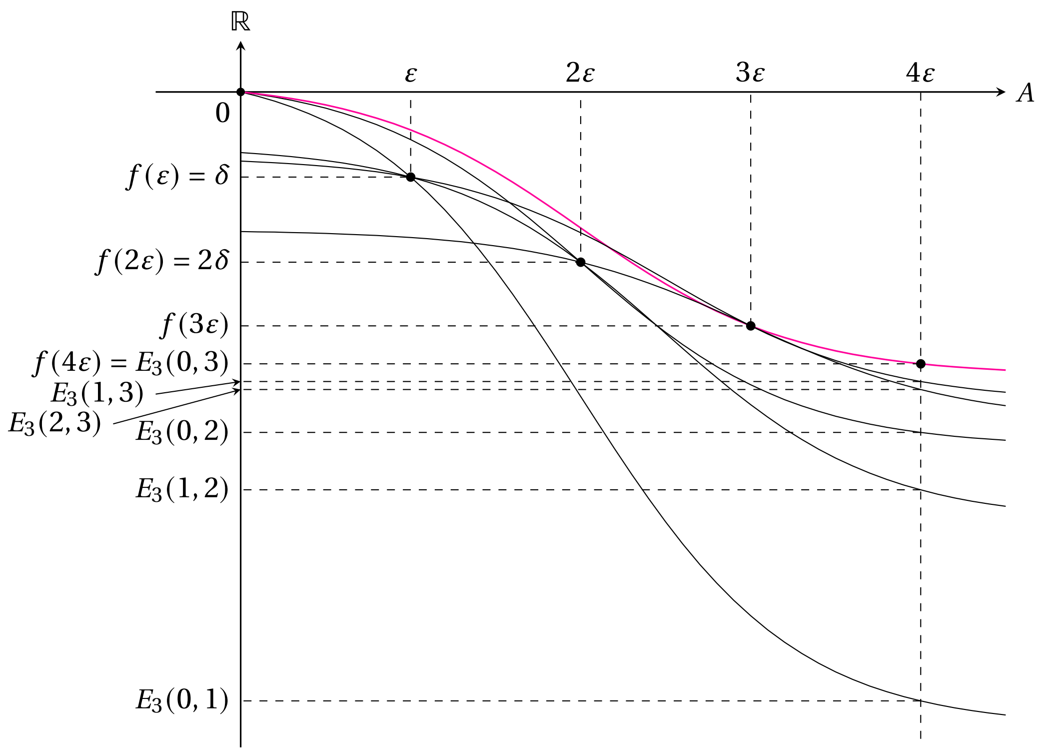
for every $k \in \mathbb{N}$ such that $0 \leq m < n \leq k$. Then we have $f((k+1)\varepsilon) = \max_{(m,n)\in P_k} E_k(m,n)$. For instance, we get $f(2\varepsilon) = \max_{(m,n)\in P_1} E_1(m,n) = E_1(0,1) = 2\delta$. Similarly, we have $f(3\varepsilon) = \max\{E_2(0,1), E_2(0,2), E_2(1,2)\}$, and

$$f(4\varepsilon) = \max\{E_3(0,1), E_3(0,2), E_3(0,3), E_3(1,2), E_3(1,3), E_3(2,3)\}.$$

The following figure represents a definition of $f(3\varepsilon)$.



Similarly, the following figure represents a definition of $f(4\varepsilon)$.



For ease of recognizing, the figures above are drawn for the large epsilon $\varepsilon = 0.8$, and $\delta = -0.5$. If $|\varepsilon|$ is significantly small, then the four points $(\varepsilon, f(\varepsilon)), (2\varepsilon, f(2\varepsilon)), (3\varepsilon, f(3\varepsilon))$ and $(4\varepsilon, f(4\varepsilon))$ are almost aligned in a straight line.

By the definition of f, we have

$$f(t) \leq \frac{\sinh(u-t)}{\sinh(u-t)+\sinh(t-s)} f(s) + \frac{\sinh(t-s)}{\sinh(u-t)+\sinh(t-s)} f(u)$$

for any $s, t, u \in A$ such that $s < t < u$. Therefore, a function f discretely simulates the extension of a (-1)-convex function to ∞ when $|\varepsilon| \ll 1$.

In what follows, we assume that $0 < \varepsilon \leq 1$. Then we expect that the following hold.

Conjecture 6.34. *For any $t \in A$,*

$$f(t) = \frac{\sinh \frac{t}{2}}{\sinh \frac{\varepsilon}{2} \cosh\left(\frac{t-\varepsilon}{2}\varepsilon\right)} \delta = \frac{\delta}{\tanh \frac{\varepsilon}{2}} \tanh \frac{t-\varepsilon}{2} + \delta.$$

In other words, for any $l \in \mathbb{N} \cup \{0\}$,

$$f(l\varepsilon) = \frac{\sinh\left(\frac{l}{2}\varepsilon\right)}{\sinh \frac{\varepsilon}{2} \cosh\left(\frac{l-1}{2}\varepsilon\right)} \delta = \frac{\delta}{\tanh \frac{\varepsilon}{2}} \tanh \frac{l\varepsilon-\varepsilon}{2} + \delta. \tag{F_l}$$

We can easily verify that the above equation holds if $l = 0, 1, 2$. This Conjecture 6.34 can be proven if the following conjecture is true:

Conjecture 6.35. *For any $k, m, n \in \mathbb{N} \cup \{0\}$ such that $k \geq 2$ and $0 \leq m < n \leq k$,*

$$\frac{1}{\sinh\left(\frac{n-m}{2}\varepsilon\right)\cosh\left(\frac{n-m}{2}\varepsilon\right)} \cdot \left(\frac{\sinh\left(\frac{k+1-m}{2}\varepsilon\right)\cosh\left(\frac{k+1-2n+m}{2}\varepsilon\right)\sinh\left(\frac{n}{2}\varepsilon\right)}{\cosh\left(\frac{n-1}{2}\varepsilon\right)} - \frac{\sinh\left(\frac{k+1-n}{2}\varepsilon\right)\cosh\left(\frac{k+1-n}{2}\varepsilon\right)\sinh\left(\frac{m}{2}\varepsilon\right)}{\cosh\left(\frac{m-1}{2}\varepsilon\right)} \right) \geq \frac{\sinh\left(\frac{k+1}{2}\varepsilon\right)}{\cosh\left(\frac{k}{2}\varepsilon\right)}.$$

Proof of Conjecture 6.34 under Conjecture 6.35. We show it by induction. Suppose that (F_l) holds for $l = 0, 1, 2, 3, \ldots, k$. Then we have

$$\begin{aligned}
E_k(m,n) &= \frac{(\sinh((k+1-n)\varepsilon) + \sinh((n-m)\varepsilon)) f(n\varepsilon) - \sinh((k+1-n)\varepsilon) f(m\varepsilon)}{\sinh((n-m)\varepsilon)} \\
&= \frac{\sinh\left(\frac{k+1-m}{2}\varepsilon\right)\cosh\left(\frac{k+1-2n+m}{2}\varepsilon\right)}{\sinh\left(\frac{n-m}{2}\varepsilon\right)\cosh\left(\frac{n-m}{2}\varepsilon\right)} f(n\varepsilon) - \frac{\sinh((k+1-n)\varepsilon)}{\sinh((n-m)\varepsilon)} f(m\varepsilon) \\
&= \frac{\sinh\left(\frac{k+1-m}{2}\varepsilon\right)\cosh\left(\frac{k+1-2n+m}{2}\varepsilon\right)}{\sinh\left(\frac{n-m}{2}\varepsilon\right)\cosh\left(\frac{n-m}{2}\varepsilon\right)} \cdot \frac{\sinh\left(\frac{n}{2}\varepsilon\right)}{\sinh\frac{\varepsilon}{2}\cosh\left(\frac{n-1}{2}\varepsilon\right)} \delta \\
&\quad - \frac{\sinh\left(\frac{k+1-n}{2}\varepsilon\right)\cosh\left(\frac{k+1-n}{2}\varepsilon\right)}{\sinh\left(\frac{n-m}{2}\varepsilon\right)\cosh\left(\frac{n-m}{2}\varepsilon\right)} \cdot \frac{\sinh\left(\frac{m}{2}\varepsilon\right)}{\sinh\frac{\varepsilon}{2}\cosh\left(\frac{m-1}{2}\varepsilon\right)} \delta \\
&= \frac{\delta}{\sinh\left(\frac{n-m}{2}\varepsilon\right)\cosh\left(\frac{n-m}{2}\varepsilon\right)\sinh\frac{\varepsilon}{2}} \\
&\quad \cdot \left(\frac{\sinh\left(\frac{k+1-m}{2}\varepsilon\right)\cosh\left(\frac{k+1-2n+m}{2}\varepsilon\right)\sinh\left(\frac{n}{2}\varepsilon\right)}{\cosh\left(\frac{n-1}{2}\varepsilon\right)} - \frac{\sinh\left(\frac{k+1-n}{2}\varepsilon\right)\cosh\left(\frac{k+1-n}{2}\varepsilon\right)\sinh\left(\frac{m}{2}\varepsilon\right)}{\cosh\left(\frac{m-1}{2}\varepsilon\right)} \right)
\end{aligned}$$

for any $(m,n) \in P_k$. In particular, we also have

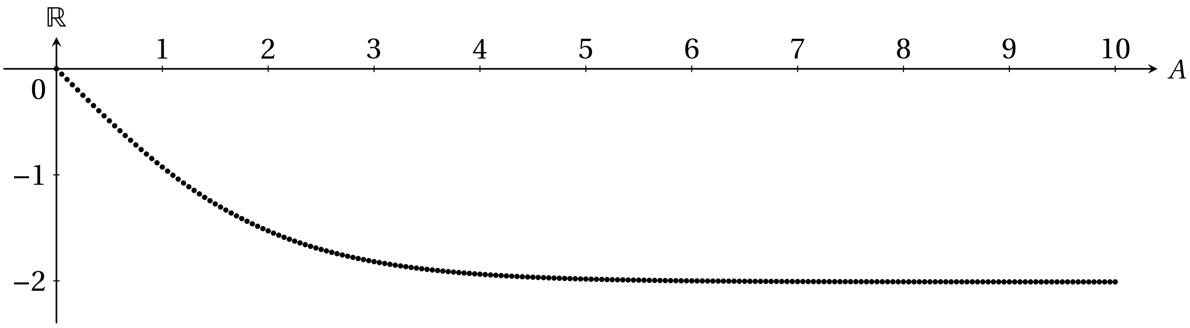
$$E_k(0,k) = \frac{\delta}{\sinh\left(\frac{k}{2}\varepsilon\right)\cosh\left(\frac{k}{2}\varepsilon\right)\sinh\frac{\varepsilon}{2}} \cdot \frac{\sinh\left(\frac{k+1}{2}\varepsilon\right)\cosh\left(\frac{k-1}{2}\varepsilon\right)\sinh\left(\frac{k}{2}\varepsilon\right)}{\cosh\left(\frac{k-1}{2}\varepsilon\right)} = \frac{\sinh\left(\frac{k+1}{2}\varepsilon\right)}{\sinh\frac{\varepsilon}{2}\cosh\left(\frac{k}{2}\varepsilon\right)}\delta.$$

Therefore, if Conjecture 6.35 is true, then $E_k(m,n) \le E_k(0,k)$ for any $(m,n) \in P_k$. Hence

$$f((k+1)\varepsilon) = \max_{(m,n)\in P_k} E_k(m,n) = E_k(0,k) = \frac{\sinh\left(\frac{k+1}{2}\varepsilon\right)}{\sinh\frac{\varepsilon}{2}\cosh\left(\frac{k}{2}\varepsilon\right)}\delta,$$

which implies the conclusion. □

We consider the case where $\varepsilon = 0.01$ and $\delta = -0.01$. Define a function $f\colon A \to \mathbb{R}$ by the same method, where $A = \{0.01k \mid k \in \mathbb{N} \cup \{0\}\}$. The following figure shows 201 points $(0, f(0))$, $(0.05, f(0.05))$, $(0.1, f(0.1)), \dots, (9.95, f(9.95))$, $(10, f(10))$.

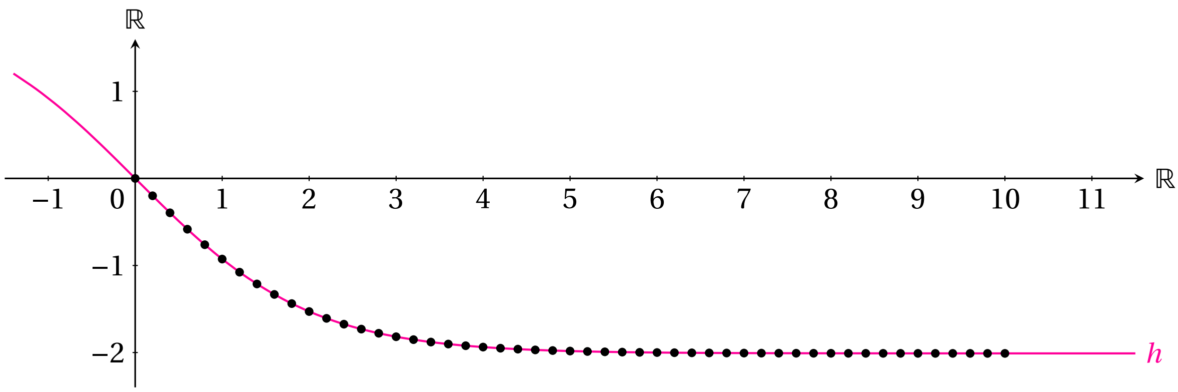


Then the shape of the graph of f is like the graph of the hyperbolic tangent function. Put $A' = \{t \in A \mid t \le 10\} = \{0.01k \mid k \in \mathbb{Z}, 0 \le k \le 1000\}$, and define $h\colon A' \to \mathbb{R}$ by

$$h(t) = \frac{\delta}{\tanh\frac{\varepsilon}{2}}\tanh\frac{t-\varepsilon}{2} + \delta = -\frac{0.01}{\tanh 0.005}\tanh\frac{t-0.01}{2} - 0.01$$

for $t \in A'$. We attempt to calculate $\max_{t\in A'}|f(t)-h(t)|$ by a computer. Then, using a quadruple-precision floating point calculation, we obtain $\max_{t\in A'}|f(t)-h(t)| \approx 2.153 \times 10^{-31}$. This result is consistent with Conjecture 6.34, since $\max_{t\in A'}|f(t)-h(t)| = 0$ if that conjecture is true.

The following figure shows a graph of h and points $(0, f(0))$, $(0.2, f(0.2)), \dots, (10, f(10))$.



From this result, we suggest Conjecture 6.32.

Chapter 7

Conclusion

In this thesis, we consider the following themes:

- Natures of κ-convex combinations on geodesic spaces;
- fixed point approximation theorems for tightly quasinonexpansive mappings on complete CAT(κ) spaces;
- resolvents of the equilibrium problem on complete CAT(κ) spaces;
- resolvents of a convex function on complete CAT(κ) spaces;
- natures of (-1)-convex functions.

First, we consider another type of convex combination which is named a κ-convex combination. We show that a κ-convex combination has good properties on CAT(κ) spaces. In particular, we obtain the following results on the unit sphere on a real Hilbert space.

Theorem 7.1. *Let $\mathcal{H}$ be a real Hilbert space and $S_{\mathcal{H}} = \{x \in \mathcal{H} \mid \|x\| = 1\}$ a unit sphere on $\mathcal{H}$ with a metric $d\colon S_{\mathcal{H}} \times S_{\mathcal{H}} \to [0,\pi]$ defined by $d(u,v) = \cos^{-1}\langle u,v\rangle$ for $u,v \in S_{\mathcal{H}}$. Then for any $x,y \in S_{\mathcal{H}}$ such that $d(x,y) < \pi$ and $\alpha \in [0,1]$,*

$$\alpha x \overset{1}{\oplus} (1-\alpha)y = \frac{\alpha x + (1-\alpha)y}{\|\alpha x + (1-\alpha)y\|}.$$

Theorem 7.2. *Let $\mathcal{H}$ and $S_{\mathcal{H}}$ be the same as Theorem 7.1. Let S be a nonempty convex subspace of $S_{\mathcal{H}}$ such that $d(u,v) < \pi$ for any $u,v \in S$. Let $\triangle(x,y,z)$ be a geodesic triangle on S such that $[x,y] \cap [y,z] \cap [z,x] = \emptyset$. For $\alpha,\beta,\gamma \in\,]0,1[$, take $p = (1-\alpha)x \overset{1}{\oplus} \alpha y$, $q = (1-\beta)y \overset{1}{\oplus} \beta z$, and $r = (1-\gamma)z \overset{1}{\oplus} \gamma x$. Then the following are equivalent:*

- *$[x,q] \cap [y,r] \cap [z,p] \neq \emptyset$;*
- *$\alpha\beta\gamma/((1-\alpha)(1-\beta)(1-\gamma)) = 1$.*

Next, we propose a notion of tightly quasinonexpansive mappings on CAT(κ) spaces. We show that every tightly quasinonexpansive mapping is quasinonexpansive, and every firmly vicinal mapping with ψ is tightly quasinonexpansive. We prove Mann type fixed point approximation theorems for vicinal mappings with ψ and tightly quasinonexpansive mappings on a complete CAT(κ) space as follows:

Theorem 7.3. *Let X be an admissible complete* CAT(κ) *space and $T\colon X \to X$ a vicinal mapping with ψ. Suppose that ψ satisfies* (P1) *or* (P2):

(P1) *ψ is nondecreasing;*
(P2) $\sup_{n\in\mathbb{N}} \psi(d(x_n, Tx_n)) < \infty$.

Let $\{\alpha_n\} \subset [0,1[$ such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by *either*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tx_n$$

for $n \in \mathbb{N}$ *or*

$$x_{n+1} = \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n)Tx_n$$

for $n \in \mathbb{N}$. *Let us denote* (a) *and* (b) *by the following conditions:*

(a) $\{Tx_n\}$ *is* κ*-bounded;*
(b) $\sup_{n\in\mathbb{N}} d(x_n, Tx_n) < D_\kappa/2$.

Then the following hold:

(i) *Suppose that* ψ *satisfies* (P1). *Then* $F(T) \neq \emptyset$ *if* (a) *and* (b) *hold. Conversely,* $F(T) \neq \emptyset$ *only if* (a) *and* (b) *hold when* T *is tightly quasinonexpansive.*
(i)′ *Suppose that* ψ *satisfies* (P2). *Then* $F(T) \neq \emptyset$ *if and only if* (a) *holds.*

Theorem 7.4. *Let* X *be an admissible complete* CAT(κ) *space and* T *a quasinonexpansive and* Δ*-demiclosed mapping from* X *into itself. Suppose that* $\{\alpha_n\}$ *and* $\{x_n\}$ *are the same as the previous theorem. Then the following hold:*

(ii) *If* T *is tightly quasinonexpansive and* $\limsup_{n\to\infty} \alpha_n < 1$, *then* $\{x_n\}$ Δ*-converges to some fixed point of* T.
(iii) *If* $\liminf_{n\in\mathbb{N}} \alpha_n(1-\alpha_n) > 0$, *then* $\{x_n\}$ Δ*-converges to some fixed point of* T.

We also prove Halpern type fixed point approximation theorems for tightly quasinonexpansive mappings on a complete CAT(κ) space as follows:

Theorem 7.5. *Let* X *be an admissible complete* CAT(κ) *space and* $T: X \to X$ *a tightly quasinonexpansive and* Δ*-demiclosed mapping. Let* $\{\alpha_n\} \subset \,]0,1[$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Let* $u, x_1 \in X$ *arbitrarily and define* $\{x_n\} \subset X$ *by*

$$x_{n+1} = \alpha_n u \overset{\kappa}{\oplus} (1-\alpha_n)Tx_n$$

for any $n \in \mathbb{N}$. *In the case where* $\kappa > 0$, *suppose that* (i) *or* (ii) *holds:*

(i) $\sup_{n\in\mathbb{N}} d(u, Tx_n) < D_\kappa/2$,
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then $\{x_n\}$ *converges to* $P_{F(T)}u$.

In Chapter 5, we study the equilibrium problem on complete CAT(κ) spaces. For a CAT(κ) space X, its nonempty closed convex subset K, and a function $\varphi\colon [0, c_\kappa(D_\kappa/2)[\, \to \mathbb{R}$, we consider a resolvent operator $R_f\colon X \to K$ for a bifunction $f\colon K\times K \to \mathbb{R}$ defined by

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) + \varphi(c_\kappa(d(x,y))) - \varphi(c_\kappa(d(x,z)))) \ge 0 \right\} \qquad (\star)$$

for each $x \in X$. We get sufficient conditions such that R_f to be a single-valued mapping as follows:

Theorem 7.6. *Let* X *be an admissible complete* CAT(κ) *space and suppose that* X *has the convex hull finite property. Let* K *be a nonempty closed convex subset of* X *and* f *a real function on* K^2 *with conditions* (E1)–(E4). *Suppose that a function* $\varphi\colon [0, c_\kappa(D_\kappa/2)[\, \to [0,\infty[$ *is strictly increasing, differentiable, and* φ' *is continuous on* $[0, c_\kappa(D_\kappa/2)[$. *In addition, if* $\kappa \le 0$, *then suppose that* φ *has the following conditions* **(d₁)** *and at least one of* **(c₁)** *and* **(c₂)***:*

(c_1) *φ' is nondecreasing;*
(c_2) *$\varphi \circ c_\kappa$ is convex on $[0, \infty[$, and $\varphi(c_\kappa(d(x, \cdot)))$ is strictly midpoint convex on K for any $x \in X$;*
(d_1) *K is bounded; otherwise, an inequality*

$$\liminf_{\substack{d(u,z)\to\infty \\ z\in K}} \frac{f(u,z)}{d(u,z)} + \lim_{d\to\infty} \frac{\varphi(c_\kappa(d))}{d} > 0$$

holds for some $u \in K$.

Otherwise, if $\kappa > 0$, then suppose that φ has the following conditions **(c_1)** *and* **(d_2)***:*

(c_1) *φ' is nondecreasing;*
(d_2) $\lim_{d\to D_\kappa/2} \varphi(c_\kappa(d)) = \infty$*, that is,* $\lim_{\lambda\nearrow 1} \varphi(\lambda/\kappa) = \infty$.

Define a set-valued mapping $R_f \colon X \to 2^K$ by the formula $(\star)$. Then the following hold:

- *R_f is well-defined as a single-valued mapping from X into K;*
- *R_f is firmly vicinal with $\varphi' \circ c_\kappa \colon [0, D_\kappa/2[\to]0, \infty[$;*
- *R_f is tightly quasinonexpansive and Δ-demiclosed;*
- *the set of all fixed points of R_f and the set of all solutions to an equilibrium problem for f are identical.*

We also consider a resolvent operator of a convex function. For a proper convex function $f \colon X \to]-\infty, \infty]$, we consider a resolvent operator $S_f \colon X \to \mathrm{dom}(f)$ defined by

$$S_f x = \operatorname*{argmin}_{y\in X} \big(f(y) + \varphi(c_\kappa(d(x, y)))\big) \qquad (\star\star)$$

for each $x \in X$. We get the following result which gives a sufficient condition such that S_f to be single-valued.

Theorem 7.7. *Let X be an admissible complete* CAT(κ) *space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Suppose that a function $\varphi \colon [0, c_\kappa(D_\kappa/2)[\to [0, \infty[$ is nondecreasing, differentiable, and φ' is continuous on $[0, c_\kappa(D_\kappa/2)[$. Furthermore, suppose that* **(c_1)** *or* **(c_3)** *hold:*

(c_1) *φ' is nondecreasing;*
(c_3) *$\varphi(c_\kappa(d(x, \cdot)))$ is strictly midpoint convex for any $x \in X$.*

Suppose the following:

- *If $\kappa \le 0$, then suppose that*

$$\liminf_{d(u,z)\to\infty} \frac{f(z)}{d(u,z)} + \lim_{d\to\infty} \frac{\varphi(c_\kappa(d))}{d} > 0$$

  for some $u \in X$.
- *If $\kappa > 0$, then suppose that* $\lim_{d\to D_\kappa/2} \varphi(c_\kappa(d)) = \infty$.

Define a set-valued mapping $S_f \colon X \to 2^{\mathrm{dom}(f)}$ by the formula $(\star\star)$. Then the following hold.

- *S_f is well-defined as a single-valued mapping from X into* $\mathrm{dom}(f)$*;*
- *S_f is firmly vicinal with $\varphi' \circ c_\kappa \colon [0, D_\kappa/2[\to]0, \infty[$;*
- *S_f is tightly quasinonexpansive and Δ-demiclosed;*
- *the set of all fixed points of S_f and the set of all minimizers of f are identical.*

We also obtain the following convergence theorem of a solution to the equilibrium problem and the convex minimization problem. In what follows, $\operatorname{Equil} f$ denotes a set of all solutions to an equilibrium problem for a bifunction f.

Theorem 7.8. *Let X, K, and f be the same as Theorem 7.6. Let $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$ be a nondecreasing and differentiable function such that φ' is continuous on $[0, c_\kappa(D_\kappa/2)[$. Let $R_f\colon X \to K$ be a resolvent well defined by the formula* ($\star$). *Let $\{\alpha_n\} \subset [0,1[$ such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Take $x_1 \in X$ arbitrarily and generate $\{x_n\} \subset X$ by either*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) R_f x_n$$

for $n \in \mathbb{N}$ or

$$x_{n+1} = \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) R_f x_n$$

for $n \in \mathbb{N}$. Let us denote (a) *and* (b) *by the following conditions:*

(a) *$\{R_f x_n\}$ is κ-bounded;*
(b) $\sup_{n\in\mathbb{N}} d(x_n, R_f x_n) < D_\kappa/2$.

Suppose that R_f is vicinal with $\psi\colon [0, D_\kappa/2[\to]0,\infty[$, and define conditions (P1) *and* (P2) *as follows:*

(P1) *ψ is nondecreasing;*
(P2) $\sup_{n\in\mathbb{N}} \psi(d(x_n, R_f x_n)) < \infty$.

Then the following hold:

(i) *Suppose that ψ satisfies* (P1). *Then* $\operatorname{Equil} f \neq \emptyset$ *if and only if* (a) *and* (b) *hold.*
(i)′ *Suppose that ψ satisfies* (P2). *Then* $\operatorname{Equil} f \neq \emptyset$ *if and only if* (a) *holds.*

Theorem 7.9. *Let X, K, f, φ, and R_f be the same as Theorem 7.8, and suppose that* $\operatorname{Equil} f \neq \emptyset$. *Let $\{\alpha_n\} \subset [0,1[$ such that* $\limsup_{n\to\infty} \alpha_n < 1$ *and* $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. *Take $x_1 \in X$ and generate $\{x_n\} \subset X$ by either*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) R_f x_n$$

for $n \in \mathbb{N}$ or

$$x_{n+1} = \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) R_f x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to some solution to the equilibrium problem for f.

Theorem 7.10. *Let X, K, f, φ, and R_f be the same as Theorem 7.8, and suppose that* $\operatorname{Equil} f \neq \emptyset$. *Let $\{\alpha_n\} \subset]0,1[$ such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Let $u, x_1 \in X$ arbitrarily and define $\{x_n\} \subset X$ by*

$$x_{n+1} = \alpha_n u \overset{\kappa}{\oplus} (1-\alpha_n) R_f x_n$$

for any $n \in \mathbb{N}$. In the case where $\kappa > 0$, suppose that (i) *or* (ii) *holds:*

(i) $\sup_{n\in\mathbb{N}} d(u, R_f x_n) < D_\kappa/2$,
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then $\{x_n\}$ converges to some solution to the equilibrium problem for f.

Theorem 7.11. *Let X and f be the same as Theorem 7.7. Let $\varphi\colon [0, c_\kappa(D_\kappa/2)[\to [0,\infty[$ be a nondecreasing and differentiable function such that φ' is continuous on $[0, c_\kappa(D_\kappa/2)[$. Let $S_f\colon X \to 2^{\operatorname{dom}(f)}$ be a resolvent well defined by the formula* ($\star\star$). *Let $\{\alpha_n\} \subset [0,1[$ such that*

$\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. *Take* $x_1 \in X$ *arbitrarily and generate* $\{x_n\} \subset X$ *by either*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) S_f x_n$$

for $n \in \mathbb{N}$ *or*

$$x_{n+1} = \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) S_f x_n$$

for $n \in \mathbb{N}$. *Let us denote* (a) *and* (b) *by the following conditions:*

(a) $\{S_f x_n\}$ *is* κ*-bounded;*
(b) $\sup_{n\in\mathbb{N}} d(x_n, S_f x_n) < D_\kappa/2$.

Suppose that S_f *is vicinal with* $\psi\colon [0, D_\kappa/2[\to]0,\infty[$, *and define conditions* (P1) *and* (P2) *as follows:*

(P1) ψ *is nondecreasing;*
(P2) $\sup_{n\in\mathbb{N}} \psi(d(x_n, S_f x_n)) < \infty$.

Then the following hold:

(i) *Suppose that* ψ *satisfies* (P1). *Then* $\operatorname{argmin} f \neq \emptyset$ *if and only if* (a) *and* (b) *hold.*
(i)′ *Suppose that* ψ *satisfies* (P2). *Then* $\operatorname{argmin} f \neq \emptyset$ *if and only if* (a) *holds.*

Theorem 7.12. *Let* X, f, φ, *and* S_f *be the same as Theorem 7.11, and suppose that* $\operatorname{argmin} f \neq \emptyset$. *Let* $\{\alpha_n\} \subset [0,1[$ *such that* $\limsup_{n\to\infty} \alpha_n < 1$ *and* $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. *Take* $x_1 \in X$ *and generate* $\{x_n\} \subset X$ *by either*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) S_f x_n$$

for $n \in \mathbb{N}$ *or*

$$x_{n+1} = \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n) S_f x_n$$

for $n \in \mathbb{N}$. *Then* $\{x_n\}$ Δ*-converges to some minimizer of* f.

Theorem 7.13. *Let* X, f, φ, *and* S_f *be the same as Theorem 7.11, and suppose that* $\operatorname{argmin} f \neq \emptyset$. *Let* $\{\alpha_n\} \subset]0,1[$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Let* $u, x_1 \in X$ *arbitrarily and define* $\{x_n\} \subset X$ *by*

$$x_{n+1} = \alpha_n u \overset{\kappa}{\oplus} (1-\alpha_n) S_f x_n$$

for any $n \in \mathbb{N}$. *In the case where* $\kappa > 0$, *suppose that* (i) *or* (ii) *holds:*

(i) $\sup_{n\in\mathbb{N}} d(u, S_f x_n) < D_\kappa/2$,
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then $\{x_n\}$ *converges to some minimizer of* f.

In Section 6.2, we study a special convex function named a (-1)-convex function. Let X be a uniquely metric space. We show that a (-1)-convex function $f\colon X \to \mathbb{R}$ is convex on $\operatorname{dom}(f)$. We also prove that a (-1)-convex function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\liminf_{d(u,v)\to\infty} \frac{f(u)}{d(u,v)} \geq 0$$

for all $v \in \mathbb{R}$, where $d(\cdot,\cdot) = |\cdot - \cdot|$. By doing numerical experiments, we consider the behavior and properties of (-1)-convex functions.

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