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Resolvents for equilibrium problems and a convergence theorem using a balanced mapping in geodesic spaces.

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Contents

Chapter 1

Introduction

Let C be a nonempty set and T a mapping of C into itself. We call $x_0 \in C$ a fixed point of T if $x_0 = Tx_0$. The theory of fixed points has been studied by many researchers. Further, we can apply it to the convex minimization problems and the equilibrium problems etc.

Various resolvent of a convex function were introduced by many researchers. In a Hilbert space H, a resolvent for convex functions is defined by

$$J_f x = \operatorname*{Argmin}_{y \in H} \left\{ f(y) + \frac{1}{2}\|y - x\|^2 \right\}.$$

In complete geodesic spaces, many researchers introduced a resolvent for convex functions [3, 24, 16]. In particular, in a CAT(−1) space X, a resolvent for convex functions is defined by Kajimura and Kimura [17] as follows:

$$R_f x = \operatorname*{Argmin}_{y \in X} \{ f(y) + \tanh d(x, y) \sinh d(x, y) \}.$$

Let C be a nonempty set and $f \colon C \times C \to \mathbb{R}$. A equilibrium problems is defined as to find $z_0 \in C$ such that $f(z_0, y) \geq 0$ for all $y \in C$. Equilibrium problems were first studied by Blum and Oettli [4] in Banach spaces. Equilibrium problems include convex minimization problems. In 2005, Combettes and Hirstoaga [8] proposed the resolvent of equilibrium problems in Hilbert spaces.

Theorem 1.1 (Combettes and Hirstoaga [8])**.** *Let H be a Hilbert space and K a nonempty closed convex subset of H. Let $f \colon K \times K \to \mathbb{R}$ and S_f the set of solutions to the equilibrium problem for f. Suppose the following conditions:*

- $f(y, y) = 0$ *for all* $y \in K$;
- $f(y, z) + f(z, y) \leq 0$ *for all* $y, z \in K$;
- $f(y, \cdot) \colon K \to \mathbb{R}$ *is lower semicontinuous and convex for every* $y \in H$;
- $f(\cdot, z) \colon K \to \mathbb{R}$ *is upper hemicontinuous for every* $y \in K$.

Then the resolvent operator J_f defined by

$$J_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z, y) + \langle z - x, y - z \rangle) \geq 0 \right\}$$

has the following properties:

(i) $D(J_f) = X$;
(ii) J_f *is single-valued and firmly nonexpansive;*
(iii) $\operatorname{Fix} J_f = S_f$;
(iv) S_f *is closed and convex.*

Recently, the resolvents of equilibrium problems were proposed in geodesic spaces. In a CAT(0) space with the convex hull finite property, it was proposed by Kimura and Kishi in

2018; see [23]. In an admissible CAT(1) space with the convex hull finite property, it was proposed by Kimura in 2021; see [22].

First, we introduce a resolvent of convex functions different from R_f and a resolvent for an equilibrium problem in a CAT(−1) space. Further, we introduce a delta-convergence sequence generated by the proximal point algorithm in a CAT(−1) space.

On the other hand, approximating of a common fixed point has been studied by many researchers. In 1992, Wittmann [45] introduced a Halpern iteration and proved convergence to a fixed point of a nonexpansive mapping in a Hilbert space. In 1997, Shioji and Takahashi [43] introduced it in a Banach space. In a CAT(0) space, Saejung [39] introduced it in a CAT(1) space and prove a convergence to a common fixed point of a nonexpansive mapping, and Kimura–Satô [30] introduced it and prove a common fixed point of a strongly quasinonexpansive and delta-demiclosed mapping. Further, in 2007, Aoyama et al. [2] introduced a Halpern iteration using a countable family of nonexpansive mappings in a Banach space.

Theorem 1.2 (Aoyama, Kimura, Takahashi and Toyoda [2]). *Let C be a nonempty closed convex subset of uniformly convex Banach space whose norm is uniformly Gâteaux differentiable. Let $S_k\colon C \to C$ a nonexpansive for $k \in \mathbb{N}$ such that $\bigcap_{k=1}^{\infty} \operatorname{Fix} S_k \neq \emptyset$. Let $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0,1]$, $\{\beta_n^k \mid k,n \in \mathbb{N}, k \leq n\} \subset [0,1]$ such that $\sum_{k=1}^{n} \beta_n^k = 1$ for $n \in \mathbb{N}$. Define $\{x_n\}$ by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) \sum_{k=1}^{n} \beta_n^k S_k x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions:

- $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1} - \alpha_n| < \infty$;
- $\lim_{n\to\infty} \beta_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \left|\beta_{n+1}^k - \beta_n^k\right| < \infty$.

Then, $x_n \to Qu$, where Q is sunny nonexpansive retraction of E onto $\bigcap_{k=1}^{\infty} \operatorname{Fix} S_k$.

In 2016, Huang and Kimura [14] introduced a Halpern iteration using a countable family of nonexpansive mappings in a CAT(0) space.

Theorem 1.3 (Huang and Kimura [14]). *Let X be a complete CAT(0) space, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k,n \in \mathbb{N}, k \leq n\} \subset]0,1[$ with $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$. Define $x_1, u \in X$ and*

$$x_{n+1} = \delta_n u \oplus (1-\delta_n) \bigoplus_{k=1}^{n} \alpha_n^k T_k x_n$$

for each $n \in \mathbb{N}$, where

$$\bigoplus_{i=1}^{n} \beta^i y_i = \begin{cases} y_1 & (n=1); \\ \beta^n y_n \oplus (1-\beta^n) \displaystyle\bigoplus_{i=1}^{n-1} \frac{\beta^i}{1-\beta^n} y_i & (n \geq 2) \end{cases}$$

for $\{y_1, y_2, \dots, y_n\} \subset X$ and $\{\beta^i \mid i = 1,2,\dots,n\} \subset]0,1[$ with $\sum_{i=1}^{n} \beta^i = 1$. Suppose the following conditions:

- $\lim_{n\to\infty} \alpha_n^k > 0$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$;
- $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then, $\{x_n\}$ converges strongly to $P_{\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k} u$, where $P_{\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k}$ is the metric projection of X onto $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$.

In geodesic spaces, the convex combination of more than three points are order-dependent. Hasegawa and Kimura [12], Kajimura et al. [15], and Kimura and Sasaki [29] introduced another definition of convex combination such that it is order-independent for more that three points.

Second, we introduce a Halpern iteration using a balanced mapping of a countable family of mappings and prove a convergence to a common fixed point in geodesic spaces.

Further, there are projection methods as fixed point approximating schemes. In 2003, Nakajo and Takahashi [37] introduced Nakajo–Takahashi projection method. In 2008, Takahashi, Takeuchi and Kubota [44] introduced the shrinking projection method. Kimura, Takahashi and Yao [34] introduced another type of a projection methods, which is called the combining projection method.

Theorem 1.4 (Kimura, Takahashi and Yao[34]). *Let C a nonempty closed convex subset C of a Hilbert space and T_j a nonexpansive mapping of C into itself for $j \in \{1, 2, \dots, N\}$ such that $\bigcap_{j=1}^{N} \operatorname{Fix} T_j \neq \emptyset$. Put $I_N = \{1, 2, \dots, N\}$. Let $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0, 1]$, $\left\{\beta_n^j \mid j \in I_N, n \in \mathbb{N}\right\} \subset [0, 1]$ such that $\sum_{j=1}^{N} \beta_n^j = 1$ for $n \in \mathbb{N}$, $\{\gamma_{n,k} \mid n, k \in \mathbb{N}, k \leq n\}$ such that $\sum_{k=1}^{n} \gamma_{n,k} = 1$ for $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0, 1]$. Define a sequence $\{x_n\}$ by $u, x_1 \in C$ and*

$$y_n^j = \alpha_n x_n + (1 - \alpha_n) T_j x_n \text{ for } j \in I_N;$$
$$C_n^j = \left\{ z \in C \,\middle|\, \|z - y_n^j\| \leq \|z - x_n\| \right\} \text{ for } j \in I_N;$$
$$x_{n+1} = \delta_n u + (1 - \delta_n) \sum_{k=1}^{n} \gamma_{n,k} \sum_{j=1}^{N} \beta_k^j P_{C_k^j} x_n$$

for each $n \in \mathbb{N}$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H. Suppose the following conditions hold:

(i) $\liminf_{n\to\infty} \alpha_n < 1$;
(ii) $\beta_n^j > 0$ *for all* $j \in I_N$ *and* $n \in \mathbb{N}$;
(iii) $\lim_{n\to\infty} \gamma_{n,k} > 0$ *for all* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\gamma_{n+1,k} - \gamma_{n,k}| < \infty$;
(iv) $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then, $\{x_n\}$ converges strongly to $P_{\bigcap_{j=1}^{N} \operatorname{Fix} T_j} u$.

Finally, we introduce projection methods using a balanced mapping of a countable family of mappings in a CAT(κ) space for $\kappa \in \mathbb{R}$.

In this thesis, we introduce a new type of resolvents for convex functions and equilibrium problems. We prove a delta-convergence theorem with the proximal point algorithm in CAT(-1) space. Further, we consider the properties of a balanced mapping of a countable family of mappings and introduce a Halpern iteration of countable family of mappings in CAT(κ) space. We also consider a projection method of a countable family of mappings, which is generated by a Halpern iteration.

Chapter 2

Preliminaries

Let (X, d) a metric space and T a mapping of X into itself. Then, the set of fixed points is denoted $\operatorname{Fix} T$. Let $\{x_n\}$ a bounded sequence of X and $x_0 \in X$. Then, x_0 is an *asymptotic center of* $\{x_n\}$ if the equation

$$\limsup_{n\to\infty} d(x_n, x_0) = \inf_{y\in X} \limsup_{n\to\infty} d(x_n, y)$$

holds. The set of all asymptotic centers of $\{x_n\}$ is denoted by $\operatorname{AC}(\{x_n\})$. Further, $\{x_n\}$ is *delta-convergent to* x_0 if for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, $\operatorname{AC}(\{x_{n_i}\}) = \{x_0\}$, which is denoted by $x_n \xrightarrow{\Delta} x_0$.

Let $x, y \in X$ and γ_{xy} a mapping of $[0, d(x, y)]$ into X. A mapping γ_{xy} is said to be a *geodesic with endpoints x and y* if $\gamma_{xy}(0) = x$, $\gamma_{xy}(d(x, y)) = y$ and $d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s - t|$ for all $s, t \in [0, d(x, y)]$. For $D \in]0, \infty]$, we call X a *D-geodesic space* if a geodesic with endpoints x and y exists for all $x, y \in X$ with $d(x, y) < D$. In what follows, when X is a D-geodesic space, every geodesic in X whose length is less than D is always supposed to be unique. We call such X a *uniquely D-geodesic space*. Then, the image of a geodesic with endpoints x and y is denoted by $[x, y]$ for all $x, y \in X$ with $d(x, y) < D$. For $x, y \in X$ with $d(x, y) < D$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(y, z) = td(x, y)$. We denote it by $z = tx \oplus (1 - t)y$. For $\kappa \in \mathbb{R}$, a 2-dimensional model space M^2_κ with a curvature κ is defined by

$$M^2_\kappa = \begin{cases} \dfrac{1}{\sqrt{\kappa}}\mathbb{S}^2 & (\kappa > 0); \\ \mathbb{E}^2 & (\kappa = 0); \\ \dfrac{1}{\sqrt{-\kappa}}\mathbb{H}^2 & (\kappa < 0), \end{cases}$$

where $\mathbb{E}^2$ is the 2-dimensional Euclidean space, $\mathbb{H}^2$ is the 2-dimensional hyperbolic space and $\mathbb{S}^2$ is the 2-dimensional unit sphere. The diameter of M^2_κ is denoted by D_κ, that is,

$$D_\kappa = \begin{cases} \dfrac{\pi}{\sqrt{\kappa}} & (\kappa > 0); \\ \infty & (\kappa \le 0). \end{cases}$$

For $x, y, z \in X$, a *geodesic triangle* $\triangle(x, y, z)$ with vertices $x, y, z \in X$ is defined by $[x, y] \cup [y, z] \cup [z, x]$. For $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, a *comparison triangle* to $\triangle(x, y, z) \subset X$ of vertices $\bar{x}, \bar{y}, \bar{z} \in M^2_\kappa$ is defined by $[\bar{x}, \bar{y}] \cup [\bar{y}, \bar{z}] \cup [\bar{z}, \bar{x}]$ with $d(x, y) = d_{M^2_\kappa}(\bar{x}, \bar{y})$, $d(y, z) = d_{M^2_\kappa}(\bar{y}, \bar{z})$ and $d(z, x) = d_{M^2_\kappa}(\bar{z}, \bar{x})$. It is denoted by $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z}) \subset M^2_\kappa$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a *comparison point* for $p \in [x, y]$ if $d(x, p) = d_{M^2_\kappa}(\bar{x}, \bar{p})$. If for all $p, q \in \triangle(x, y, z)$ and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$, the inequality $d(p, q) \le d_{M^2_\kappa}(\bar{p}, \bar{q})$ holds for all triangles in X, then we call X a $\operatorname{CAT}(\kappa)$ space. We call X an *admissible* $\operatorname{CAT}(\kappa)$ space if X is a $\operatorname{CAT}(\kappa)$ space and $d(x, y) < D_\kappa/2$ for every $x, y \in X$. A nonempty subset C of X is called *κ-bounded* if $\sup_{x,y\in C} d(x, y) < D_\kappa/2$. Let $\{x_n\} \subset X$. If

$$\inf_{y\in X} \limsup_{n\to\infty} d(x_n, y) < \frac{D_\kappa}{2},$$

a sequence $\{x_n\}$ is called a *κ-bounded sequence*. In general, for real numbers κ, κ' with $\kappa' < \kappa$, a CAT(κ') space is a CAT(κ) space; see [5].

We define a function c_κ of $\mathbb{R}$ into itself by

$$c_\kappa(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\kappa^{n-1}t^{2n}}{(2n)!} = \begin{cases} \dfrac{1-\cos(\sqrt{\kappa}t)}{\kappa} & (\kappa > 0); \\ \dfrac{t^2}{2} & (\kappa = 0); \\ \dfrac{\cosh(\sqrt{-\kappa}t)-1}{-\kappa} & (\kappa < 0) \end{cases}$$

for $t \in \mathbb{R}$. Then it follows that

$$c_\kappa'(t) = \begin{cases} \dfrac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}} & (\kappa > 0); \\ t & (\kappa = 0); \\ \dfrac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases}$$

and

$$c_\kappa''(t) = \begin{cases} \cos(\sqrt{\kappa}t) & (\kappa > 0); \\ 1 & (\kappa = 0); \\ \cosh(\sqrt{-\kappa}t) & (\kappa < 0) \end{cases}$$

for $t \in \mathbb{R}$. Put $\phi_\kappa(x, y) = c_\kappa(d(x, y))$. Let $t \in [0, 1]$ and $d \in [0, D_\kappa[$. We define $(t)_d^\kappa \in [0, 1]$ and $A_\kappa \in \,]0, \infty]$ by

$$(t)_d^\kappa = \begin{cases} \dfrac{c_\kappa'(td)}{c_\kappa'(d)} & (d \in \,]0, D_\kappa[); \\ t & (d = 0) \end{cases}$$

and

$$A_\kappa = \begin{cases} \dfrac{1}{\kappa} & (\kappa > 0); \\ \infty & (\kappa \leq 0); \end{cases}$$

Then the following conditions hold:

- $c_\kappa(\cdot)\colon [0, D_\kappa[\,\to [0, A_\kappa[$ is continuous, convex, increasing and bijective;
- $c_\kappa^{-1}(\cdot)\colon [0, A_\kappa[\,\to [0, D_\kappa[$ is continuous and increasing;
- c_κ' is increasing;
- $c_\kappa(0) = c_\kappa'(0) = 0$ and $c_\kappa''(0) = 1$;
- $c_\kappa''(x) > 0$ for all $x \in [0, D_\kappa/2[$;
- $1 - \kappa c_\kappa(a) = c_\kappa''(a)$ for all $a \in [0, D_\kappa/2[$;
- $c_\kappa(a) = 2c_\kappa'(a/2)^2$ for all $a \in [0, D_\kappa/2[$;
- $c_\kappa'(a) + c_\kappa'(b) = 2c_\kappa'((a+b)/2)c_\kappa''((a-b)/2)$ for all $a, b \in [0, D_\kappa/2[$;
- $c_\kappa'(a) - c_\kappa'(b) = 2c_\kappa''((a+b)/2)c_\kappa'((a-b)/2)$ for all $a, b \in [0, D_\kappa/2[$;
- $c_\kappa'(2a) = 2c_\kappa'(a)c_\kappa''(a)$ for all $a \in [0, D_\kappa/2[$;

see [19, 33].

Lemma 2.1 (Kimura and Sudo [33]). *Let $\kappa \in \mathbb{R}$. Then,*

$$1 - \frac{1}{2(1/2)_d^\kappa} = \kappa c_\kappa\left(\frac{d}{2}\right)$$

for each $d \in [0, D_\kappa[$.

Lemma 2.2 (Kimura and Sudo [33]). *Let $\kappa \in \mathbb{R}$ and $t \in [0,1]$. Then,*

$$\lim_{t \searrow 0} \frac{1 - (1-t)_d^\kappa}{(t)_d^\kappa} = c_\kappa''(d)$$

for each $d \in [0, D_\kappa[$.

Theorem 2.1 (Bridson and Haeflinger [5], Kimura and Sudo [33]). *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then*

$$\begin{aligned}\phi_\kappa(tx \oplus (1-t)y, z) \le (t)_d^\kappa \phi_\kappa(x,z) + (1-t)_d^\kappa \phi_\kappa(y,z) \\ - (t)_d^\kappa \phi_\kappa(x, tx \oplus (1-t)y) - (1-t)_d^\kappa \phi_\kappa(y, tx \oplus (1-t)y)\end{aligned}$$

holds for $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ and $t \in [0,1]$, where $d = d(x,y)$.

Writing this theorem for each curvature, we get the following:

- if $\kappa = 1$, then
$$\cos d(tx \oplus (1-t)y, z) \sin d(x,y) \ge \cos d(x,z) \sin td(x,y) + \cos d(y,z) \sin(1-t)d(x,y);$$
- if $\kappa = 0$, then
$$d(tx \oplus (1-t)y, z)^2 \le td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2;$$
- if $\kappa = -1$, then
$$\cosh d(tx \oplus (1-t)y, z) \sinh d(x,y) \le \cosh d(x,z) \sinh td(x,y) + \cosh d(y,z) \sinh(1-t)d(x,y).$$

Corollary 2.1. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$\phi_\kappa(tx \oplus (1-t)y, z) \le t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z)$$

holds for $x, y, z \in X$ and $t \in [0,1]$.

Lemma 2.3 (Bačák [3]). *Let X be a* CAT(0) *space and $x_1, x_2, x_3, x_4 \in X$. Then*

$$d(x_1,x_2)^2 + d(x_3,x_4)^2 \le d(x_2,x_3)^2 + d(x_4,x_1)^2 + 2d(x_1,x_3)d(x_2,x_4)$$

holds.

Lemma 2.4 (Kajimura and Kimura [17]). *Let X be a complete* CAT(-1) *space. Then,*

$$\cosh d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \cosh \frac{d(x,y)}{2} \le \frac{1}{2}\cosh d(x,z) + \frac{1}{2}\cosh d(y,z)$$

for $x, y, z \in X$.

Let X be a geodesic space and f a function of X into $\mathbb{R}$. Then, $\operatorname{Argmax}_{x \in X} f(x)$ is the set of all maximizers of f and $\operatorname{Argmin}_{x \in X} f(x)$ is the set of all minimizers of f.

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and T a mapping of X into itself. Then, a mapping T is *nonexpansive* if for every $x, y \in X$, the inequality $d(Tx, Ty) \le d(x,y)$ holds. We call a mapping T *quasinonexpansive mapping* if $\operatorname{Fix} T \ne \emptyset$ and the inequality $d(Tx, z) \le d(x,z)$

holds for all $x \in X$ and $z \in \operatorname{Fix} T$. Further, we call a mapping T *strongly quasinonexpansive* if T is quasinonexpansive, and for every $z \in \operatorname{Fix} T$ and $\{x_n\} \subset X$ satisfying $\sup_{n \in \mathbb{N}} d(x_n, z) < D_\kappa/2$ and $\lim_{n\to\infty}(d(x_n, z) - d(Tx_n, z)) = 0$, it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. A mapping T is said to be *delta-demiclosed* if for every $\{x_n\} \subset X$ satisfying $x_n \stackrel{\Delta}{\rightharpoonup} x_0 \in X$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, it follows that $x_0 \in \operatorname{Fix} T$. A mapping T satisfies *condition (D)* if for every $\{x_n\} \subset$ satisfying $x_n \to x_0 \in X$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, it implies $x_0 \in \operatorname{Fix} T$.

Lemma 2.5. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$, and T a quasinonexpansive of X into itself. Then,* $\operatorname{Fix} T$ *is closed and convex.*

Proof. We first show $\operatorname{Fix} T$ is closed. Let $\{x_n\} \subset \operatorname{Fix} T$ with $x_n \to x_0 \in X$. Since T is quasinonexpansive, we get $d(x_n, Tx_0) \le d(x_n, x_0)$. Letting $n \to \infty$, we have $x_0 \in \operatorname{Fix} T$. Next, we show $\operatorname{Fix} T$ is convex. Let $x, y \in \operatorname{Fix} T$ and $t \in [0, 1]$. Put $w = tx \oplus (1-t)y$ and $d = d(x, y)$. Then, we get

$$\begin{aligned} \phi_\kappa(w, Tw) &\le (t)_d^\kappa \phi_\kappa(x, Tw) + (1-t)_d^\kappa \phi_\kappa(y, Tw) - (t)_d^\kappa \phi_\kappa(x, w) - (1-t)_d^\kappa \phi_\kappa(y, w) \\ &\le (t)_d^\kappa \phi_\kappa(x, w) + (1-t)_d^\kappa \phi_\kappa(y, w) - (t)_d^\kappa \phi_\kappa(x, w) - (1-t)_d^\kappa \phi_\kappa(y, w) \\ &= 0 \end{aligned}$$

and hence $w \in \operatorname{Fix} T$. This implies that $\operatorname{Fix} T$ is convex. Consequently, we complete the proof. □

Lemma 2.6 (Dhompongsa, Kirk and Sims [9], Espínola and Fernández-León [10])**.** *Let X be a complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $\{x_n\}$ a κ-bounded sequence of X. Then the asymptotic center of $\{x_n\}$ consists of one point.*

Lemma 2.7 (Bačák [3], Espínola, and Fernández–León [10])**.** *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $\{x_n\}$ a κ-bounded sequence of X. Then, $\{x_n\}$ has a delta-convergent subsequence.*

Lemma 2.8 (Kirk and Panyanak [35], Espínola, Fernández and León [10])**.** *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $\{x_n\} \subset X$ a κ-bounded sequence. If $x_n \stackrel{\Delta}{\rightharpoonup} x_0 \in X$, then*

$$x_0 \in \bigcap_{k=1}^{\infty} \operatorname{clco}\{x_k, x_{k+1}, \dots\}.$$

Lemma 2.9 (He, Fang, López and Li [13])**.** *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, $\{x_n\} \subset X$ a κ-bounded sequence such that $x_n \stackrel{\Delta}{\rightharpoonup} x_0 \in X$. Then*

$$d(x_0, z) \le \liminf_{n\to\infty} d(x_n, z)$$

holds for $z \in X$.

Lemma 2.10 (Kimura [20], Kimura and Satô [30])**.** *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $\{x_n\}$ a sequence of X such that $x_n \stackrel{\Delta}{\rightharpoonup} x_0 \in X$ and $\lim_{n\to\infty} d(x_n, z) = d(x_0, z)$ for some $z \in X$. Then, $x_n \to x_0$.*

In the following a theorem and lemmas, we show that there exists a metric projection in a CAT(κ) space for $\kappa \in \mathbb{R}$. Further, we consider its properties.

Theorem 2.2. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, C a nonempty closed convex subset of X. Then, there exists a unique point $p_x \in C$ such that $\{p_x\} = \operatorname{Argmin}_{y \in C} d(x, y)$.*

Proof. Let $x \in X$ and put $M = \inf_{y \in C} \phi_\kappa(x, y)$. Let $\{y_n\} \subset C$ with

$$\phi_\kappa(x, y_n) \le M + \frac{1}{n}.$$

Then, $\lim_{n\to\infty} \phi_\kappa(x, y_n) = M$. Let $m, n \in \mathbb{N}$ with $n \le m$ and put $d = d(y_m, y_n)$. By Theorem 2.1, we get

$$\begin{aligned}
M &\le \phi_\kappa\left(x, \frac{1}{2}y_m \oplus \frac{1}{2}y_n\right) \\
&\le (1/2)_d^\kappa \phi_\kappa(x, y_m) + (1/2)_d^\kappa \phi_\kappa(x, y_n) - (1/2)_d^\kappa \phi_\kappa\left(y_m, \frac{1}{2}y_m \oplus \frac{1}{2}y_n\right) - (1/2)_d^\kappa \phi_\kappa\left(y_n, \frac{1}{2}y_m \oplus \frac{1}{2}y_n\right) \\
&= (1/2)_d^\kappa \phi_\kappa(x, y_m) + (1/2)_d^\kappa \phi_\kappa(x, y_n) - 2(1/2)_d^\kappa c_\kappa\left(\frac{d}{2}\right) \\
&\le (1/2)_d^\kappa\left(M + \frac{1}{m}\right) + (1/2)_d^\kappa\left(M + \frac{1}{n}\right) - 2(1/2)_d^\kappa c_\kappa\left(\frac{d}{2}\right) \\
&\le 2(1/2)_d^\kappa\left(M + \frac{1}{n}\right) - 2(1/2)_d^\kappa c_\kappa\left(\frac{d(y_m, y_n)}{2}\right)
\end{aligned}$$

and hence

$$c_\kappa\left(\frac{d}{2}\right) \le M + \frac{1}{n} - \frac{1}{2(1/2)_d^\kappa}M = \left(1 - \frac{1}{2(1/2)_d^\kappa}\right)M + \frac{1}{n}.$$

By Lemma 2.1, we get

$$c_\kappa\left(\frac{d(y_m, y_n)}{2}\right) \le \kappa c_\kappa\left(\frac{d(y_m, y_n)}{2}\right)M + \frac{1}{n}$$

and hence

$$(1 - \kappa M)c_\kappa\left(\frac{d(y_m, y_n)}{2}\right) \le \frac{1}{n}.$$

Therefore, it follows that

$$d(y_m, y_n) \le 2c_\kappa^{-1}\left(\frac{1}{(1 - \kappa M)n}\right)$$

and thus $\{y_n\}$ is a Cauchy sequence. Since X is complete and C is closed, there exists p_x such that $y_n \to p_x \in C$. Then, $M = \lim_{n\to\infty} \phi_\kappa(x, y_n) = \phi_\kappa(x, p_x)$. Let $q_x \in C$ with $M = \phi_\kappa(x, q_x)$. Put $d' = d(p_x, q_x)$. Then, we obtain

$$\begin{aligned}
M = \phi_\kappa(x, p_x) &\le \phi_\kappa\left(x, \frac{1}{2}p_x \oplus \frac{1}{2}q_x\right) \\
&\le (1/2)_{d'}^\kappa \phi_\kappa(x, p_x) + (1/2)_{d'}^\kappa \phi_\kappa(x, q_x) \\
&\quad - (1/2)_{d'}^\kappa \phi_\kappa\left(p_x, \frac{1}{2}p_x \oplus \frac{1}{2}q_x\right) - (1/2)_{d'}^\kappa\left(q_x, \frac{1}{2}p_x \oplus \frac{1}{2}q_x\right) \\
&= (1/2)_{d'}^\kappa \phi_\kappa(x, p_x) + (1/2)_{d'}^\kappa \phi_\kappa(x, q_x) - 2(1/2)_{d'}^\kappa c_\kappa\left(\frac{d(p_x, q_x)}{2}\right) \\
&= 2(1/2)_{d'}^\kappa M - 2(1/2)_{d'}^\kappa c_\kappa\left(\frac{d(p_x, q_x)}{2}\right)
\end{aligned}$$

and thus

$$c_\kappa\left(\frac{d(p_x, q_x)}{2}\right) \le \left(1 - \frac{1}{2(1/2)_{d'}^\kappa}\right)M = \kappa M c_\kappa\left(\frac{d(p_x, q_x)}{2}\right)$$

and hence

$$(1 - \kappa M)c_\kappa\left(\frac{d(p_x, q_x)}{2}\right) \le 0.$$

Since $1 - \kappa M > 0$, we get $p_x = q_x$. Consequently, we complete the proof. □

We define the *metric projection P_C of X onto C* by

$$P_C x = p_x = \operatorname*{Argmin}_{y \in C} d(x, y)$$

for each $x \in X$.

Lemma 2.11. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, C a nonempty closed convex subset of X, $x \in X$ and $y \in C$. Then*

$$\phi_\kappa(P_C x, y) + c''_\kappa(d(y, P_C x))\phi_\kappa(x, P_C x) \leq \phi_\kappa(x, y)$$

holds.

Proof. Let $x \in X$, $y \in C$ and $t \in]0, 1[$. Put $d = d(y, P_C x)$. By Theorem 2.1, we get

$$\begin{aligned}
\phi_\kappa(x, P_C x) &\leq \phi_\kappa(x, ty \oplus (1-t)P_C x) \\
&\leq (t)_d^\kappa \phi_\kappa(x, y) + (1-t)_d^\kappa \phi_\kappa(x, P_C x) - (t)_d^\kappa (y, ty \oplus (1-t)P_C x) \\
&\quad - (1-t)_d^\kappa (P_C x, ty \oplus (1-t)P_C x) \\
&\leq (t)_d^\kappa \phi_\kappa(x, y) + (1-t)_d^\kappa \phi_\kappa(x, P_C x) - (t)_d^\kappa (y, ty \oplus (1-t)P_C x)
\end{aligned}$$

and hence

$$\left(1 - (1-t)_d^\kappa\right) \phi_\kappa(x, P_C x) \leq (t)_d^\kappa \phi_\kappa(x, y) - (t)_d^\kappa \phi_\kappa(y, ty \oplus (1-t)P_C x).$$

Dividing by $(t)_d^\kappa > 0$, we have

$$\frac{1 - (1-t)_d^\kappa}{(t)_d^\kappa} \phi_\kappa(x, P_C x) \leq \phi_\kappa(x, y) - \phi_\kappa(y, ty \oplus (1-t)P_C x).$$

By Lemma 2.2, letting $t \searrow 0$, we have

$$c''_\kappa(d(y, P_C x))\phi_\kappa(x, P_C x) \leq \phi_\kappa(x, y) - \phi_\kappa(y, P_C x)$$

and we get the desired result. □

Writing this lemma for each curvature, we obtain the following:

- if $\kappa > 0$,
$$\cos d(x, P_C x) \cos d(P_C x, y) \geq \cos d(x, y);$$
- if $\kappa = 0$,
$$d(x, P_C x)^2 + d(P_C x, y)^2 \leq d(x, y)^2;$$
- if $\kappa < 0$,
$$\cosh d(x, P_C x) \cosh d(P_C x, y) \leq \cosh d(x, y).$$

Lemma 2.12. *Let X be a complete* CAT(0) *space and C a nonempty closed convex subset of X. Then, the metric projection P_C of X onto C is nonexpansive.*

Proof. Let $x, y \in X$. If $x = y$, we get

$$d(P_C x, P_C y) \leq d(x, y)$$

obviously. Suppose $x \neq y$. Then, we get

$$d(x, P_C x)^2 + d(P_C x, P_C y)^2 \leq d(x, P_C y)^2$$

for $x, y \in X$. Similarly, we get

$$d(y, P_C y)^2 + d(P_C y, P_C x)^2 \le d(y, P_C x)^2.$$

Adding these inequalities and using Lemma 2.3, we get

$$\begin{aligned} 2d(P_C x, P_C y)^2 &\le d(x, P_C y)^2 + d(P_C x, y)^2 - d(x, P_C x)^2 - d(y, P_C y)^2 \\ &\le 2d(P_C x, P_C y) d(x, y) \end{aligned}$$

and hence $d(P_C x, P_C y) \le d(x, y)$. Therefore, P_C is nonexpansive. Then, we get the desired result. □

Lemma 2.13. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and C a nonempty closed convex subset of X. Then, the metric projection P_C is strongly quasinonexpansive and delta-demiclosed.*

Proof. Let $x \in X$ and $p \in C$. Then, we get

$$0 \le c''_\kappa(d(p, P_C x))\phi_\kappa(x, P_C x) \le \phi_\kappa(x, p) - \phi_\kappa(P_C x, p)$$

and thus $d(P_C x, p) \le d(x, p)$. This implies that P_C is quasinonexpansive. Let $\{x_n\} \subset X$ with $\sup_{n \in \mathbb{N}} d(x_n, p) < D_\kappa/2$ and

$$\lim_{n \to \infty} (d(x_n, p) - d(P_C x_n, p)) = 0.$$

By Lemma 2.11, we get

$$0 \le c''_\kappa(d(p, P_C x_n))\phi_\kappa(x_n, P_C x_n) \le \phi_\kappa(x_n, p) - \phi_\kappa(P_C x_n, p).$$

Since $c''_\kappa(d(p, P_C x_n)) > 0$, letting $n \to \infty$, we get $\lim_{n \to \infty} d(x_n, P_C x_n) = 0$. This implies that P_C is strongly quasinonexpansive. We next show P_C is delta-demiclosed. Let $\{x_n\} \subset X$ with $x_n \overset{\Delta}{\rightharpoonup} x_0 \in X$ and $\lim_{n \to \infty} d(x_n, P_C x_n) = 0$. Then, it follows that $P_C x_n \overset{\Delta}{\rightharpoonup} x_0 \in X$. By Lemma 2.8, we get $x_0 \in C = \operatorname{Fix} P_C$ and thus P_C is delta-demiclosed. Consequently, we complete the proof. □

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$. We define *κ-convex combination* by

$$\begin{aligned} \alpha x \overset{\kappa}{\oplus} (1-\alpha) y &= \operatorname*{Argmin}_{u \in X} \{\alpha \phi_\kappa(x, u) + (1-\alpha)\phi_\kappa(y, u)\} \\ &= \begin{cases} \dfrac{1}{d(x,y)} t_\kappa^{-1}\left(\dfrac{\alpha c'_\kappa(d(x,y))}{1-\alpha+\alpha c''_\kappa d(x,y)}\right) \oplus \dfrac{1}{d(x,y)} t_\kappa^{-1}\left(\dfrac{(1-\alpha) c'_\kappa(d(x,y))}{\alpha+(1-\alpha) c''_\kappa(d(x,y))}\right) & (x \ne y); \\ x & (x = y) \end{cases} \end{aligned}$$

for $x, y \in X$ with $d(x, y) < D_\kappa/2$ and $\alpha \in [0, 1]$, where

$$t_\kappa(a) = \begin{cases} \dfrac{\tan(\sqrt{\kappa} a)}{\sqrt{\kappa}} & (\kappa > 0); \\ a & (\kappa = 0); \\ \dfrac{\tanh(\sqrt{-\kappa} a)}{\sqrt{-\kappa}} & (\kappa < 0). \end{cases}$$

Then, we know that $\alpha u \overset{\kappa}{\oplus} (1-\alpha) v = tu \oplus (1-t) v$, where

$$\alpha = \frac{c'_\kappa(td(u,v))}{c'_\kappa(td(u,v)) + c'_\kappa((1-t)d(u,v))};$$

see [26, 27, 33].

Lemma 2.14 (Kimura and Sasaki [26], Kimura and Sasaki [27], Kimura and Sudo [33]). *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \le t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z)$$

holds for $x, y, z \in X$ and $t \in [0,1]$.

Lemma 2.15 (Kimura and Sasaki [26]). *Let X be an admissible* CAT(1) *space. Then for $u, y, z \in X$ and $\alpha \in]0,1[$, the inequality*

$$\begin{aligned}
&1 - \cos d(\alpha u \overset{1}{\oplus} (1-\alpha)y, z) \\
&\le (1-\beta)(1 - \cos d(y,z)) \\
&\quad + \beta\left(1 - \frac{\left(1 - \alpha + \sqrt{\alpha^2 + 2\alpha(1-\alpha)\cos d(u,y) + (1-\alpha)^2}\right)\cos d(u,z)}{\alpha + 2(1-\alpha)\cos d(u,y)}\right)
\end{aligned}$$

holds where

$$\beta = 1 - \frac{1-\alpha}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cos d(u,y) + (1-\alpha)^2}}.$$

Lemma 2.16 (Kimura and Sasaki [27]). *Let X be a* CAT(-1) *space. Then for $u, y, z \in X$ and $\alpha \in]0,1[$, the inequality*

$$\begin{aligned}
&\cosh d(\alpha u \overset{-1}{\oplus} (1-\alpha)y, z) - 1 \\
&\le (1-\beta)(\cosh d(y,z) - 1) \\
&\quad + \beta\left(\frac{\left(1 - \alpha + \sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(u,y) + (1-\alpha)^2}\right)\cosh d(u,z)}{\alpha + 2(1-\alpha)\cosh d(u,y)} - 1\right)
\end{aligned}$$

holds where

$$\beta = 1 - \frac{1-\alpha}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(u,y) + (1-\alpha)^2}}.$$

The following lemmas are important to prove convergence theorems.

Lemma 2.17 (Aoyama, Kimura, Takahashi and Toyoda [2]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of $[0,1]$ with $\sum_{n=1}^\infty \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^\infty u_n < \infty$ and $\{t_n\}$ a real numbers with $\limsup_{n\to\infty} t_n \le 0$. Suppose that $s_{n+1} \le (1-\alpha_n)s_n + \alpha_n t_n + u_n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_n = 0$.*

Lemma 2.18 (Xu [46]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that $a_{n+1} \le a_n + b_n$ for all $n \in \mathbb{N}$. If $\sum_{k=1}^\infty b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.19 (Aoyama, Kimura and Kohsaka [1], Saejung and Yotkaew [40]). *Let $\{a_n\} \subset [0,\infty[$, $\{t_n\} \subset \mathbb{R}$ and $\{\beta_n\} \subset]0,1[$ such that $\sum_{n=1}^\infty \beta_n = \infty$. Suppose that*

$$a_{n+1} \le (1-\beta_n)a_n + \beta_n t_n$$

for all $n \in \mathbb{N}$. If $\liminf_{i\to\infty}\left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \ge 0$ implies $\limsup_{i\to\infty} t_{\varphi(i)} \le 0$ for all nondecreasing function φ of $\mathbb{N}$ into itself such that $\lim_{i\to\infty}\varphi(i) = \infty$. Then, $\lim_{n\to\infty} a_n = 0$.

Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$, $\{C_n\} \subset 2^X$ a sequence of nonempty closed convex subset of X. Define $x \in d\text{-}\mathrm{Li}_n C_n$ by there exists $\{x_n\} \subset X$ such that $x_n \to x$ and $x_n \in C_n$ for all $n \in \mathbb{N}$ and define $y \in \Delta_\kappa\text{-}\mathrm{Ls}_n C_n$ by there exists $\{y_i\} \subset X$ and $\{n_i\} \subset \mathbb{N}$ such that

$$\inf_{x\in X}\limsup_{i\to\infty} d(y_i, x) < \frac{D_\kappa}{2},$$

$\mathrm{AC}(\{y_i\}) = \{y\}$ and $y_i \in C_{n_i}$ for $i \in \mathbb{N}$. If a subset C_0 of X satisfies that

$$d\text{-}\mathrm{Li}_n C_n = C_0 = \Delta_\kappa\text{-}\mathrm{Ls}_n C_n,$$

it is said that $\{C_n\}$ is Δ_κ-*Mosco convergent to* C_0; see details [20, 30]. The following theorem is important to prove a convergence theorem with the shrinking projection method.

Theorem 2.3 (Kimura [20], Kimura and Satô [30])**.** *Let X be an admissible complete* $\mathrm{CAT}(\kappa)$ *space for $\kappa \in \mathbb{R}$, $\{C_n\} \subset 2^X$ a sequence of nonempty closed convex subset of X such that $C_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$. Then, $\{C_n\}$ is Δ_κ-Mosco convergent to C_0 if and only if $\{P_{C_n}x\}$ is convergent to $P_{C_0}x$ for each $x \in X$, where P_K is the metric projection of X onto a nonempty closed convex subset K of X.*

Theorem 2.4 (Kimura [20], Kimura and Satô [30])**.** *Let X be an admissible complete* $\mathrm{CAT}(\kappa)$ *space for $\kappa \in \mathbb{R}$, $\{C_n\} \subset 2^X$ a sequence of nonempty closed convex subset of X such that $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$ and $C_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$. Then, $\{C_n\}$ is Δ_κ-Mosco convergent to C_0.*

Chapter 3

Resolvent operators

In this chapter, we introduce a resolvent for convex function in a CAT(−1) space. Further, we introduce an equilibrium problem in a CAT(−1) space having the convex hull finite property.

Let X be a geodesic space and f a function of X into $]-\infty, \infty]$. A function f is *proper* if the set $\{x \in X \mid f(x) < \infty\}$ is nonempty. A function f is said to be *lower semicontinuous* if the set $\{x \in X \mid f(x) \le a\}$ is closed for all $a \in \mathbb{R}$. If f is continuous, then it is lower semicontinuous. A function f is said to be *convex* if

$$f(\alpha x \oplus (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y)$$

holds for all $x, y \in X$ and $\alpha \in]0,1[$.

Lemma 3.1 (Mayer [36])**.** *Let X be a complete* CAT(0) *space and f a lower semicontinuous convex function from X into $\mathbb{R}$. Then there exists a nonnegative real number c which depends on only f such that*

$$\liminf_{d(u,v)\to\infty} \frac{f(u)}{d(u,v)} \ge -c$$

for all $v \in X$.

Lemma 3.2 (Kajimura and Kimura [17])**.** *Let X be a complete* CAT(0) *space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Suppose that $f(x) \to \infty$ wherever $d(x,p) \to \infty$ for some $p \in X$. Then $\operatorname{Argmin}_X f$ is nonempty. If*

$$f\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) < \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

holds for all $x, y \in X$ with $x \ne y$, then $\operatorname{Argmin}_X f$ consists of one point.

Let X be a complete CAT(0) space and C a subset of X. Then C is said to be *delta-compact* if every net $\{x_\alpha\}$ of C has a subnet delta-converging to a point in C. It is known that every bounded closed convex set in CAT(0) space is delta-compact; see [35]. Let E be a subset of X. Then *a convex hull of E* is defined by

$$\operatorname{co} E = \bigcup_{n=0}^{\infty} X_n,$$

where $X_0 = E$ and $X_n = \{t u_{n-1} \oplus (1-t) v_{n-1} \mid u_{n-1}, v_{n-1} \in X_{n-1}, t \in [0,1]\}$. A complete CAT(0) space has *the convex hull finite property* if every continuous mapping T of $\operatorname{clco} E$ into itself has a fixed point for all finite subsets E of X, where $\operatorname{clco} E$ is the closure of $\operatorname{co} E$; see [42].

The following lemmas show that the KKM lemma holds on a complete CAT(0) space having the convex hull finite property.

Lemma 3.3 (Niculescu and Rovenţa [38])**.** *Let X be a complete* CAT(0) *space having the convex hull finite property and C a nonempty subset of X. Suppose that for every $x \in C$, there exists a closed subset $M(x)$ of C such that $\operatorname{co} E \subset \bigcup_{x\in E} M(x)$ for all finite subsets E of C. Then $\bigcap_{i=1}^{n} M(y_i)$ is nonempty for every finite subset $\{y_1, y_2, \ldots, y_n\}$ of X.*

Lemma 3.4 (Kimura and Kishi [23]). *Let X be a complete* CAT(0) *space having the convex hull finite property and C a delta-compact subset of X. Then if $\{F_i\}_{i\in I}$ is a family of delta-closed subsets of C with the finite intersection property, then $\bigcap_{i\in I} F_i$ is nonempty.*

3.1 Convex function

Lemma 3.5. *Let X be a complete* CAT(−1) *space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Then the function g from X into $]-\infty, \infty]$ defined by*

$$g(x) = f(x) + \cosh d(x, p)$$

for $x \in X$ is convex for all $p \in X$.

Proof. By the assumption of f, we know that f is convex. We show

$$\cosh d(\cdot, p)$$

is convex for all $p \in X$. Let $t \in \mathbb{R}$ and $t \geq 0$. Then, we have

$$(\cosh t)'' = \cosh t.$$

Thus $\cosh t$ is convex for $t \geq 0$. Let $x, y \in X$ and $\alpha \in]0, 1[$. Since $\cosh t$ is increasing for $t \geq 0$, and $d(\cdot, p)$ and $\cosh t$ are convex, we get

$$\begin{aligned}\cosh d(\alpha x \oplus (1-\alpha)y, p) &\leq \cosh(\alpha d(x, p) + (1-\alpha)d(y, p))\\ &\leq \alpha \cosh d(x, p) + (1-\alpha)\cosh d(y, p).\end{aligned}$$

Then, it follows that

$$\begin{aligned}g(\alpha x \oplus (1-\alpha)y) &= f(\alpha x \oplus (1-\alpha)y) + \cosh d(\alpha x \oplus (1-\alpha)y, p)\\ &\leq \alpha f(x) + (1-\alpha)f(y) + \alpha \cosh d(x, p) + (1-\alpha)\cosh d(y, p)\\ &= \alpha g(x) + (1-\alpha)g(y)\end{aligned}$$

for all $p \in X$. Thus g is convex. Then, we complete the proof. □

Lemma 3.6. *Let X be a complete* CAT(−1) *space and f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$. For $p \in X$, define a function $g\colon X \to]-\infty, \infty]$ by*

$$g(\cdot) = f(\cdot) + \cosh d(\cdot, p).$$

Then the inequality

$$g\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) < \frac{1}{2}g(x) + \frac{1}{2}g(y)$$

holds wherever $x \neq y$.

Proof. Let $x, y \in X$ with $x \neq y$ and let $p \in X$ with $p \neq \frac{1}{2}x \oplus \frac{1}{2}y$. By Lemma 2.4, we have

$$\begin{aligned}g\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) &= f\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) + \cosh d\left(\frac{1}{2}x \oplus \frac{1}{2}y, p\right)\\ &\leq \frac{1}{2}f(x) + \frac{1}{2}f(y) + \cosh d\left(\frac{1}{2}x \oplus \frac{1}{2}y, p\right)\\ &< \frac{1}{2}f(x) + \frac{1}{2}f(y) + \cosh d\left(\frac{1}{2}x \oplus \frac{1}{2}y, p\right)\cosh\frac{d(x, y)}{2}\\ &\leq \frac{1}{2}f(x) + \frac{1}{2}f(y) + \frac{1}{2}\cosh d(x, p) + \frac{1}{2}\cosh d(y, p)\end{aligned}$$

$$= \frac{1}{2}g(x) + \frac{1}{2}g(y).$$

If $p = \frac{1}{2}x \oplus \frac{1}{2}y$, then it is obvious to hold the inequality of the conclusion. Then, we get the desired result. □

Theorem 3.1. *Let X be a complete* CAT(-1) *space and f a proper lower semicontinuous convex function of X into $]-\infty,\infty]$. For $p \in X$, define a function $g\colon X \to]-\infty,\infty]$ by*

$$g(\cdot) = f(\cdot) + \cosh d(\cdot, p).$$

Then $\mathrm{Argmin}_X g$ *consists of one point.*

Proof. By Lemma 3.5, g is a proper lower semicontinuous convex function of X into $]-\infty,\infty]$. From Lemma 3.1, we get

$$\begin{aligned}
\liminf_{d(x,p)\to\infty} \frac{g(x)}{d(x,p)} &= \liminf_{d(x,p)\to\infty} \frac{f(x) + \cosh d(x,p)}{d(x,p)} \\
&\geq \liminf_{d(x,p)\to\infty} \frac{f(x)}{d(x,p)} + \liminf_{d(x,p)\to\infty} \frac{\cosh d(x,p)}{d(x,p)} \\
&\geq -c + \liminf_{d(x,p)\to\infty} \frac{\cosh d(x,p)}{d(x,p)}
\end{aligned}$$

and hence $g(x) \to \infty$ as $d(x,p) \to \infty$. By Lemma 3.2 and Lemma 3.6, $\mathrm{Argmin}_X g$ consists of one point. Consequently, we complete the proof. □

Let f be a proper lower semicontinuous convex function of X into $]-\infty,\infty]$. Then the resolvent L_f of f is defined by

$$L_f x = \operatorname*{Argmin}_{y\in X}\{f(y) + \cosh d(x,y)\}$$

for all $x \in X$. From Theorem 3.1, L_f is defined as a single-valued mapping. Let C a nonempty closed convex subset of X. If f is the indicator function i_C of C which is defined by

$$i_C(x) = \begin{cases} 0 & (x \in C); \\ \infty & (x \in X \setminus C). \end{cases}$$

Then we know L_{i_C} is the metric projection P_C onto C. Indeed, by the definition of i_C, we get

$$L_{i_C} x = \operatorname*{Argmin}_{y\in X}\{i_C(y) + \cosh d(x,y)\} = \operatorname*{Argmin}_{y\in C} \cosh d(x,y).$$

Since $\cosh t$ is increasing for all nonnegative real number t, we get the conclusion.

Theorem 3.2. *Let X be a complete* CAT(-1) *space, f a proper lower semicontinuous convex function of X into $]-\infty,\infty]$, and L_f a resolvent of f. Then the following conditions hold:*

(i) *the inequality*

$$(\cosh d(x, L_f x) + \cosh d(y, L_f y)) \cosh d(L_f x, L_f y) \leq \cosh d(L_f x, y) + \cosh d(x, L_f y)$$

holds for $x, y \in X$;

(ii) $\mathrm{Fix}\, L_f = \mathrm{Argmin}_X f$.

Proof. (i) Let $x, y \in X$. If $L_f x = L_f y$, we get the desired inequality obviously. Suppose $L_f x \neq L_f y$. Let $t \in]0,1[$ and put $z = tL_f x \oplus (1-t)L_f y$. By the definition of L_f and convexity of f, we get

$$f(L_f y) + \cosh d(y, L_f y) \leq f(z) + \cosh d(y,z) \leq tf(L_f x) + (1-t)f(L_f y) + \cosh d(y,z)$$

and hence

$$t(f(L_f y) - f(L_f x)) \le \cosh d(y,z) - \cosh d(y, L_f y).$$

Putting $D = d(L_f x, L_f y)$ and multipling $(\sinh D)/t$, we have

$$\begin{aligned}
&(f(L_f y) - f(L_f x)) \sinh D \\
&\le \frac{1}{t}(\cosh d(y,z) \sinh D - \cosh d(y, L_f y) \sinh D) \\
&\le \frac{1}{t}(\cosh d(y, L_f x) \sinh tD + \cosh d(y, L_f y) \sinh(1-t)D - \cosh d(y, L_f y) \sinh D) \\
&= \frac{1}{t}(\cosh d(y, L_f x) \sinh tD - \cosh d(y, L_f y)(\sinh D - \sinh(1-t)D)) \\
&= \frac{1}{t}\left(\cosh d(y, L_f x) \sinh tD - 2\cosh d(y, L_f y) \cosh\left(\left(1-\frac{t}{2}\right)D\right) \sinh\left(\frac{t}{2}D\right)\right) \\
&= \frac{2}{t} \sinh\left(\frac{t}{2}D\right)\left(\cosh d(y, L_f x) \cosh\left(\frac{t}{2}D\right) - \cosh d(y, L_f y) \cosh\left(\left(1-\frac{t}{2}\right)D\right)\right) \\
&= D\frac{2}{tD} \sinh\left(\frac{tD}{2}\right)\left(\cosh d(y, L_f x) \cosh\left(\frac{t}{2}D\right) - \cosh d(y, L_f y) \cosh\left(\left(1-\frac{t}{2}\right)D\right)\right).
\end{aligned}$$

Letting $t \searrow 0$, we get

$$(f(L_f y) - f(L_f x)) \sinh D \le D(\cosh d(y, L_f x) - \cosh d(y, L_f y) \cosh D). \tag{3.1}$$

Similarly, it holds that

$$(f(L_f x) - f(L_f y)) \sinh D \le D(\cosh d(L_f y, x) - \cosh d(x, L_f x) \cosh D). \tag{3.2}$$

Adding (3.1) and (3.2), we get

$$(\cosh d(x, L_f x) + \cosh d(y, L_f y)) \cosh d(L_f x, L_f y) \le \cosh d(L_f x, y) + \cosh d(x, L_f y).$$

(ii) Let $z \in \operatorname{Argmin}_X f$. Then we have

$$f(z) + \cosh d(z,z) = f(z) + 1 \le f(y) + 1 \le f(y) + \cosh d(y,z)$$

for all $y \in X$. Thus we get

$$f(z) + \cosh d(z,z) = \inf_{y \in X}(f(y) + \cosh d(y,z))$$

and hence $z = L_f z$. Therefore, this equality implies that $z \in \operatorname{Fix} L_f$. Inversely, let $z \in \operatorname{Fix} L_f$ and $t \in]0,1[$. Then we have

$$\begin{aligned}
f(z) &\le f(z) + \cosh d(z,z) \\
&= f(L_f z) + \cosh d(L_f z, z) \\
&\le f(ty \oplus (1-t)L_f z) + \cosh d(ty \oplus (1-t)L_f z, z) \\
&\le tf(y) + (1-t)f(L_f z) + \cosh d(ty \oplus (1-t)L_f z, z) \\
&= tf(y) + (1-t)f(z) + \cosh d(ty \oplus (1-t)z, z) \\
&= tf(y) + (1-t)f(z) + \cosh td(y,z)
\end{aligned}$$

and hence

$$f(z) \le f(y) + \frac{\cosh td(y,z)}{td(y,z)} d(y,z)$$

for all $y \in X \setminus \{z\}$. Letting $t \searrow 0$, we get $f(z) \le f(y)$ for all $y \in X \setminus \{z\}$. Therefore, this inequality implies that $z \in \operatorname{Argmin}_X f$. Then, we complete the proof. □

3.2 Equilibrium problem

Condition 3.1. Let X be a complete CAT(-1) space and K a nonempty closed convex subset of X. We suppose that a bifunction $f\colon K \times K \to \mathbb{R}$ satisfies the following conditions:

(i) $f(x,x) = 0$ for all $x \in K$;
(ii) $f(x,y) + f(y,x) \leq 0$ for all $x, y \in K$;
(iii) for every $x \in K$, $f(x,\cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex;
(iv) for every $y, z \in K$, $\limsup_{t \searrow 0} f(ty \oplus (1-t)z, y) \leq f(z,y)$.

The set of solutions to the equilibrium problem for f is denoted by $\operatorname{Equil} f$, that is,

$$\operatorname{Equil} f = \left\{ z \in K \;\middle|\; \inf_{y \in K} f(z,y) \geq 0 \right\}.$$

In the following a theorem and lemmas, we introduce a resolvent for equilibrium problems and prove its well-definedness in a complete CAT(-1) space having the convex hull finite property.

Lemma 3.7. *Let X be a complete* CAT(-1) *space having the convex hull finite property, K a nonempty closed convex subset of X, and C a nonempty delta-compact closed convex subset of K. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies* Condition 3.1. *Then, there exists $z \in C$ such that*

$$f(y,z) \leq \cosh d(x,y) - \cosh d(x,z)$$

for all $x \in X$ and $y \in C$.

Proof. For arbitrarily fixed $x \in X$, let

$$h(y,u) = f(y,u) + \cosh d(x,u) - \cosh d(x,y)$$

for $u, y \in K$. Put

$$M(y) = \{u \in C \mid h(y,u) \leq 0\}$$

for all $y \in C$. Then $M(y)$ is delta-closed. We show that $\bigcap_{y \in C} M(y)$ is nonempty. Let $\{y_i\}_{i \in N}$ be a subset of C with a finite index set N and I a nonempty subset of N. Put $D = \{y_i\}_{i \in I}$. Let $p \in \operatorname{co} D$ arbitrarily, where $\operatorname{co} D$ is defined by

$$\operatorname{co} D = \bigcup_{n=0}^{\infty} F_n$$

wherever $F_0 = D$ and $F_n = \{tu_{n-1} \oplus (1-t)v_{n-1} \mid u_{n-1}, v_{n-1} \in F_{n-1}, t \in [0,1]\}$ for $n \in \mathbb{N}$. We first show the following inequality by induction. For $n \in \mathbb{N} \cup \{0\}$, if $p \in F_n$, then there exists $\left\{\mu_j \in \mathbb{R} \;\middle|\; j \in I\right\}$ such that

$$h(y_i, p) \leq \sum_{j \in I} \mu_j h(y_i, y_j) \tag{3.3}$$

where $\sum_{j \in I} \mu_j = 1$ and $\mu_j \geq 0$. If $n = 0$, then for $p \in F_0$, we get $p = y_{i_0}$ for some $i_0 \in I$. Put

$$\mu_j = \begin{cases} 1 & (j = i_0); \\ 0 & (j \neq i_0). \end{cases}$$

Then, we get

$$h(y_i, p) = h(y_i, y_{i_0}) = \sum_{j \in I} \mu_j h(y_i, y_j).$$

Suppose (3.3) holds for fixed $n \in \mathbb{N}$. Suppose $p \in F_{n+1}$. Then there exists $u, v \in F_n$ and $t \in [0,1]$ such that $p = tu \oplus (1-t)v$ and we get

$$h(y_i, p) \le th(y_i, u) + (1-t)h(y_i, v).$$

Hence there exist $\lambda_i, \nu_i \in [0,1]$ such that

$$\begin{aligned} th(y_i, u) + (1-t)h(y_i, v) &\le t \sum_{j \in I} \lambda_j h(y_i, y_j) + (1-t) \sum_{j \in I} \nu_j h(y_i, y_j) \\ &= \sum_{j \in I} (t\lambda_j + (1-t)\nu_j) h(y_i, y_j) \end{aligned}$$

and $\sum_{j \in I} \lambda_j = \sum_{j \in I} \nu_j = 1$. Put $\mu_j = t\lambda_j + (1-t)\nu_j$. Then we have (3.3) holds for $n+1$. Therefore it holds for every $n \in \mathbb{N}$. Suppose that $h(y_i, p) > 0$ for all $i \in I$. Then we get

$$0 < \sum_{i \in I} \mu_i h(y_i, p) \le \sum_{i \in I} \sum_{j \in I} \mu_i \mu_j h(y_i, y_j) = \frac{1}{2} \sum_{i \in I} \sum_{j \in I} \mu_i \mu_j (h(y_i, y_j) + h(y_j, y_i)) \le 0$$

and this is a contradiction. We have there exists $i \in I$ such that $h(y_i, p) \le 0$ and $p \in M(y_i)$. Since $p \in \operatorname{co} D$ is arbitrary, we obtain $\operatorname{co} D \subset \bigcup_{i \in I} M(y_i)$. By Lemma 3.3, we get $\bigcap_{i \in N} M(y_i)$ is nonempty. Using Lemma 3.4, we obtain $\bigcap_{y \in C} M(y)$ is nonempty. Therefore we can take $z \in \bigcap_{y \in C} M(y)$, which satisfies the desired result. □

Lemma 3.8. *Let X be a complete* CAT(−1) *space having the convex hull finite property, K a nonempty closed convex subset of X, and C a nonempty delta-compact and closed convex subset of K. Suppose that $f: K \times K \to \mathbb{R}$ satisfies* Condition 3.1. *Then for $x \in X$ and $z \in C$, the following* (1) *and* (2) *are equivalent:*

(1) $f(y, z) \le \cosh d(x, y) - \cosh d(x, z)$ *for all* $y \in C$;
(2) $0 \le f(z, y) + \cosh d(x, y) - \cosh d(x, z)$ *for all* $y \in C$.

Proof. If the statement (2) holds, by Condition 3.1, we get $f(z, y) \le -f(y, z)$ and hence we get the statement (1). Inversely, suppose the statement (1) holds. Put $\tau = ty \oplus (1-t)z$ for $t \in]0,1[$. By Condition 3.1 and Lemma 3.7, we get

$$\begin{aligned} 0 &= f(\tau, \tau) \\ &\le tf(\tau, y) + (1-t)f(\tau, z) \\ &\le tf(\tau, y) + (1-t)(\cosh d(x, \tau) - \cosh d(x, z)) \\ &\le tf(\tau, y) + (1-t)(t \cosh d(x, y) + (1-t)\cosh d(x, z) - \cosh d(x, z)) \\ &\le tf(\tau, y) + (1-t)t(\cosh d(x, y) - \cosh d(x, z)). \end{aligned}$$

Dividing both sides by $t > 0$, we get

$$0 \le f(\tau, y) + (1-t)(\cosh d(x, y) - \cosh d(x, z)).$$

By (iv) of condition 3.1, letting $t \searrow 0$, we have

$$0 \le f(z, y) + \cosh d(x, y) - \cosh d(x, z)$$

and hence we get the statement (2). □

Lemma 3.9. *Let X be a complete* CAT(−1) *space having the convex hull finite property, K a nonempty closed convex subset of X, and C a nonempty delta-compact closed convex subset of K. Suppose that $f: K \times K \to \mathbb{R}$ satisfies* Condition 3.1. *Then there exists $z \in C$ such that*

$$0 \le f(z, y) + \cosh d(x, y) - \cosh d(x, z)$$

for all $x \in X$ and $y \in C$.

Proof. By Lemma 3.7, there exists $z \in C$ such that

$$f(y,z) \le \cosh d(x,y) - \cosh d(x,z)$$

for all $y \in C$. Further, by Lemma 3.8, we get

$$0 \le f(z,y) + \cosh d(x,y) - \cosh d(x,z)$$

for all $y \in C$. Consequently, we obtain the desired result. □

Theorem 3.3. *Let X be a complete* CAT(-1) *space having the convex hull finite property and K a nonempty closed convex subset of X. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies* Condition 3.1. *Define a set-valued mapping $L_f\colon X \to 2^K$ by*

$$L_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K}(f(z,y) + \cosh d(x,y) - \cosh d(x,z)) \ge 0 \right\}$$

for all $x \in X$. Then the following conditions hold:

(i) $D(L_f) = X$;

(ii) L_f *is single-valued and the inequality*

$$(\cosh d(x, L_f x) + \cosh d(y, L_f y)) \cosh d(L_f x, L_f y) \le \cosh d(L_f x, y) + \cosh d(x, L_f y)$$

for $x, y \in X$.

(iii) $\operatorname{Equil} f = \operatorname{Fix} L_f$, *and thus* $\operatorname{Equil} f$ *is closed and convex.*

Proof. (i) Fix $x \in K$. We show that there exists $z \in K$ such that

$$0 \le f(z,y) + \cosh d(x,y) - \cosh d(x,z)$$

for all $y \in K$. Fix $a \in K$. Then for $w \in K$, we get

$$\begin{aligned} f(w,a) + \cosh d(x,a) - \cosh d(x,w) &\le -f(a,w) + \cosh d(x,a) - \cosh d(x,w) \\ &= \cosh d(x,a) - (f(a,w) + \cosh d(x,w)). \end{aligned}$$

By Lemma 3.1, there exists $c \ge 0$ such that

$$\liminf_{d(a,w)\to\infty} \frac{f(a,w)}{d(a,w)} \ge -c.$$

Then, we get

$$\liminf_{d(a,w)\to\infty} \frac{f(a,w) + \cosh d(x,w)}{d(a,w)} \ge -c + \liminf_{d(a,w)\to\infty} \frac{\cosh d(x,w)}{d(a,w)} = \infty$$

and hence $f(a,w) + \cosh d(x,z) \to \infty$ as $d(a,w) \to \infty$. This implies that

$$f(w,a) + \cosh d(x,a) - \cosh d(x,w) \to -\infty$$

as $d(a,w) \to \infty$. Then, we can take large $R > 0$ such that

$$f(w,a) + \cosh d(x,a) - \cosh d(x,w) \le 0$$

wherever $d(w,a) = R$ for $w \in K$. Let $C = \{w \in K \mid d(w,a) \le R\}$. Then C is bounded, closed and convex. Hence C is Δ-compact. By Lemma 3.9, there exists $z_0 \in C$ such that

$$0 \le f(z_0,y) + \cosh d(x,y) - \cosh d(x,z_0)$$

for all $y \in C$. We next show

$$0 \leq f(z_0, y) + \cosh d(x, y) - \cosh d(x, z_0)$$

for each $y \in K \setminus C$. Let $y \in K \setminus C$. Then, we have $d(a, y) > R$. Let

$$u_0 = \begin{cases} a & (d(a, z_0) = R), \\ z_0 & (d(a, z_0) < R). \end{cases}$$

Then, we get $d(a, u_0) < R$. In fact, if $d(a, z_0) = R$, then we have $d(a, u_0) = d(a, a) = 0$. On the other hand, if $d(a, z_0) < R$, then we have $d(a, u_0) = d(a, z_0) < R$. Since $d(a, y) > R$ and $d(a, u_0) < R$, we can take sufficiently small $t_0 \in]0, 1[$ satisfying $t_0 d(a, y) + (1 - t_0)d(a, u_0) < R$. Then, we get

$$d(a, t_0 y \oplus (1 - t_0)u_0) \leq t_0 d(a, y) + (1 - t_0)d(a, u_0) < R$$

and hence $d(a, t_0 y \oplus (1 - t_0)u_0) < R$. Since K is convex, we get $t_0 y \oplus (1 - t_0)u_0 \in K$. This implies that $t_0 y \oplus (1 - t_0)u_0 \in C$. Therefore

$$\begin{aligned} 0 &\leq f(z_0, t_0 y \oplus (1 - t_0)u_0) + \cosh d(x, t_0 y \oplus (1 - t_0)u_0) - \cosh d(x, z_0) \\ &\leq t_0(f(z_0, y) + \cosh d(x, y) - \cosh d(x, z_0)) + (1 - t_0)(f(z_0, u_0) + \cosh d(x, u_0) - \cosh d(x, z_0)). \end{aligned}$$

Further, we get

$$f(z_0, u_0) + \cosh d(x, u_0) - \cosh d(x, z_0) \leq 0.$$

Indeed, if $d(a, z_0) = R$, we get

$$f(z_0, u_0) + \cosh d(x, u_0) - \cosh d(x, z_0) = f(z_0, a) + \cosh d(x, a) - \cosh d(x, z_0) \leq 0.$$

On the other hand, if $d(a, z_0) < R$, then $u_0 = z_0$ and thus we get

$$f(z_0, u_0) + \cosh d(x, u_0) - \cosh d(x, z_0) = 0.$$

Then, we get

$$f(z_0, y) + \cosh d(x, y) - \cosh d(x, z_0) \geq -\frac{1 - t_0}{t_0}(\cosh d(z_0, u_0) + \cosh d(x, u_0) - \cosh d(x, z_0)) \geq 0.$$

Since $y \in K \setminus C$ is arbitrary, we get

$$f(z_0, y) + \cosh d(x, y) - \cosh d(x, z) \geq 0$$

for all $y \in K$.

(ii) Let $x \in X$, $w \in K$ and $z \in L_f x$ with $w \neq z$. Put $\tau_t = tw \oplus (1 - t)z$ for $t \in]0, 1[$. Then, we get

$$\begin{aligned} 0 &\leq f(z, \tau_t) + \cosh d(x, \tau_t) - \cosh d(x, z) \\ &\leq t f(z, w) + \cosh d(x, \tau_t) - \cosh d(x, z) \\ &= t f(z, w) + \frac{1}{\sinh d(w, z)}(\cosh d(x, \tau_t) \sinh d(w, z) - \cosh d(x, z) \sinh d(w, z)) \\ &\leq t f(z, w) \\ &\quad + \frac{\cosh d(x, w) \sinh t d(w, z) + \cosh d(x, z) \sinh (1 - t) d(w, z) - \cosh d(x, z) \sinh d(z, w)}{\sinh d(z, w)}. \end{aligned}$$

Putting

$$L(t) = \cosh d(x, w) \sinh t d(w, z) + \cosh d(x, z) \sinh (1 - t) d(w, z)$$

and dividing both sides by $t > 0$, we get

$$0 \leq f(z,w) + \frac{1}{t \sinh d(z,w)}(L(t) - \cosh d(x,z) \sinh d(z,w))$$

Letting $t \searrow 0$, we get

$$\begin{aligned}
0 &\leq f(z,w) + \frac{1}{\sinh d(z,w)} \lim_{t \searrow 0} \frac{L(t) - \cosh d(x,z) \sinh d(z,w)}{t} \\
&= f(z,w) + \frac{1}{\sinh d(z,w)} \lim_{t \searrow 0} \frac{d}{dt}(L(t) - \cosh d(x,z) \sinh d(z,w)) \\
&= f(z,w) + \frac{d(z,w)}{\sinh d(z,w)} \lim_{t \searrow 0} (\cosh d(x,w) \cosh td(z,w) - \cosh d(x,z) \cosh(1-t)d(z,w)) \\
&= f(z,w) + \frac{d(z,w)}{\sinh d(z,w)}(\cosh d(x,w) - \cosh d(x,z) \cosh d(z,w)).
\end{aligned}$$

Fix $x_1, x_2 \in X$. Let $z_1 \in L_f x_1$ and $z_2 \in L_f x_2$. We suppose $z_1 \neq z_2$. Then, we get

$$0 \leq f(z_1,z_2) + \frac{d(z_1,z_2)}{\sinh d(z_1,z_2)}(\cosh d(x_1,z_2) - \cosh d(x_1,z_1) \cosh d(z_1,z_2)). \tag{3.4}$$

Similarly, it holds that

$$0 \leq f(z_2,z_1) + \frac{d(z_1,z_2)}{\sinh d(z_1,z_2)}(\cosh d(x_2,z_1) - \cosh d(x_2,z_2) \cosh d(z_1,z_2)). \tag{3.5}$$

Adding (3.4) and (3.5), we have

$$\begin{aligned}
0 &\leq f(z_1,z_2) + f(z_2,z_1) + \frac{d(z_1,z_2)}{\sinh d(z_1,z_2)} \cosh d(x_1,z_2) - \cosh d(x_1,z_1) \cosh d(z_1,z_2) \\
&\quad + \frac{d(z_1,z_2)}{\sinh d(z_1,z_2)}(\cosh d(x_2,z_1) - \cosh d(x_2,z_2) \cosh d(z_1,z_2)) \\
&\leq \frac{d(z_1,z_2)}{\sinh d(z_1,z_2)}(\cosh d(x_1,z_2) - \cosh d(x_1,z_1) \cosh d(z_1,z_2)) \\
&\quad + \frac{d(z_1,z_2)}{\sinh d(z_1,z_2)}(\cosh d(x_2,z_1) - \cosh d(x_2,z_2) \cosh d(z_1,z_2)).
\end{aligned}$$

Since $t/(\sinh t) > 0$ for $t > 0$, we get

$$(\cosh d(x_1,z_1) + \cosh d(x_2,z_2)) \cosh d(z_1,z_2) \leq \cosh d(x_1,z_2) + \cosh d(x_2,z_1).$$

Notice that this inequality obviously holds when $z_1 = z_2$. Using this inequality, we show that L_f is a singleton. If $x = x_1 = x_2$, we get

$$(\cosh d(x,z_1) + \cosh d(x,z_2)) \cosh d(z_1,z_2) \leq \cosh d(x,z_2) + \cosh d(x,z_1).$$

and hence $\cosh d(z_1,z_2) \leq 1$. This implies $z_1 = z_2$. Therefore L_f is a singleton and we get the desired inequality.

(iii) Let $z \in \operatorname{Equil} f$. Then we get

$$\inf_{y \in K}(f(z,y) + \cosh d(z,y) - \cosh d(z,z)) = \inf_{y \in K}(f(z,y) + \cosh d(z,y) - 1) \geq \inf_{y \in K} f(z,y) \geq 0$$

and hence $z \in \operatorname{Fix} L_f$. On the other hand, let $z \in \operatorname{Fix} L_f$ and $w \in K$ with $L_f z \neq w$. Then we get

$$0 \leq f(L_f z,w) + \frac{d(L_f z,w)}{\sinh d(L_f z,w)}(\cosh d(L_f z,w) - \cosh d(L_f z,z) \cosh d(z,w))$$

$$
\begin{aligned}
&\leq f(z,w) + \frac{d(z,w)}{\sinh d(z,w)}(\cosh d(z,w) - \cosh d(z,z)\cosh d(z,w)) \\
&= f(z,w).
\end{aligned}
$$

This implies $z \in \operatorname{Equil} f$. Therefore we get $\operatorname{Equil} f = \operatorname{Fix} L_f$. Since L_f is quasinonexpansive, we obtain that $\operatorname{Equil} f$ is closed and convex. □

Let $f\colon K \times K \to \mathbb{R}$ satisfy Condition 3.1. Then the resolvent L_f of an equilibrium problem is defined by

$$
L_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K}(f(z,y) + \cosh d(x,y) - \cosh d(x,z)) \geq 0 \right\}
$$

for all $x \in X$. By Theorem 3.3, a mapping L_f is single-valued.

Chapter 4

A delta-convergence theorem with the proximal point algorithm

In this chapter, we prove an approximation theorem to a solution to an equilibrium problem using the proximal point algorithm in a CAT(−1) space.

The following a theorem is necessary to prove a delta-convergent theorem with the proximal point algorithm.

Theorem 4.1 (Kajimura and Kimura [18]). *Let X be a complete* CAT(−1) *space, $\{z_n\}$ a bounded sequence in X, $\{\beta_n\}$ a sequence of positive real numbers with $\sum_{n=1}^{\infty} \beta_n = \infty$ and*

$$g(y) = \limsup_{n\to\infty} \frac{1}{\sum_{l=1}^{n} \beta_l} \sum_{k=1}^{n} \beta_k \cosh d(y, z_k)$$

for $y \in X$. Then, $\mathrm{Argmin}_X g$ *consists of one point.*

In the following lemmas, we obtain some properties of a resolvent for equilibrium problems.

Lemma 4.1. *Let X be a complete* CAT(−1) *space having the convex hull finite property and K a nonempty closed convex subset of X. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies* Condition 3.1. *Let $L_{\lambda f}$ be a resolvent of λf for $\lambda > 0$. Then the inequality*

$$0 \le f(L_{\lambda f}x, w) + \frac{d(L_{\lambda f}x, w)}{\lambda \sinh d(L_{\lambda f}x, w)}(\cosh d(x, w) - \cosh d(x, L_{\lambda f}x) \cosh d(w, L_{\lambda f}x))$$

holds for $x \in X$ and $w \in K$ with $w \ne L_{\lambda f}x$.

Proof. Let $x \in X$ and $w \in K$ with $w \ne L_{\lambda f}x$. Put $\tau = tw \oplus (1-t)L_{\lambda f}x \in K$ for $t \in \,]0,1[$. Then, we get

$$\begin{aligned} 0 &\le \lambda f(L_{\lambda f}x, \tau) + \cosh d(x, \tau) - \cosh d(x, L_{\lambda f}x) \\ &\le \lambda t f(L_{\lambda f}x, w) + \cosh d(x, \tau) - \cosh d(x, L_{\lambda f}x) \\ &\le \lambda t f(L_{\lambda f}x, w) + \frac{L(t) - \cosh d(x, L_{\lambda f}x) \sinh d(L_{\lambda f}x, w)}{\sinh d(L_{\lambda f}x, w)}, \end{aligned}$$

where

$$L(t) = \cosh d(x, w) \sinh t d(L_{\lambda f}x, w) + \cosh d(x, L_{\lambda f}x) \sinh(1-t) d(L_{\lambda f}x, w).$$

Dividing by $\lambda t > 0$ and letting $t \searrow 0$, we obtain

$$\begin{aligned} 0 &\le f(L_{\lambda f}x, w) + \frac{1}{\lambda \sinh d(L_{\lambda f}x, w)} \lim_{t\searrow 0} \frac{L(t) - \cosh d(x, L_{\lambda f}x) \sinh d(L_{\lambda f}x, w)}{t} \\ &= f(L_{\lambda f}x, w) + \frac{1}{\lambda \sinh d(L_{\lambda f}x, w)} \lim_{t\searrow 0} \frac{d}{dt}(L(t) - \cosh d(x, L_{\lambda f}x) \sinh d(L_{\lambda f}x, w)) \end{aligned}$$

$$= f(L_{\lambda f}x, w) + \frac{d(L_{\lambda f}x, w)}{\lambda \sinh d(L_{\lambda f}x, w)}(\cosh d(x, w) - \cosh d(x, L_{\lambda f}x) \cosh d(L_{\lambda f}x, w))$$

and hence we get the desired result. □

Corollary 4.1. *Let X be a complete* CAT(−1) *space having the convex hull finite property and K a nonempty closed convex subset of X. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies* Condition 3.1*. Let $L_{\lambda f}$ a resolvent of λf for $\lambda > 0$. Then the following inequalities hold:*

$$(\mu \cosh d(x, L_{\lambda f}x) + \lambda \cosh d(y, L_{\mu f}y)) \cosh d(L_{\lambda f}x, L_{\mu f}y) \leq \mu \cosh d(x, L_{\mu f}y) + \lambda \cosh d(L_{\lambda f}x, y)$$

and

$$(\lambda + \mu) \cosh d(L_{\lambda f}x, L_{\mu f}y) \leq \mu \cosh d(x, L_{\mu f}y) + \lambda \cosh d(L_{\lambda f}x, y)$$

for all $x, y \in X$ and $\lambda, \mu > 0$.

Proof. Let $x, y \in X$ and $\lambda, \mu > 0$ with $D = d(L_{\lambda f}x, L_{\mu f}y) > 0$. By Lemma 4.1, we get

$$0 \leq f(L_{\lambda f}x, L_{\mu f}y) + \frac{D}{\lambda \sinh D}(\cosh d(x, L_{\mu f}y) - \cosh d(x, L_{\lambda f}x) \cosh D). \tag{4.1}$$

Similarly, it holds that

$$0 \leq f(L_{\mu f}y, L_{\lambda f}x) + \frac{D}{\mu \sinh D}(\cosh d(L_{\lambda f}x, y) - \cosh d(y, L_{\mu f}y) \cosh D). \tag{4.2}$$

From Condition 3.1, adding (4.1) and (4.2), we get

$$\begin{aligned} 0 &\leq f(L_{\lambda f}x, L_{\mu f}y) + f(L_{\mu f}y, L_{\lambda f}x) + \frac{D}{\lambda \sinh D}(\cosh d(x, L_{\mu f}y) - \cosh d(x, L_{\lambda f}x) \cosh D) \\ &\quad + \frac{D}{\mu \sinh D}(\cosh d(L_{\lambda f}x, y) - \cosh d(y, L_{\mu f}y) \cosh D) \\ &\leq \frac{D}{\sinh D}\left(\frac{\cosh d(x, L_{\mu f}y) - \cosh d(x, L_{\lambda f}x) \cosh D}{\lambda} + \frac{\cosh d(x, L_{\mu f}y) - \cosh d(y, L_{\mu f}y) \cosh D}{\mu}\right) \end{aligned}$$

Since $t/(\sinh t) > 0$ for $t > 0$, we get

$$(\mu \cosh d(x, L_{\lambda f}x) + \lambda \cosh d(y, L_{\mu f}y)) \cosh D \leq \mu \cosh d(x, L_{\mu f}y) + \lambda \cosh d(L_{\lambda f}x, y).$$

Since $\cosh t \geq 1$ for $t \geq 0$, we get

$$(\lambda + \mu) \cosh D \leq \mu \cosh d(x, L_{\mu f}y) + \lambda \cosh d(L_{\lambda f}x, y).$$

If $D = 0$, the inequalities obviously hold. It completes the proof. □

Corollary 4.2. *Let X be a complete* CAT(−1) *space having the convex hull finite property and K a nonempty closed convex subset of X. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies* Condition 3.1 *and that* Equil f *is nonempty. Let $L_{\lambda f}$ a resolvent of λf for $\lambda > 0$. Then the following inequality holds:*

$$\cosh d(x, L_{\lambda f}x) \cosh d(L_{\lambda f}x, z) \leq \cosh d(x, z)$$

for all $x \in X$ and $z \in$ Equil f.

Proof. Let $x \in X$, $z \in \operatorname{Equil} f$ and $\lambda > 0$. By Corollary 4.1, we get

$$(\cosh d(x, L_{\lambda f}x) + \lambda \cosh d(z, L_f z)) \cosh d(L_{\lambda f}x, z) \leq \cosh d(x, z) + \lambda \cosh d(L_{\lambda f}x, z)$$

and hence

$$(\cosh d(x, L_{\lambda f}x) + \lambda) \cosh d(L_{\lambda f}x, z) \leq \cosh d(x, z) + \lambda \cosh d(L_{\lambda f}x, z).$$

Therefore, we have

$$\cosh d(x, L_{\lambda f}x) \cosh d(L_{\lambda f}x, z) \leq \cosh d(x, z)$$

and get the desired result. □

Lemma 4.2. *Let X be a complete* CAT(−1) *space having the convex hull finite property and K a nonempty closed convex subset of X. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies* Condition 3.1. *Let $\{\lambda_n\} \subset]0, \infty[$ such that $\limsup_{n\to\infty} \lambda_n > 0$, $L_{\lambda_n f}$ a resolvent of $\lambda_n f$, and $\{x_n\}$ a bounded sequence of X such that $x_n \stackrel{\Delta}{\rightharpoonup} x_0 \in X$ and $\lim_{n\to\infty} d(x_n, L_{\lambda_n f}x_n) = 0$. Then $x_0 \in \operatorname{Equil} f$.*

Proof. Put $\lambda_0 = \limsup_{n\to\infty} \lambda_n$. By Corollary 4.1, we get

$$\cosh d(L_{\lambda_n f}x_n, L_f x_0) \leq \frac{\lambda_n}{1+\lambda_n} \cosh d(L_{\lambda_n f}x_n, x_0) + \frac{1}{1+\lambda_n} \cosh d(x_n, L_f x_0).$$

For $y \in K$, we get

$$d(L_{\lambda_n f}x_n, y) \leq d(L_{\lambda_n f}x_n, x_n) + d(x_n, y) \leq 2d(L_{\lambda_n f}x_n, x_n) + d(L_{\lambda_n f}x_n, y).$$

Since $d(x_n, L_{\lambda_n f}x_n) \to 0$, letting $n \to \infty$, we get

$$\limsup_{n\to\infty} d(L_{\lambda_n f}x_n, y) = \limsup_{n\to\infty} d(x_n, y).$$

Suppose $\lambda_0 = \infty$. Then we take a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that $\lim_{i\to\infty} \lambda_{n_i} = \infty$. It implies that

$$\begin{aligned}
\limsup_{i\to\infty} \left(\cosh d(x_{n_i}, L_f x_0)\right) &= \limsup_{i\to\infty} \left(\cosh d(L_{\lambda_{n_i} f}x_{n_i}, L_f x_0)\right) \\
&\leq \limsup_{i\to\infty} \left(\cosh d(L_{\lambda_{n_i} f}x_{n_i}, x_0)\right) \\
&= \limsup_{i\to\infty} \left(\cosh d(x_{n_i}, x_0)\right).
\end{aligned}$$

Since x_0 is an asymptotic center of $\{x_{n_i}\}$, we get $x_0 = L_f x_0$ and hence $x_0 \in \operatorname{Equil} f$. We next suppose $\lambda_0 < \infty$. Taking a subsequence $\{\lambda_{n_i}\} \subset \{\lambda_n\}$ such that $\lim_{i\to\infty} \lambda_{n_i} = \lambda_0$, we get

$$\begin{aligned}
&\limsup_{i\to\infty} \left(\cosh d(x_{n_i}, L_f x_0)\right) \\
&= \limsup_{i\to\infty} \left(\cosh d(L_{\lambda_{n_i} f}x_{n_i}, L_f x_0)\right) \\
&\leq \frac{\lambda_0}{1+\lambda_0} \limsup_{i\to\infty} \left(\cosh d(L_{\lambda_{n_i} f}x_{n_i}, x_0)\right) + \frac{1}{1+\lambda_0} \limsup_{i\to\infty} \left(\cosh d(x_{n_i}, L_f x_0)\right) \\
&= \frac{\lambda_0}{1+\lambda_0} \limsup_{i\to\infty} \left(\cosh d(x_{n_i}, x_0)\right) + \frac{1}{1+\lambda_0} \limsup_{i\to\infty} \left(\cosh d(x_{n_i}, L_f x_0)\right)
\end{aligned}$$

and hence

$$\limsup_{i\to\infty} \left(\cosh d(x_{n_i}, L_f x_0)\right) \leq \limsup_{i\to\infty} \left(\cosh d(x_{n_i}, x_0)\right).$$

Since x_0 is an asymptotic center of $\{x_{n_i}\}$, we get $x_0 = L_f x_0$ and hence $x_0 \in \operatorname{Equil} f$. Consequently, we complete the proof. □

We next show a delta-convergence theorem with the proximal point algorithm in a CAT(−1) space having the convex hull finite property.

Theorem 4.2. *Let X be a complete* $\mathrm{CAT}(-1)$ *space having the convex hull finite property, K a nonempty closed convex subset of X, $f\colon K \times K \to \mathbb{R}$ satisfying* Condition 3.1 *and $\{\lambda_n\} \subset\,]0,\infty[$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. For given $x_1 \in X$, define $\{x_n\}$ by*

$$x_{n+1} = L_{\lambda_n f} x_n = \left\{ z \in K \;\middle|\; \inf_{y \in K} (\lambda_n f(z,y) + \cosh d(x_n, y) - \cosh d(x_n, z)) \geq 0 \right\}$$

for all $n \in \mathbb{N}$. Then, the following hold:

(i) $\operatorname{Equil} f$ *is nonempty if and only if $\{x_n\}$ is bounded;*

(ii) *if* $\operatorname{Equil} f \neq \emptyset$ *and* $\liminf_{n\to\infty} \lambda_n > 0$, $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \operatorname{Equil} f$.

Proof. (i) We first suppose that $\operatorname{Equil} f$ is nonempty and show that $\{x_n\}$ is bounded. Let $u \in \operatorname{Equil} f$. Since $L_{\lambda_n f}$ is quasinonexpansive with $\operatorname{Fix} L_{\lambda_n f} = \operatorname{Equil} f$, we get

$$d(x_{n+1}, u) = d(L_{\lambda_n f} x_n, u) \leq d(x_n, u)$$

and hence $\{d(x_n, u)\}$ is nonincreasing and $\{x_n\}$ is bounded for $n \in \mathbb{N}$. We next suppose $\{x_n\}$ is bounded and show that $\operatorname{Equil} f$ is nonempty. For $k \in \mathbb{N}$ with $k \leq n$, by Corollary 4.1 we get

$$(1 + \lambda_k) \cosh d(L_{\lambda_k f} x_k, L_f y) \leq \lambda_k \cosh d(L_{\lambda_k f} x_k, y) + \cosh d(x_k, L_f y)$$

and hence

$$\lambda_k \cosh d(x_{k+1}, L_f y) \leq \lambda_k \cosh d(x_{k+1}, y) + (\cosh d(x_k, L_f y) - \cosh d(x_{k+1}, L_f y))$$

for all $y \in X$. Adding both sides of the inequality above from $k = 1$ to $k = n$ and dividing both sides by $\sum_{l=1}^{n} \lambda_l$, we get

$$\begin{aligned}
&\frac{1}{\sum_{l=1}^{n} \lambda_l} \sum_{k=1}^{n} \lambda_k \cosh d(x_{k+1}, L_f y) \\
&\leq \frac{1}{\sum_{l=1}^{n} \lambda_l} \sum_{k=1}^{n} \lambda_k \cosh d(x_{k+1}, y) + \frac{1}{\sum_{l=1}^{n} \lambda_l} (\cosh d(x_1, L_f y) - \cosh d(x_{n+1}, L_f y)) \\
&\leq \frac{1}{\sum_{l=1}^{n} \lambda_l} \sum_{k=1}^{n} \lambda_k \cosh d(x_{k+1}, y) + \frac{1}{\sum_{l=1}^{n} \lambda_l} \cosh d(x_1, y).
\end{aligned}$$

By Theorem 4.1, we know that $\operatorname{Argmin}_X g$ consists of one point, where

$$g(z) = \limsup_{n\to\infty} \frac{1}{\sum_{l=1}^{n} \lambda_l} \sum_{k=1}^{n} \lambda_k \cosh d(x_{k+1}, z)$$

for all $z \in X$. Let $p \in \operatorname{Argmin}_X g$. Since $\sum_{l=1}^{\infty} \lambda_l = \infty$, letting $n \to \infty$, we get

$$g(L_f p) \leq g(p) \leq g(L_f p)$$

and hence $p = L_f p$. This implies that $p \in \operatorname{Equil} f$.

(ii) Suppose $\operatorname{Equil} f$ is nonempty and $\liminf_{n\to\infty} \lambda_n > 0$. Let $p \in \operatorname{Equil} f$. Since $L_{\lambda_n f}$ and $\operatorname{Equil} f$ is nonempty, $L_{\lambda_n f}$ is quasinonexpansive. Then, we get

$$0 \leq d(x_{n+1}, p) = d(L_{\lambda_n f} x_n, p) \leq d(x_n, p)$$

and hence $\{d(x_n, p)\}$ is nonincreasing. Then, there exists $\lim_{n\to\infty} d(x_n, p)$. Put $c_p = \lim_{n\to\infty} d(x_n, p)$. By Corolally 4.1, we have

$$1 \leq \cosh d(x_n, L_{\lambda_n f} x_n) \leq \frac{\cosh d(x_n, p)}{\cosh d(x_{n+1}, p)}.$$

Letting $n \to \infty$, we obtain

$$1 \leq \lim_{n\to\infty} (\cosh d(x_n, L_{\lambda_n f} x_n)) \leq \frac{\cosh c_p}{\cosh c_p} = 1$$

and hence $d(x_n, L_{\lambda_n f} x_n) \to 0$. Put $\mathrm{AC}(\{x_n\}) = \{x_0\}$. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. Since $\{x_{n_i}\}$ is bounded and by Lemma 2.6, $\mathrm{AC}(\{x_{n_i}\}) = \{y_0\}$. Further, there exists a subseqence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \xrightarrow{\Delta} z_0 \in X$. Then, $\mathrm{AC}(\{x_{n_{i_j}}\}) = \{z_0\}$. By Lemma 4.2, we get $z_0 \in \operatorname{Equil} f$. Then, we get

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, z_0) &= \limsup_{j\to\infty} d(x_{n_{i_j}}, z_0) \\
&\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, y_0) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, y_0) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, x_0) \\
&\leq \limsup_{n\to\infty} d(x_n, x_0) \leq \limsup_{n\to\infty} d(x_n, z_0)
\end{aligned}$$

and hence $y_0 = x_0 = z_0 \in \operatorname{Equil} f$. Therefore $x_n \xrightarrow{\Delta} x_0 \in \operatorname{Equil} f$. Consequently, we get the desired result. □

We can prove the following theorem easily by (ii) of Theorem 4.2. However, we give this different proof.

Theorem 4.3. *Let X be a complete* CAT(−1) *space having the convex hull finite property, K a nonempty closed convex subset of X, $f: K \times K \to \mathbb{R}$ satisfying* Condition 3.1 *and $\{\lambda_n\} \subset]0, \infty[$ such that $\inf_{n\in\mathbb{N}} \lambda_n > 0$. For given $x_1 \in X$, define $\{x_n\}$ by*

$$x_{n+1} = L_{\lambda_n f} x_n = \left\{ z \in K \,\middle|\, \inf_{y\in K} (\lambda_n f(z, y) + \cosh d(x_n, y) - \cosh d(x_n, z)) \geq 0 \right\}$$

for all $n \in \mathbb{N}$. If $\operatorname{Equil} f \neq \emptyset$, $x_n \xrightarrow{\Delta} x_0 \in \operatorname{Equil} f$.

Proof. Suppose $\operatorname{Equil} f \neq \emptyset$ and $p \in \operatorname{Equil} f$. Since $L_{\lambda_n f}$ is quasinonexpansive, we get $d(x_{n+1}, p) \leq d(x_n, p)$ and hence there exists $\lim_{n\to\infty} d(x_n, p)$. Put

$$c_p = \lim_{n\to\infty} d(x_n, p).$$

By Corollary 4.2, we obtain

$$1 \leq \cosh d(x_n, x_{n+1}) = \cosh d(x_n, L_{\lambda_n f} x_n) \leq \frac{\cosh d(x_n, p)}{\cosh d(L_{\lambda_n f} x_n, p)} = \frac{\cosh d(x_n, p)}{\cosh d(x_{n+1}, p)} \to \frac{\cosh c_p}{\cosh c_p} = 1$$

and hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $w \in K$ with $w \neq x_{n+1} = L_{\lambda_n f} x_n$. By Lemma 4.1, we have

$$0 \leq f(L_{\lambda_n f} x_n, w) + \frac{d(L_{\lambda_n f} x_n, w)}{\lambda_n \sinh d(L_{\lambda_n f} x_n, w)} (\cosh d(x_n, w) - \cosh d(x, L_{\lambda_n f} x_n) \cosh d(L_{\lambda_n f} x_n, w)).$$

Since $\inf_{n\in\mathbb{N}} \lambda_n > 0$ and $d/\sinh d$ is bounded for $d \in]0, \infty[$, there exists $M > 0$ such that

$$d(L_{\lambda_n f} x_n, w) \leq \lambda_n M \sinh d(L_{\lambda_n f} x_n, w).$$

for every $n \in \mathbb{N}$. Hence, we get

$$\begin{aligned}0 &\le f(L_{\lambda_n f} x_n, w) + \frac{d(L_{\lambda_n f} x_n, w)}{\lambda_n \sinh d(L_{\lambda_n f} x_n, w)} (\cosh d(x_n, w) - \cosh d(x_n, L_{\lambda_n f} x_n) \cosh d(L_{\lambda_n f} x_n, w)) \\ &\le f(L_{\lambda_n f} x_n, w) + M |\cosh d(x_n, w) - \cosh d(x_n, L_{\lambda_n f} x_n) \cosh d(L_{\lambda_n f} x_n, w)| \\ &\le -f(w, L_{\lambda_n f} x_n) + M |\cosh d(x_n, w) - \cosh d(x_n, L_{\lambda_n f} x_n) \cosh d(L_{\lambda_n f} x_n, w)|\end{aligned}$$

and thus

$$\begin{aligned}&f(w, x_{n+1}) \\ &\le M |\cosh d(x_n, w) - \cosh d(x_n, L_{\lambda_n f} x_n) \cosh d(L_{\lambda_n f} x_n, w)| \\ &\le M \Big(|\cosh d(x_n, w) - \cosh d(L_{\lambda_n f} x_n, w)| + \cosh d(L_{\lambda_n f} x_n, w) |1 - \cosh d(x_n, L_{\lambda_n f} x_n)| \Big) \\ &= M \left(|\cosh d(x_n, w) - \cosh d(x_{n+1}, w)| + \cosh d(x_{n+1}, w) |1 - \cosh d(x_n, x_{n+1})| \right).\end{aligned}$$

Letting $n \to \infty$, we get

$$\limsup_{n \to \infty} f(w, x_n) \le 0.$$

for all $w \in K$. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Since $\{x_{n_i}\} \subset K$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} x_0$ By Lemma 2.8, we have $x_0 \in K$. Since $f(w, \cdot) \colon K \to \mathbb{R}$ is lower semicontinuous, we have

$$f(w, x_0) \le \liminf_{j \to \infty} f(w, x_{n_{i_j}}) \le \limsup_{j \to \infty} f(w, x_{n_{i_j}}) \le 0$$

and hence $f(w, x_0) \le 0$ for all $w \in K$. Fix $u \in K$ arbitrarily and let $t \in]0, 1[$. Put $\tau = tu \oplus (1-t) x_0 \in K$. By (i) of Condition 3.1, we obtain

$$0 = f(\tau, \tau) = f(\tau, tu \oplus (1-t) x_0) \le t f(\tau, u) + (1-t) f(\tau, x_0) \le t f(\tau, u)$$

and thus $f(tu \oplus (1-t) x_0, u) \ge 0$. By (iv) of Condition 3.1, we get

$$f(x_0, u) \ge \limsup_{t \searrow 0} f(tu \oplus (1-t) x_0, u) \ge 0.$$

Since $u \in K$ is arbitrary, we get $x_0 \in \operatorname{Equil} f$. Put $\operatorname{AC}(\{x_n\}) = \{x_0\}$ and $\operatorname{AC}(\{x_{n_i}\}) = \{y_0\}$. Since $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} x_0$, $\operatorname{AC}(\{x_{n_{i_j}}\}) = \{x_0\}$. Then, we get

$$\begin{aligned}\limsup_{n \to \infty} d(x_n, z_0) &= \limsup_{j \to \infty} d(x_{n_{i_j}}, z_0) \\ &\le \limsup_{j \to \infty} d(x_{n_{i_j}}, y_0) \\ &\le \limsup_{i \to \infty} d(x_{n_i}, y_0) \\ &\le \limsup_{i \to \infty} d(x_{n_i}, x_0) \\ &\le \limsup_{n \to \infty} d(x_n, x_0) \le \limsup_{n \to \infty} d(x_n, z_0)\end{aligned}$$

and hence $y_0 = x_0 = z_0 \in \operatorname{Equil} f$. Therefore $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \operatorname{Equil} f$. Consequently, we get the desired result. □

Chapter 5

Properties of balanced mappings

In this chapter, we consider properties of a balanced mapping and prove a delta-convergence theorem in a CAT(κ) space for $\kappa \in \mathbb{R}$.

Let X be an admissible complete CAT(κ) space, T_k a mapping of X into itself for $k \in \mathbb{N}$, and $\{\alpha^k \mid k \in \{1, 2, \dots, N\}\} \subset]0, 1[$ such that $\sum_{k=1}^{N} \alpha^k = 1$ for $n \in \{1, 2, \dots, N\}$. A *balanced mapping U of $\{T_k\}$ and $\{\alpha^k\}$ of X into itself* is defined by

$$Ux = \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y) = \begin{cases} \operatorname*{Argmax}_{y\in X} \sum_{k=1}^{N} \alpha^k \cos\left(\sqrt{\kappa} d(T_k x, y)\right) & (\kappa > 0); \\ \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{N} \alpha^k d(T_k x, y)^2 & (\kappa = 0); \\ \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{N} \alpha^k \cosh\left(\sqrt{-\kappa} d(T_k x, y)\right) & (\kappa < 0) \end{cases}$$

for each $x \in X$; see [12, 15, 29]. In the following theorem, we know the properties of a balanced mapping.

Theorem 5.1 ([12, 15, 29])**.** *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, T_k a strongly qusinonexpansive and delta-demiclosed mapping for $k \in \{1, 2, \dots, N\}$ such that $\bigcap_{k=1}^{N} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha^k \mid k \in \{1, 2, \dots, N\}\} \subset]0, 1[$ such that $\sum_{k=1}^{N} \alpha^k = 1$, and U a balanced mapping of $\{T_k\}$ and $\{\alpha^k\}$. Then, the following conditions hold:*

(i) *U is single-valued;*

(ii) *the inequality*

$$\phi_\kappa(Ux, y) \le \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y) - c''_\kappa(d(Ux, y)) \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, Ux)$$

holds for $x, y \in X$;

(iii) $\operatorname{Fix} U = \bigcap_{k=1}^{N} \operatorname{Fix} T_k$;

(iv) *U is strongly quasinonexpansive and delta-demiclosed.*

Proof. (i) Let $x \in X$. Put $g_x(y) = \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y)$ and $M = \inf_{y\in X} g_x(y)$. Let $\{y_n\} \subset X$ with

$$\phi_\kappa(x, y_n) \le M + \frac{1}{n}.$$

Then, $\lim_{n\to\infty} \phi_\kappa(x, y_n) = M$. Let $m, n \in \mathbb{N}$ with $n \le m$. Put $d = d(y_m, y_n)$. By Theorem 2.1, we get

$$M \le g_x\left(\frac{1}{2} y_m \oplus \frac{1}{2} y_m\right)$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{N} \alpha^k \phi_\kappa\left(T_k x, \frac{1}{2}y_m \oplus \frac{1}{2}y_n\right) \\
&\leq (1/2)_d^\kappa \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y_m) + (1/2)_d^\kappa \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y_n) \\
&\quad - (1/2)_d^\kappa \phi_\kappa\left(y_m, \frac{1}{2}y_m \oplus \frac{1}{2}y_n\right) - (1/2)_d^\kappa \phi_\kappa\left(y_n, \frac{1}{2}y_m \oplus \frac{1}{2}y_n\right) \\
&= (1/2)_d^\kappa \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y_m) + (1/2)_d^\kappa \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y_n) - 2(1/2)_d^\kappa c_\kappa\left(\frac{d(y_m, y_n)}{2}\right) \\
&\leq (1/2)_d^\kappa \left(M + \frac{1}{m}\right) + (1/2)_d^\kappa \left(M + \frac{1}{n}\right) - 2(1/2)_d^\kappa c_\kappa\left(\frac{d(y_m, y_n)}{2}\right) \\
&\leq 2(1/2)_d^\kappa \left(M + \frac{1}{n}\right) - 2(1/2)_d^\kappa c_\kappa\left(\frac{d(y_m, y_n)}{2}\right)
\end{aligned}
$$

and hence

$$
c_\kappa\left(\frac{d(y_m, y_n)}{2}\right) \leq M + \frac{1}{n} - \frac{1}{2(1/2)_d^\kappa} M = \left(1 - \frac{1}{2(1/2)_d^\kappa}\right) M + \frac{1}{n}.
$$

By Lemma 2.1, we get

$$
c_\kappa\left(\frac{d(y_m, y_n)}{2}\right) \leq \kappa c_\kappa\left(\frac{d(y_m, y_n)}{2}\right) M + \frac{1}{n}
$$

and hence

$$
(1 - \kappa M) c_\kappa\left(\frac{d(y_m, y_n)}{2}\right) \leq \frac{1}{n}.
$$

Therefore, it follows that

$$
d(y_m, y_n) \leq 2c_\kappa^{-1}\left(\frac{1}{(1 - \kappa M)n}\right)
$$

and thus $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists y_x such that $y_n \to y_x \in X$. Then, $M = \lim_{n\to\infty} g_x(y_n) = g_x(y_x)$. Let $z_x \in X$ with $M = g_x(z_x)$. Put $d = d(y_x, z_x)$. Then, we obtain

$$
\begin{aligned}
M = g_x(y_x) &= \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y_x) \\
&\leq \sum_{k=1}^{N} \alpha^k \phi_\kappa\left(T_k x, \frac{1}{2}y_x \oplus \frac{1}{2}z_x\right) \\
&\leq (1/2)_d^\kappa \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y_x) + (1/2)_d^\kappa \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, z_x) \\
&\quad - (1/2)_d^\kappa \phi_\kappa\left(y_x, \frac{1}{2}y_x \oplus \frac{1}{2}z_x\right) - (1/2)_d^\kappa \left(z_x, \frac{1}{2}y_x \oplus \frac{1}{2}z_x\right) \\
&= (1/2)_d^\kappa \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y_x) + (1/2)_d^\kappa \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, z_x) - 2(1/2)_d^\kappa c_\kappa\left(\frac{d(y_x, z_x)}{2}\right) \\
&= 2(1/2)_d^\kappa M - 2(1/2)_d^\kappa c_\kappa\left(\frac{d(y_x, z_x)}{2}\right)
\end{aligned}
$$

and hence

$$c_\kappa\left(\frac{d(y_x,z_x)}{2}\right) \le \left(1-\frac{1}{2(1/2)_d^\kappa}\right)M = \kappa M c_\kappa\left(\frac{d(y_x,z_x)}{2}\right)$$

and hence

$$(1-\kappa M)c_\kappa\left(\frac{d(y_x,z_x)}{2}\right) \le 0.$$

Since $1-\kappa M > 0$, we get $y_x = z_x$. Consequently, we complete the proof.

(ii) Let $x, y \in X$ and $t \in [0,1]$ and put $d = d(Ux, y)$. By the definition of U and Theorem 2.1, we get

$$\begin{aligned}\sum_{k=1}^N \alpha^k \phi_\kappa(T_k x, Ux) &\le \sum_{k=1}^N \alpha^k \phi_\kappa(T_k x, ty \oplus (1-t)Ux)\\ &\le (t)_d^\kappa \sum_{k=1}^N \alpha^k \phi_\kappa(T_k x, y) + (1-t)_d^\kappa \sum_{k=1}^N \alpha^k \phi_\kappa(T_k x, Ux)\\ &\quad - (t)_d^\kappa \phi_\kappa(y, ty \oplus (1-t)Ux) - (1-t)_d^\kappa \phi_\kappa(Ux, ty\oplus(1-t)Ux)\end{aligned}$$

and hence

$$\left(1-(t)_d^\kappa\right)\sum_{k=1}^N \alpha^k \phi_\kappa(T_k x, Ux) \le (t)_d^\kappa \sum_{k=1}^N \alpha^k \phi_\kappa(T_k x, y) - (t)_d^\kappa \phi_\kappa(y, ty\oplus(1-t)Ux).$$

By Lemma 2.2, dividing by $(t)_d^\kappa > 0$ and letting $t \searrow 0$, we obtain

$$c_\kappa''(d(Ux,y))\sum_{k=1}^N \alpha^k \phi_\kappa(T_k x, Ux) \le \sum_{k=1}^N \alpha^k \phi_\kappa(T_k x, y) - \phi_\kappa(Ux, y)$$

and get the desired result.

(iii) Let $z \in \bigcap_{k=1}^N \operatorname{Fix} T_k$. Then, we have

$$Uz = \operatorname*{Argmin}_{y\in X} \sum_{k=1}^N \alpha^k \phi_\kappa(T_k z, y) = \operatorname*{Argmin}_{y\in X} \sum_{k=1}^N \alpha^k \phi_\kappa(z, y) = z$$

and hence $z \in \operatorname{Fix} U$. Inversely, let $z \in \operatorname{Fix} U$ and $w \in \bigcap_{k=1}^N \operatorname{Fix} T_k \subset \operatorname{Fix} U$. Then, it follows that

$$0 \le c_\kappa''(d(Uz,w))\sum_{k=1}^N \alpha^k \phi_\kappa(T_k z, z) \le \sum_{k=1}^N \alpha^k \phi_\kappa(T_k z, w) - \phi_\kappa(z,w) \le \phi_\kappa(z,w) - \phi_\kappa(z,w) = 0.$$

Since $c_\kappa''(d(Uz,w)) > 0$, we get $\phi_\kappa(T_k z, z) = 0$ for $k \in \{1,2,\dots,N\}$. This implies that $z \in \bigcap_{k=1}^N \operatorname{Fix} T_k$. Therefore, $\operatorname{Fix} U = \bigcap_{k=1}^N \operatorname{Fix} T_k$.

(iv) Let $x \in X$ and $z \in \operatorname{Fix} U$. Then, we obtain

$$0 \le c_\kappa''(d(Ux,z))\sum_{k=1}^N \alpha^k \phi_\kappa(T_k x, Ux) \le \sum_{k=1}^N \alpha^k \phi_\kappa(T_k x, z) - \phi_\kappa(Ux,z) \le \phi_\kappa(x,z) - \phi_\kappa(Ux,z)$$

and hence $\phi_\kappa(Ux,p) \le \phi_\kappa(x,p)$. Therefore, U is quasinonexpansive. Let $p \in \operatorname{Fix} U$ and $\{x_n\} \subset X$ with $\sup_{n\in\mathbb{N}} d(x_n,p) < D_\kappa/2$ and $\lim_{n\to\infty}(d(x_n,p) - d(Ux_n,p)) = 0$. Then, we get

$$0 \le c_\kappa''(d(Ux_n,p))\sum_{k=1}^N \alpha^k \phi_\kappa(T_k x_n, Ux_n) \le \sum_{k=1}^N \alpha^k \phi_\kappa(T_k x_n, p) - \phi_\kappa(Ux_n,p) \le \phi_\kappa(x_n,p) - \phi_\kappa(Ux_n,p)$$

Fix $i \in \{1, 2, \ldots, N\}$. Then, we get

$$0 \le \alpha^i c''_\kappa(d(Ux_n, p))\phi_\kappa(T_i x_n, Ux_n) \le \phi_\kappa(x_n, p) - \phi_\kappa(Ux_n, p).$$

Since $\sup_{n\in\mathbb{N}} d(Ux_n, p) < D_\kappa/2$, there exists $L > 0$ such that $L \le c''_\kappa(d(Ux_n, p))$. Then, it follows that

$$0 \le \alpha^i L\phi_\kappa(T_k x_n, Ux_n) \le \phi_\kappa(x_n, p) - \phi_\kappa(Ux_n, p)$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(T_i x_n, Ux_n) = 0$. Further, we have

$$0 \le d(x_n, p) - d(T_i x_n, p) \le d(x_n, p) - (d(Ux_n, p) - d(T_i x_n, Ux_n))$$

and letting $n \to \infty$, we get $\lim_{n\to\infty}(d(x_n, p) - d(T_i x_n, p)) = 0$. Since T_i is strongly quasinonexpansive, we obtain $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$. Further, it follows that

$$d(x_n, Ux_n) \le d(x_n, T_i x_n) + d(T_i x_n, Ux_n) \to 0$$

as $n \to \infty$ and hence $\lim_{n\to\infty} d(x_n, Ux_n) = 0$. This means that U is strongly quasinonexpansive. We next show U is delta-demiclosed. Let $z \in \operatorname{Fix} U$, and $\{x_n\} \subset X$ a κ-bounded sequence with $x_n \overset{\Delta}{\rightharpoonup} x_0 \in X$ and $\lim_{n\to\infty} d(x_n, Ux_n) = 0$. Then, it follows that

$$\begin{aligned}
0 \le c''_\kappa(d(Ux_n, z)) \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x_n, Ux_n) &\le \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x_n, z) - \phi_\kappa(Ux_n, z) \\
&\le \phi_\kappa(x_n, z) - \phi_\kappa(Ux_n, z) \\
&= 2\left(c'_\kappa\left(\frac{d(x_n, z)}{2}\right)^2 - c'_\kappa\left(\frac{d(Ux_n, z)}{2}\right)^2 \right).
\end{aligned}$$

and hence

$$0 \le c''_\kappa(d(Ux_n, z)) \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x_n, Ux_n) \le 2\left(c'_\kappa\left(\frac{d(x_n, z)}{2}\right)^2 - c'_\kappa\left(\frac{d(Ux_n, z)}{2}\right)^2 \right). \tag{5.1}$$

Further, we have

$$\begin{aligned}
&c'_\kappa\left(\frac{d(x_n, z)}{2}\right)^2 - c'_\kappa\left(\frac{d(Ux_n, z)}{2}\right)^2 \\
&= \left(c'_\kappa\left(\frac{d(x_n, z)}{2}\right) + c'_\kappa\left(\frac{d(Ux_n, z)}{2}\right) \right)\left(c'_\kappa\left(\frac{d(x_n, z)}{2}\right) - c'_\kappa\left(\frac{d(Ux_n, z)}{2}\right) \right) \\
&= 4c'_\kappa\left(\frac{d(x_n, z) + d(Ux_n, z)}{4}\right) c''_\kappa\left(\frac{d(x_n, z) - d(Ux_n, z)}{4}\right) \\
&\quad \times c''_\kappa\left(\frac{d(x_n, z) + d(Ux_n, z)}{4}\right) c'_\kappa\left(\frac{d(x_n, z) - d(Ux_n, z)}{4}\right) \\
&= c'_\kappa\left(\frac{d(x_n, z) - d(Ux_n, z)}{2}\right) c'_\kappa\left(\frac{d(x_n, z) + d(Ux_n, z)}{2}\right) \\
&\le c'_\kappa\left(\frac{d(x_n, Ux_n)}{2}\right) c'_\kappa\left(\frac{d(x_n, z) + d(Ux_n, z)}{2}\right).
\end{aligned}$$

Since $\{x_n\}$ is κ-bounded, letting $n \to \infty$, we get

$$\lim_{n\to\infty} \left(c'_\kappa\left(\frac{d(x_n, z)}{2}\right)^2 - c'_\kappa\left(\frac{d(Ux_n, z)}{2}\right)^2 \right) = 0.$$

Fix $i \in \{1, 2, \dots, N\}$ arbitrarily. By (5.1), it follows that

$$\begin{aligned}
0 &\leq c''_\kappa(d(Ux_n, z))\alpha^i \phi_\kappa(T_i x_n, Ux_n) \\
&\leq c''_\kappa(d(Ux_n, z)) \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x_n, Ux_n) \leq 2\left(c'_\kappa\left(\frac{d(x_n, z)}{2}\right)^2 - c'_\kappa\left(\frac{d(Ux_n, z)}{2}\right)^2\right).
\end{aligned}$$

Since $\{x_n\}$ is κ-bounded, letting $n \to \infty$, we get $\lim_{n\to\infty} d(T_i x_n, Ux_n) = 0$. Then, we get

$$0 \leq d(x_n, T_i x_n) \leq d(x_n, Ux_n) + d(Ux_n, T_i x_n)$$

and letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$. By delta-demiclosedness of T_i, we have $x_0 \in \operatorname{Fix} T_i$. Since $i \in \{1, 2, \dots, N\}$ is arbitrary, we get $x_0 \in \bigcap_{k=1}^{N} \operatorname{Fix} T_i = \operatorname{Fix} U$. This implies that U is delta-demiclosed. Consequently, we complete the proof. □

Writing (ii) of Theorem 5.1 for each curvature, we obtain the following:

- if $\kappa > 0$,
$$\sum_{k=1}^{N} \alpha^k \cos d(T_k x, Ux) \cos d(Ux, y) \geq \sum_{k=1}^{N} \alpha^k \cos d(T_k x, y);$$
- if $\kappa = 0$,
$$\sum_{k=1}^{N} \alpha^k d(T_k x, Ux)^2 \leq \sum_{k=1}^{N} \alpha^k d(T_k x, y)^2 - d(Ux, y)^2;$$
- if $\kappa < 0$,
$$\sum_{k=1}^{N} \alpha^k \cosh d(T_k x, Ux) \cosh d(Ux, y) \leq \sum_{k=1}^{N} \alpha^k \cosh d(T_k x, y).$$

Lemma 5.1. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, T_k a mapping of X into itself for $k \in \{1, 2, \dots, N, N+1\}$, $\{\alpha^k\}, \{\beta^k\} \subset\,]0, 1[$ such that $\sum_{k=1}^{N} \alpha^k = \sum_{k=1}^{N+1} \beta^k = 1$. Put*

$$Ux = \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, y)$$

and

$$Vx = \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{N+1} \beta^k \phi_\kappa(T_k x, y)$$

for each $x \in X$. Let

$$M = \max_{k=1,2,\dots,N,N+1} \{d(T_k x, Ux), d(T_k x, Vx)\} < \frac{D_\kappa}{2}.$$

Then the inequality

$$\begin{aligned}
&c'_\kappa\left(\frac{d(Ux, Vx)}{2}\right) \\
&\leq \frac{1}{2L_\kappa} \sum_{k=1}^{N} |\beta^k - \alpha^k| \left(c'_\kappa\left(\frac{d(T_k x, Ux) + d(T_k x, Vx)}{2}\right) + c'_\kappa\left(\frac{d(T_{N+1} x, Ux) + d(T_{N+1} x, Vx)}{2}\right)\right)
\end{aligned}$$

holds for $x \in X$, where

$$L_\kappa = \begin{cases} \cos M & (\kappa > 0); \\ 1 & (\kappa \leq 0). \end{cases}$$

Proof. Let $x \in X$. If $d(Ux, Vx) = 0$, we get the desired inequality obviously. Suppose $d(Ux, Vx) > 0$. By Theorem 5.1, we get

$$\begin{aligned}\phi_\kappa(Ux, Vx) &\leq \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, Vx) - c''_\kappa(d(Ux, Vx)) \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, Ux) \\ &= \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, Vx) - (1 - \kappa\phi_\kappa(Ux, Vx)) \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, Ux) \\ &= \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, Vx) - \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, Ux) + \kappa\phi_\kappa(Ux, Vx) \sum_{k=1}^{N} \alpha^k \phi_\kappa(T_k x, Ux)\end{aligned}$$

and hence

$$\sum_{k=1}^{N} \alpha^k (1 - \kappa\phi_\kappa(T_k x, Ux))\phi_\kappa(Ux, Vx) \leq \sum_{k=1}^{N} \alpha^k (\phi_\kappa(T_k x, Vx) - \phi_\kappa(T_k x, Ux)).$$

Then, it follows that

$$\sum_{k=1}^{N} \alpha^k c''_\kappa(d(T_k x, Ux))\phi_\kappa(Ux, Vx) \leq \sum_{k=1}^{N} \alpha^k (\phi_\kappa(T_k x, Vx) - \phi_\kappa(T_k x, Ux)).$$

Dividing by $\sum_{k=1}^{N} \alpha^k c''_\kappa(d(T_k x, Ux))$, we get

$$\phi_\kappa(Ux, Vx) \leq \frac{\sum_{k=1}^{N} \alpha^k (\phi_\kappa(T_k x, Vx) - \phi_\kappa(T_k x, Ux))}{\sum_{k=1}^{N} \alpha^k c''_\kappa(d(T_k x, Ux))} \leq \frac{\sum_{k=1}^{N} \alpha^k (\phi_\kappa(T_k x, Vx) - \phi_\kappa(T_k x, Ux))}{L_\kappa}.$$

and hence

$$\phi_\kappa(Ux, Vx) \leq \frac{\sum_{k=1}^{N} \alpha^k (\phi_\kappa(T_k x, Vx) - \phi_\kappa(T_k x, Ux))}{L_\kappa}. \tag{5.2}$$

Similarly, we obtain

$$\phi_\kappa(Ux, Vx) \leq \frac{\sum_{k=1}^{N+1} \beta^k (\phi_\kappa(T_k x, Ux) - \phi_\kappa(T_k x, Vx))}{L_\kappa}. \tag{5.3}$$

Adding (5.2) and (5.3), we have

$$\begin{aligned}&2\phi_\kappa(Ux, Vx) \\ &\leq \frac{\sum_{k=1}^{N} \alpha^k (\phi_\kappa(T_k x, Vx) - \phi_\kappa(T_k x, Ux))}{L_\kappa} + \frac{\sum_{k=1}^{N+1} \beta^k (\phi_\kappa(T_k x, Ux) - \phi_\kappa(T_k x, Vx))}{L_\kappa} \\ &= \frac{\sum_{k=1}^{N} (\beta^k - \alpha^k)(\phi_\kappa(T_k x, Vx) - \phi_\kappa(T_k x, Ux))}{L_\kappa} + \frac{\beta^{N+1}(\phi_\kappa(T_{N+1} x, Ux) - \phi_\kappa(T_{N+1}, Vx)}{L_\kappa} \\ &\leq \frac{\sum_{k=1}^{N} |\beta^k - \alpha^k||\phi_\kappa(T_k x, Vx) - \phi_\kappa(T_k x, Ux)|}{L_\kappa} + \frac{(1 - \sum_{k=1}^{N} \beta^k)(\phi_\kappa(T_{N+1} x, Ux) - \phi_\kappa(T_{N+1}, Vx))}{L_\kappa} \\ &\leq \frac{\sum_{k=1}^{N} |\beta^k - \alpha^k||\phi_\kappa(T_k x, Vx) - \phi_\kappa(T_k x, Ux)|}{L_\kappa} + \frac{\sum_{k=1}^{N} |\beta^k - \alpha^k||\phi_\kappa(T_{N+1} x, Ux) - \phi_\kappa(T_{N+1}, Vx)|}{L_\kappa}.\end{aligned}$$

Fix $i \in \{1, 2, \dots, N, N+1\}$ arbitrarily. Since $\phi_\kappa(u, v) = 2c'_\kappa(d(u, v)/2)^2$ for $u, v \in X$, it follows that

$$|\phi_\kappa(T_i x, Ux) - \phi_\kappa(T_i x, Vx)|$$

$$
\begin{aligned}
&= 2\left|c'_\kappa\left(\frac{d(T_ix,Vx)}{2}\right)^2 - c'_\kappa\left(\frac{d(T_ix,Ux)}{2}\right)^2\right| \\
&= 2\left|\left(c'_\kappa\left(\frac{d(T_ix,Vx)}{2}\right) + c'_\kappa\left(\frac{d(T_ix,Ux)}{2}\right)\right)\left(c'_\kappa\left(\frac{d(T_ix,Vx)}{2}\right) - c'_\kappa\left(\frac{d(T_ix,Ux)}{2}\right)\right)\right| \\
&= 8\left|c'_\kappa\left(\frac{d(T_ix,Vx)+d(T_ix,Ux)}{4}\right)c''_\kappa\left(\frac{d(T_ix,Vx)-d(T_ix,Ux)}{4}\right)\right| \\
&\quad\times\left|c''_\kappa\left(\frac{d(T_ix,Vx)+d(T_ix,Ux)}{4}\right)c'_\kappa\left(\frac{d(T_ix,Vx)-d(T_ix,Ux)}{4}\right)\right| \\
&= 2\left|c'_\kappa\left(\frac{d(T_ix,Vx)-d(T_ix,Ux)}{2}\right)c'_\kappa\left(\frac{d(T_ix,Vx)+d(T_ix,Ux)}{2}\right)\right| \\
&\leq 2c'_\kappa\left(\frac{d(Vx,Ux)}{2}\right)c'_\kappa\left(\frac{d(T_ix,Vx)+d(T_ix,Ux)}{2}\right).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
4c'_\kappa\left(\frac{d(Ux,Vx)}{2}\right)^2 &= 2\phi_\kappa(Ux,Vx) \\
&\leq \frac{2}{L_\kappa}\sum_{k=1}^{N}|\beta^k-\alpha^k|c'_\kappa\left(\frac{d(Vx,Ux)}{2}\right)c'_\kappa\left(\frac{d(T_kx,Vx)+d(T_kx,Ux)}{2}\right) \\
&\quad+\frac{2}{L_\kappa}\sum_{k=1}^{N}|\beta^k-\alpha^k|c'_\kappa\left(\frac{d(Vx,Ux)}{2}\right)c'_\kappa\left(\frac{d(T_{N+1}x,Vx)+d(T_kx,Ux)}{2}\right)
\end{aligned}
$$

Dividing by $c'_\kappa((d(Ux,Vx))/2) > 0$, we get

$$
\begin{aligned}
c'_\kappa\left(\frac{d(Ux,Vx)}{2}\right) &\leq \frac{1}{2L_\kappa}\sum_{k=1}^{N}|\beta^k-\alpha^k|c'_\kappa\left(\frac{d(T_kx,Vx)+d(T_kx,Ux)}{2}\right) \\
&\quad+\frac{1}{2L_\kappa}\sum_{k=1}^{N}|\beta^k-\alpha^k|c'_\kappa\left(\frac{d(T_{N+1}x,Vx)+d(T_{N+1}x,Ux)}{2}\right).
\end{aligned}
$$

Consequently, we complete the proof. □

We can prove the following lemma in a similar way to Lemma 5.1.

Lemma 5.2 ([11], [28], [41]). *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, T_k a mapping of X into itself for $k \in \{1,2,\ldots,N\}$, $\{\alpha^k\},\{\beta^k\} \subset]0,1[$ such that $\sum_{k=1}^{N}\alpha^k = \sum_{k=1}^{N}\beta^k = 1$. Put*

$$
Ux = \operatorname*{Argmin}_{y\in X}\sum_{k=1}^{N}\alpha^k\phi_\kappa(T_kx,y)
$$

and

$$
Vx = \operatorname*{Argmin}_{y\in X}\sum_{k=1}^{N}\beta^k\phi_\kappa(T_kx,y)
$$

for each $x \in X$. Let

$$
M' = \max_{k=1,2,\ldots,N}\{d(T_kx,Ux),d(T_kx,Vx)\} < \frac{D_\kappa}{2}.
$$

Then the inequality

$$c'_\kappa\left(\frac{d(Ux,Vx)}{2}\right) \le \frac{1}{2L_\kappa}\sum_{k=1}^{N}|\beta^k-\alpha^k|' c'_\kappa\left(\frac{d(T_kx,Ux)+d(T_kx,Vx)}{2}\right)$$

holds for $x \in X$, where

$$L_\kappa = \begin{cases}\cos M' & (\kappa > 0);\\ 1 & (\kappa \le 0).\end{cases}$$

Writing Lemma 5.1 for each curvature, we obtain the following:

- if $\kappa > 0$, then

$$\begin{aligned}&\sin\left(\frac{d(Ux,Vx)}{2}\right)\\ &\le \frac{1}{2\cos M}\sum_{k=1}^{N}|\beta^k-\alpha^k|\left(\sin\frac{d(T_kx,Ux)+d(T_kx,Vx)}{2}+\sin\frac{d(T_{N+1}x,Ux)+d(T_{N+1}x,Vx)}{2}\right);\end{aligned}$$

- if $\kappa = 0$, then

$$d(Ux,Vx) \le \frac{1}{2}\sum_{k=1}^{N}|\beta^k-\alpha^k|(d(T_kx,Ux)+d(T_kx,Vx)+d(T_{N+1}x,Ux)+d(T_{N+1}x,Vx));$$

- if $\kappa < 0$, then

$$\begin{aligned}&\sinh\left(\frac{d(Ux,Vx)}{2}\right)\\ &\le \frac{1}{2}\sum_{k=1}^{N}|\beta^k-\alpha^k|\left(\sinh\frac{d(T_kx,Ux)+d(T_kx,Vx)}{2}+\sinh\frac{d(T_{N+1},Ux)+d(T_{N+1}x,Vx)}{2}\right)\end{aligned}$$

Remark 5.1. Let H be a Hilbert space, $\{x_1,x_2,\dots,x_n\}$ for $n \in \mathbb{N}$, and $\{\alpha_n \mid n \in \mathbb{N}\} \subset]0,1[$ such that $\sum_{k=1}^{n}\alpha_n = 1$ for all $n \in \mathbb{N}$. Then, we know

$$\operatorname*{Argmin}_{y\in H}\sum_{k=1}^{n}\alpha_k\|x_k-y\|^2 = \sum_{k=1}^{n}\alpha_k x_k.$$

To show this, we show the inequality

$$\left\|\sum_{k=1}^{n}\alpha_k x_k\right\|^2 = \sum_{k=1}^{n}\alpha_k\|x_k\|^2 - \sum_{k=2}^{n}\sum_{i=1}^{k-1}\alpha_k\alpha_i\|x_k-x_i\|^2 \tag{5.4}$$

for all $y \in H$ by induction. If $n = 2$, it is clear from the parallelogram law in a Hilbert space. Suppose the inequality

$$\left\|\sum_{k=1}^{j}\alpha_k x_k\right\|^2 = \sum_{k=1}^{j}\alpha_k\|x_k\|^2 - \sum_{k=2}^{n}\sum_{i=1}^{k-1}\alpha_k\alpha_i\|x_k-x_i\|^2$$

holds for fixed $j \ge 3$ and $\sum_{k=1}^{j+1}\alpha_k = 1$. Then, we get

$$\left\|\sum_{k=1}^{j+1}\alpha_k x_k\right\|^2$$

$$
\begin{aligned}
&= \left\| \alpha_{j+1} x_{j+1} + \sum_{k=1}^{j} \alpha_k x_k \right\|^2 \\
&= \left\| \alpha_{j+1} x_{j+1} + (1-\alpha_{j+1}) \sum_{k=1}^{j} \frac{\alpha_k}{1-\alpha_{j+1}} x_k \right\|^2 \\
&= \alpha_{j+1} \|x_{j+1}\|^2 + (1-\alpha_{j+1}) \left\| \sum_{k=1}^{j} \frac{\alpha_k}{1-\alpha_{j+1}} x_k \right\|^2 - \alpha_{j+1}(1-\alpha_{j+1}) \left\| x_{k+1} - \sum_{k=1}^{j} \frac{\alpha_k}{1-\alpha_{j+1}} x_k \right\|^2 \\
&= \alpha_{j+1} \|x_{j+1}\|^2 + (1-\alpha_{j+1}) \left(\sum_{k=1}^{j} \frac{\alpha_k}{1-\alpha_{j+1}} \|x_k\|^2 - \sum_{k=2}^{j} \sum_{i=1}^{k-1} \frac{\alpha_k \alpha_i}{(1-\alpha_{j+1})^2} \|x_k - x_i\|^2 \right) \\
&\quad - \alpha_{j+1}(1-\alpha_{j+1}) \left\| \sum_{k=1}^{j} \frac{\alpha_k}{1-\alpha_{j+1}} (x_k - x_{j+1}) \right\|^2 \\
&= \sum_{k=1}^{j+1} \alpha_k \|x_k\|^2 - \frac{1}{1-\alpha_{j+1}} \sum_{k=2}^{j} \sum_{i=1}^{k-1} \alpha_k \alpha_i \|x_k - x_i\|^2 \\
&\quad - \alpha_{j+1}(1-\alpha_{j+1}) \left(\sum_{k=1}^{j} \frac{\alpha_k}{1-\alpha_{j+1}} \|x_k - x_{j+1}\|^2 - \sum_{k=2}^{j} \sum_{i=1}^{k-1} \frac{\alpha_k \alpha_i}{(1-\alpha_{j+1})^2} \|x_k - x_i\|^2 \right) \\
&= \sum_{k=1}^{j+1} \alpha_k \|x_k\|^2 - \frac{1}{1-\alpha_{j+1}} \sum_{k=2}^{j} \sum_{i=1}^{k-1} \alpha_k \alpha_i \|x_k - x_i\|^2 - \sum_{k=1}^{j} \alpha_k \alpha_{j+1} \|x_k - x_{j+1}\|^2 \\
&\quad + \frac{\alpha_{j+1}}{1-\alpha_{j+1}} \sum_{k=2}^{j} \sum_{i=1}^{k-1} \alpha_k \alpha_i \|x_k - x_i\|^2 \\
&= \sum_{k=1}^{j+1} \alpha_k \|x_k\|^2 - \sum_{k=2}^{j} \sum_{i=1}^{k-1} \alpha_k \alpha_i \|x_k - x_i\|^2 - \sum_{k=1}^{j} \alpha_k \alpha_{j+1} \|x_k - x_{j+1}\|^2 \\
&= \sum_{k=1}^{j+1} \alpha_k \|x_k\|^2 - \sum_{k=2}^{j+1} \sum_{i=1}^{k-1} \alpha_k \alpha_i \|x_k - x_i\|^2
\end{aligned}
$$

and hence we get (5.4) when $n = j+1$. By induction, we get (5.4) for every $n \in \mathbb{N}$. Then we get

$$
\begin{aligned}
\sum_{k=1}^{n} \alpha_k \|x_k - y\|^2 &= \left\| \sum_{k=1}^{n} \alpha_k (x_k - y) \right\|^2 + \sum_{k=2}^{n} \sum_{i=1}^{k-1} \alpha_k \alpha_i \|x_k - x_i\|^2 \\
&= \left\| \sum_{k=1}^{n} \alpha_k x_k - y \right\|^2 + \sum_{k=2}^{n} \sum_{i=1}^{k-1} \alpha_k \alpha_i \|x_k - x_i\|^2
\end{aligned}
$$

and hence

$$
\operatorname*{Argmin}_{y \in H} \sum_{k=1}^{n} \alpha_k \|x_k - y\|^2 = \sum_{k=1}^{n} \alpha_k x_k.
$$

Remark 5.2. Let X be a complete CAT(0) space, $\{\alpha^i \mid i \in \{1, 2, \dots, N\}\} \subset]0,1[$ with $\sum_{i=1}^{N} \alpha^i = 1$, and $\{x_1, x_2, \dots, x_N\} \subset X$. By induction, it is defined by

$$
\bigoplus_{i=1}^{N} \alpha^i x_i = \begin{cases} x_1 & (N = 1); \\ \alpha^N x_N \oplus (1-\alpha^N) \displaystyle\bigoplus_{i=1}^{N-1} \frac{\alpha^i}{1-\alpha^N} x_i & (N \geq 2); \end{cases}
$$

see details [6]. In 2014, Chidume et.al. [7] claimed the following inequality:

$$d\left(\bigoplus_{i=1}^{N} \alpha^i x_i, z\right)^2 \leq \sum_{i=1}^{N} \alpha^i d(x_i, z)^2 - \sum_{i=2}^{N}\sum_{j=1}^{i-1} \alpha^i \alpha^j d(x_i, x_j)^2 \tag{5.5}$$

for $z \in X$, $\{x_1, x_2, \dots, x_N\} \subset X$, and $\{\alpha^i \mid i \in \{1, 2, \dots, N\}\} \subset \,]0, 1[$ with $\sum_{i=1}^{N} \alpha^i = 1$. However, this inequality does not hold in general. Indeed, let X be a uniquely geodesic space and $Y \subset X$ satisfying $Y = [x_1, z] \cup [x_2, z] \cup [x_3, z]$ for $x_1, x_2, x_3 \in X$ with

$$d = d(x_1, z) = d(x_2, z) = d(x_3, z),$$

where $z = (1/2)x_2 \oplus (1/2)x_3$. Then, Y is a complete CAT(0) space [3, 5]. Let $\alpha_i = 1/3$ for $i \in \{1, 2, 3\}$ and

$$p = \bigoplus_{i=1}^{3} \alpha_i x_i = \bigoplus_{i=1}^{3} \frac{1}{3} x_i = \frac{1}{3} x_3 \oplus \frac{2}{3}\left(\frac{1}{2} x_2 \oplus \frac{1}{2} x_1\right).$$

Then,

$$d\left(\bigoplus_{i=1}^{3} \alpha_i x_i, z\right) = d\left(\bigoplus_{i=1}^{3} \frac{1}{3} x_i, z\right) = d(p, z) = \frac{1}{3} d(x_3, z).$$

On the other hand, it follows that

$$\begin{aligned}
&\sum_{i=1}^{3} \alpha_i d(x_i, z)^2 - \sum_{i=2}^{3}\sum_{j=1}^{i-1} \alpha_i \alpha_j d(x_i, x_j)^2 \\
&= \frac{1}{3} d(x_1, z)^2 + \frac{1}{3} d(x_2, z)^2 + \frac{1}{3} d(x_3, z)^2 - \frac{1}{9} d(x_1, x_2) - \frac{1}{9} d(x_2, x_3) - \frac{1}{9} d(x_3, x_1) \\
&= d(x_3, z) - \frac{4}{9} d(x_1, z)^2 - \frac{4}{9} d(x_3, z)^2 - \frac{4}{9} d(x_3, z)^2 \\
&= -\frac{1}{3} d(x_3, z)
\end{aligned}$$

and hence

$$d\left(\bigoplus_{i=1}^{3} \alpha_i x_i, z\right) > \sum_{i=1}^{3} \alpha_i d(x_i, z)^2 - \sum_{i=2}^{3}\sum_{j=1}^{i-1} \alpha_i \alpha_j d(x_i, x_j)^2.$$

It is a counterexample to the inequality (5.5).

We prove a delta-convergence theorem using a finite family of nonexpansive mappings in a CAT(κ) space for $\kappa \in \mathbb{R}$.

Theorem 5.2 (Kimura [21], Kimura and Sasaki [29])**.** *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, T_k a nonexpansive mapping for $k \in \{1, 2, \dots, N\}$ such that $F = \bigcap_{k=1}^{N} \operatorname{Fix} T_i \neq \emptyset$, $\{\alpha_n^k \mid n \in \mathbb{N}, k \in \{0, 1, \dots, N\}\} \subset [a, b] \subset \,]0, 1[$ such that $\sum_{k=0}^{N} \alpha_n^k = 1$. Define $\{x_n\} \subset X$ by $x_1 \in X$ and*

$$x_{n+1} = \operatorname*{Argmin}_{y \in X} \left\{ \alpha_n^0 \phi_\kappa(x_n, y) + \sum_{k=1}^{N} \alpha_n^k \phi_\kappa(T_k x_n, y) \right\}$$

for each $n \in \mathbb{N}$. Then, $x_n \stackrel{\Delta}{\rightharpoonup} x_0 \in \bigcap_{k=1}^{N} \operatorname{Fix} T_k$.

Proof. For each $n \in \mathbb{N}$, define a mapping $U_n \colon X \to X$ by

$$U_n x = \operatorname*{Argmin}_{y \in X} \left\{ \alpha_n^0 \phi_\kappa(x_n, y) + \sum_{i=1}^{N} \alpha_n^i \phi_\kappa(T_k x_n, y) \right\}$$

for $x \in X$. Since the set of fixed points of the identity mapping is the whole space and it is quasinonexpansive, by Theorem 5.1, we get $\operatorname{Fix} U_n = F$ and U_n is quasinonexpansive. Then, we get $y_n = U_n x_n$ for $n \in \mathbb{N}$. Let $p \in \bigcap_{k=1}^N \operatorname{Fix} T_k$. Since U_n is quasinonexpansive for $n \in \mathbb{N}$, we get

$$d(x_{n+1}, p) = d(U_n x_n, p) \leq d(x_n, p)$$

and hence $\{d(x_n, p)\}$ is nonincreasing. Then, there exists a limit of $\{d(x_n, p)\}$. Put

$$c_p = \lim_{n\to\infty} d(x_n, p).$$

Let $t \in]0, 1[$. Put $w = tp \oplus (1-t)U_n x_n$ and $d = d(U_n x_n, p)$. By definition of U_n, it follows that

$$\begin{aligned}
0 \leq & \alpha_n^0 \phi_\kappa(x_n, U_n x_n) + \sum_{k=1}^N \alpha_n^k \phi_\kappa(T_k x_n, U_n x_n) \\
& \leq \alpha_n^0 \phi_\kappa(x_n, w) + \sum_{k=1}^N \alpha_n^k \phi_\kappa(T_k x_n, w) \\
& \leq \alpha_n^0((t)_d^\kappa \phi_\kappa(x_n, p) + (1-t)_d^\kappa \phi_\kappa(x_n, U_n x_n) - (t)_d^\kappa \phi_\kappa(p, w) - (1-t)_d^\kappa \phi_\kappa(U_n x_n, , w)) \\
& \quad + \sum_{k=1}^N \alpha_n^k((t)_d^\kappa \phi_\kappa(T_k x_n, p) + (1-t)_d^\kappa \phi_\kappa(T_k x_n, U_n x_n) - (t)_d^\kappa \phi_\kappa(p, w) - (1-t)_d^\kappa \phi_\kappa \phi_\kappa(U_n x_n, w)) \\
& \leq \alpha_n^0((t)_d^\kappa \phi_\kappa(x_n, p) + (1-t)_d^\kappa \phi_\kappa(x_n, U_n x_n) - (t)_d^\kappa \phi_\kappa(p, w)) \\
& \quad + \sum_{k=1}^N \alpha_n^k((t)_d^\kappa \phi_\kappa(T_k x_n, p) + (1-t)_d^\kappa \phi_\kappa(T_k x_n, U_n x_n) - (t)_d^\kappa \phi_\kappa(p, w)) \\
& = (t)_d^\kappa \left(\alpha_n^0 \phi_\kappa(x_n, p) + \sum_{k=1}^N \alpha_n^k \phi_\kappa(T_k x_n, p) \right) \\
& \quad + (1-t)_d^\kappa \left(\alpha_n^0 \phi_\kappa(x_n, U_n x_n) + \sum_{k=1}^N \alpha_n^k \phi_\kappa(T_k x_n, U_n x_n) \right) - (t)_d^\kappa \phi_\kappa(p, w) \\
& \leq (t)_d^\kappa \phi_\kappa(x_n, p) + (1-t)_d^\kappa \left(\alpha_n^0 \phi_\kappa(x_n, U_n x_n) + \sum_{k=1}^N \alpha_n^k \phi_\kappa(T_k x_n, U_n x_n) \right) - (t)_d^\kappa \phi_\kappa(p, w)
\end{aligned}$$

and hence

$$0 \leq (1 - (1-t)_d^\kappa) \left(\alpha_n^0 \phi_\kappa(x_n, U_n x_n) + \sum_{k=1}^N \alpha_n^k \phi_\kappa(T_k x_n, U_n x_n) \right) \leq (t)_d^\kappa (\phi_\kappa(x_n, p) - \phi_\kappa(U_n x_n, p)).$$

Dividing by $(t)_d^\kappa$, it follows that

$$0 \leq \frac{1-(1-t)_d^\kappa}{(t)_d^\kappa} \left(\alpha_n^0 \phi_\kappa(x_n, U_n x_n) + \sum_{k=1}^N \alpha_n^k \phi_\kappa(T_k x_n, U_n x_n) \right) \leq \phi_\kappa(x_n, p) - \phi_\kappa(p, w).$$

By Lemma 2.2, letting $t \searrow 0$, we get

$$0 \leq c_\kappa''(d(U_n x_n, p)) \left(\alpha_n^0 \phi_\kappa(x_n, U_n x_n) + \sum_{k=1}^N \alpha_n^k \phi_\kappa(T_k x_n, U_n x_n) \right) \leq \phi_\kappa(x_n, p) - \phi_\kappa(U_n x_n, p). \quad (5.6)$$

Then, it follows that

$$0 \leq a c_\kappa''(d(x_{n+1}, p)) \phi_\kappa(x_n, U_n x_n) \leq \phi_\kappa(x_n, p) - \phi_\kappa(x_{n+1}, p).$$

Letting $n \to \infty$, we have

$$0 \leq ac''_\kappa(c_p) \lim_{n\to\infty} \phi_\kappa(x_n, U_n x_n) \leq c_\kappa(c_p) - c_\kappa(c_p) = 0.$$

Since $c''_\kappa(c_p) > 0$ and $a > 0$, we get $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. Fix $k \in \{0, 1, \dots, N\}$ arbitrarily. By (5.6), we obtain

$$\begin{aligned} 0 \leq \alpha_n^k c''_\kappa(d(x_{n+1}, p))\phi_\kappa(T_k x_n, U_n x_n) &\leq c''_\kappa(d(x_{n+1}, p)) \sum_{k=1}^N \alpha_n^k \phi_\kappa(T_k x_n, U_n x_n) \\ &\leq \phi_\kappa(x_n, p) - \phi_\kappa(U_n x_n, p) \end{aligned}$$

and thus

$$0 \leq ac''_\kappa(d(x_{n+1}, p))\phi_\kappa(T_k x_n, U_n x_n) \leq \phi_\kappa(x_n, p) - \phi_\kappa(x_{n+1}, p).$$

Letting $n \to \infty$, we have

$$0 \leq ac''_\kappa(c_p) \lim_{n\to\infty} \phi_\kappa(T_k x_n, U_n x_n) \leq c_\kappa(c_p) - c_\kappa(c_p) = 0$$

and hence $\lim_{n\to\infty} d(T_k x_n, U_n x_n) = 0$. Then, we obtain

$$d(x_n, T_k x_n) \leq d(x_n, U_n x_n) + d(U_n x_n, T_k x_n).$$

Letting $n \to \infty$, it follows that $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$.

Take a subsequence $\{x_{n_i}\} \subset \{x_n\}$ arbitrarily with $\mathrm{AC}(\{x_{n_i}\}) = \{y_0\}$. Then, there exists a subsequence $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that $x_{n_{i_j}} \xrightarrow{\Delta} z_0 \in X$. Then, it follows that

$$\begin{aligned} \limsup_{j\to\infty} d(x_{n_{i_j}}, T_k z_0) &\leq \limsup_{j\to\infty} (d(x_{n_{i_j}}, T_k x_{n_{i_j}}) + d(T_k x_{n_{i_j}}, T_k z_0)) \\ &= \limsup_{j\to\infty} d(T_k x_{n_{i_j}}, T_k z_0) \\ &\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, z_0) \end{aligned}$$

and hence $z_0 \in \operatorname{Fix} T_k$. Since $k \in \{1, 2, \dots, N\}$ is arbitrary, we get $z_0 \in \bigcap_{k=1}^N \operatorname{Fix} T_k$. Put $\mathrm{AC}(\{x_n\}) = \{x_0\}$. Then, we get

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, z_0) &= \limsup_{j\to\infty} d(x_{n_{i_j}}, z_0) \\ &\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, y_0) \\ &\leq \limsup_{i\to\infty} d(x_{n_i}, y_0) \\ &\leq \limsup_{i\to\infty} d(x_{n_i}, x_0) \\ &\leq \limsup_{n\to\infty} d(x_n, x_0) \leq \limsup_{n\to\infty} d(x_n, z_0) \end{aligned}$$

and hence $x_0 = y_0 = z_0 \in F$. Consequently, we get the desired result. □

Chapter 6

Convergence theorems using a balanced mapping of countable family of mappings

In this chapter, we consider properties of a balanced mapping of a countable family of mappings and prove an approximation theorem of a common fixed point in a CAT(κ) space for $\kappa \in \mathbb{R}$.

6.1 In a CAT(0) space

Lemma 6.1 ([12]). *Let X be a complete* CAT(0) *space, T_k a nonexpansive mapping of X into itself for $k = 1, 2, \dots, N$, $\{\alpha^k\} \subset]0, 1[$ such that $\sum_{k=1}^{N} \alpha^k = 1$ for $n \in \mathbb{N}$, and U a balanced mapping of $\{T_k\}$ and $\{\alpha^k\}$. Then, U is nonexpansive.*

Lemma 6.2. *Let X be a complete* CAT(0) *space, C a nonempty bounded subset of X, T_k a quasinonexpansive mapping of X into itself for $k \in \mathbb{N}$ with $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset]0, 1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$, then*

$$\sum_{n=1}^{\infty} \sup_{x \in C} d(U_{n+1}x, U_n x) < \infty.$$

Proof. Let $x \in C$ and $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. By Lemma 5.1, we get

$$\begin{aligned}
&d(U_n x, U_{n+1} x) \\
&\leq \frac{1}{2} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| (d(T_k x, U_n x) + d(T_k x, U_{n+1} x) + d(T_{n+1} x, U_n x) + d(T_{n+1} x, U_{n+1} x)) \\
&\leq 4 d(x, z) \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| \leq 4M \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k|,
\end{aligned}$$

where $M = \sup_{x \in C} d(x, z)$. Since $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$, we get

$$\sum_{n=1}^{\infty} \sup_{x \in C} d(U_n x, U_{n+1} x) < \infty.$$

Consequently, we complete the proof. $\square$

By Lemma 6.2, we can prove the following corollary easily.

Corollary 6.1. *Let X be a complete* CAT(0) *space, T_k a quasinonexpansive mapping of X into itself with $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$ and, $\{\alpha_n^k \mid n,k \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \sum_{k=1}^{n} \left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$, then $\sum_{n=1}^{\infty} d(U_{n+1}x, U_n x) < \infty$ and $\{U_n x\}$ is a Cauchy sequence for each $x \in X$.*

By Corollary 6.1, there exists a limit of $\{U_n x\}$. In the following lemma, we consider its properties.

Lemma 6.3. *Let X be a complete* CAT(0) *space, C a nonempty bounded subset of X, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ with $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$ and, $\{\alpha_n^k \mid n,k \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. $n \in \mathbb{N}$. Suppose the following conditions hold:*

- $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$;
- $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Put $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Then, the following conditions hold:

(i) $\lim_{n\to\infty} \sup_{x\in C} d(U_n x, Ux) = 0$;
(ii) *U is nonexpansive;*
(iii) $\operatorname{Fix} U = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$.

Proof. (i) Let $m,n \in \mathbb{N}$ such that $n \leq m$ and $x \in X$. Then, we get

$$\begin{aligned} d(U_m x, U_n x) &\leq d(U_m x, U_{n+1} x) + d(U_{n+1} x, U_n x) \\ &\leq d(U_m x, U_{n+2} x) + d(U_{n+2} x, U_{n+1} x) + d(U_{n+1} x, U_n x) \\ &\leq \cdots \\ &\leq \sum_{l=n}^{m-1} d(U_l x, U_{l+1} x) \leq \sum_{l=n}^{\infty} d(U_l x, U_{l+1} x) \end{aligned}$$

and hence

$$d(U_m x, U_n x) \leq \sum_{l=n}^{\infty} d(U_l x, U_{l+1} x). \tag{6.1}$$

By (6.1) and Corollary 6.1, letting $m \to \infty$, we get

$$\sup_{x\in C} d(Ux, U_n x) \leq \sum_{l=n}^{\infty} \sup_{x\in C} d(U_l x, U_{l+1} x).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} \sup_{x\in C} d(Ux, U_n x) = 0$.

(ii) Let $x, y \in X$. Since U_n is nonexpansive for $n \in \mathbb{N}$, we get

$$d(Ux, Uy) = \lim_{n\to\infty} d(U_n x, U_n y) \leq \lim_{n\to\infty} d(x,y) = d(x,y)$$

and hence U is a nonexpansive mapping of X into itself.

(iii) Let $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k = \operatorname{Fix} U_n$ for $n \in \mathbb{N}$. Then, we get

$$Uz = \lim_{n\to\infty} U_n z = \lim_{n\to\infty} z = z$$

and thus $z \in \operatorname{Fix} U$. On the other hand, let $z \in \operatorname{Fix} U$ and $w \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k = \operatorname{Fix} U_n$ for $n \in \mathbb{N}$. By (ii) of Theorem 5.1, we get

$$0 \leq \sum_{k=1}^{n} \alpha_n^k d(T_k z, U_n z)^2 \leq \sum_{k=1}^{n} \alpha_n^k d(T_k z, U_n w)^2 - d(U_n z, U_n w)^2$$

$$= \sum_{k=1}^{n} \alpha_n^k d(T_k z, w)^2 - d(U_n z, w)^2$$
$$\leq d(z, w)^2 - d(U_n z, w)^2.$$

Fix $j \in \mathbb{N}$ arbitrarily. Then, we have

$$0 \leq \alpha_n^j d(T_j z, U_n z)^2 \leq \sum_{k=1}^{n} \alpha_n^k d(T_k z, U_n z)^2 \leq d(z, w)^2 - d(U_n z, w)^2.$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(T_j z, U_n z) = 0$. Then, it follows that

$$d(T_j z, z) = d(T_j z, Uz) = \lim_{n\to\infty} d(T_j z, U_n z) = 0$$

and hence $z \in \operatorname{Fix} T_j$. Since $j \in \mathbb{N}$ is arbitrary, we get $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. Therefore we get $\operatorname{Fix} U = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$ and complete the proof. □

Lemma 6.4. *Let X be a complete* CAT(0) *space, C a nonempty bounded subset of X, T_k a quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ with $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$ and, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset]0, 1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. $n \in \mathbb{N}$. Suppose the following conditions hold:*

- $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$;*
- $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Put $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Then, U is quasinonexpansive and delta-demiclosed.

Proof. Let $x \in X$ and $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. Since U_n is quasinonexpansive for $n \in \mathbb{N}$, we get

$$d(Ux, z) = \lim_{n\to\infty} d(U_n x, z) \leq \lim_{n\to\infty} d(x, z) = d(x, z)$$

and hence U is quasinonexpansive. Next, we show that U is delta-demiclosed. Let $\{x_j\} \subset X$ such that $x_j \xrightarrow{\Delta} x_0 \in X$ and $\lim_{j\to\infty} d(x_j, Ux_j) = 0$. Let $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. By (ii) of Theorem 5.1, we get

$$0 \leq \sum_{i=1}^{n} \alpha_n^i d(T_i x_j, U_n x_j)^2 \leq \sum_{i=1}^{n} \alpha_n^i d(T_i x_j, z)^2 - d(U_n x_j, z)^2 \leq d(x_j, z)^2 - d(U_n x_j, z)^2.$$

Fix $i_0 \in \mathbb{N}$ arbitrarily. Then, it follows that

$$0 \leq \alpha_n^{i_0} d(T_{i_0} x_j, U_n x_j)^2 \leq \sum_{i=1}^{n} \alpha_n^i d(T_i x_j, U_n x_j)^2$$
$$\leq d(x_j, z)^2 - d(U_n x_j, z)^2 \leq d(x_j, U_n x_j)(d(x_j, z) + d(U_n x_j, z)).$$

Since $\lim_{n\to\infty} \alpha_n^{i_0} = \alpha^{i_0} > 0$, letting $n \to \infty$, it follows that

$$0 \leq \alpha^{i_0} d(T_{i_0} x_j, Ux_j)^2 \leq d(x_j, Ux_j)(d(x_j, z) + d(Ux_j, z)).$$

Letting $j \to \infty$, we get $\lim_{j\to\infty} d(T_{i_0} x_j, Ux_j) = 0$. Then, we have

$$0 \leq d(x_j, T_{i_0} x_j) \leq d(x_j, Ux_j) + d(Ux_j, T_{i_0} x_j)$$

and letting $j \to \infty$, we get $\lim_{j\to\infty} d(x_j, T_{i_0} x_j) = 0$. By delta-demiclosedness of T_{i_0}, we obtain $x_0 \in \operatorname{Fix} T_{i_0}$. Since $i_0 \in \mathbb{N}$ is arbitrary, we get $x_0 \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k = \operatorname{Fix} U$ and we get the desired result. □

6.1.1 Halpern and Mann iteration

First, we introduce a Halpern iteration using a balanced mapping of a countable family of mappings.

Theorem 6.1. *Let X be a complete* CAT(0) *space, $\{T_k \mid k \in \mathbb{N}\}$ a nonexpansive mapping of X into itself such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for all $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0,1]$. Define a sequence $\{x_n\}$ by $u, x_1 \in X$ and*

$$x_{n+1} = \delta_n u \oplus (1 - \delta) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$;*
- $\lim_{n\to\infty} \delta_n = 0$*, $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.*

Then, $\{x_n\}$ is convergent to $P_F u$.

Proof. Let $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k = \operatorname{Fix} U_n$ for $n \in \mathbb{N}$. Then, we get

$$\begin{aligned}
d(x_{n+1}, z) &\leq \delta_n d(u, z) + (1 - \delta_n) d(U_n x_n, z) \\
&\leq \delta_n d(u, z) + (1 - \delta_n) d(x_n, z) \\
&\leq \max\{d(u, z), d(x_n, z)\} \\
&\leq \max\{d(u, z), d(x_1, z)\}.
\end{aligned}$$

and hence $\{x_n\}$, $\{U_n x_n\}$ is bounded for all $n \in \mathbb{N}$. Put $M = \max\{d(u, z), d(x_1, z)\}$. Let C be a bounded subset of X including $\{x_n\}$. Then, we get

$$\begin{aligned}
d(x_{n+2}, x_{n+1}) &= d(\delta_{n+1} u \oplus (1 - \delta_{n+1}) U_{n+1} x_{n+1}, \delta_n u \oplus (1 - \delta_n) U_n x_n) \\
&\leq d(\delta_{n+1} u \oplus (1 - \delta_{n+1}) U_{n+1} x_{n+1}, \delta_n u \oplus (1 - \delta_n) U_{n+1} x_{n+1}) \\
&\quad + d(\delta_n u \oplus (1 - \delta_n) U_{n+1} x_{n+1}, \delta_n u \oplus (1 - \delta_n) U_n x_n) \\
&\leq |\delta_{n+1} - \delta_n| d(U_{n+1} x_{n+1}, u) + (1 - \delta_n) d(U_{n+1} x_{n+1}, U_n x_n) \\
&\leq |\delta_{n+1} - \delta_n| d(U_{n+1} x_{n+1}, u) \\
&\quad + (1 - \delta_n)(d(U_{n+1} x_{n+1}, U_n x_{n+1}) + d(U_n x_{n+1}, U_n x_n)) \\
&\leq |\delta_{n+1} - \delta_n| d(U_{n+1} x_{n+1}, u) \\
&\quad + (1 - \delta_n) d(x_{n+1}, x_n) + (1 - \delta_n) d(U_{n+1} x_{n+1}, U_n x_{n+1}) \\
&\leq |\delta_{n+1} - \delta_n| d(U_{n+1} x_{n+1}, u) + (1 - \delta_n) d(x_{n+1}, x_n) + d(U_{n+1} x_{n+1}, U_n x_{n+1}) \\
&\leq (1 - \delta_n) d(x_{n+1}, x_n) + |\delta_{n+1} - \delta_n| d(U_{n+1} x_{n+1}, u) + \sup_{x \in C} d(U_{n+1} x, U_n x) \\
&\leq (1 - \delta_n) d(x_{n+1}, x_n) + 2M |\delta_{n+1} - \delta_n| + \sup_{x \in C} d(U_{n+1} x, U_n x)
\end{aligned}$$

for all $n \in \mathbb{N}$. By Lemma 2.17 and Lemma 6.2, we get $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Further, we get

$$\begin{aligned}
d(x_n, U_n x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, U_n x_n) \\
&= d(x_n, x_{n+1}) + d(\delta_n u \oplus (1 - \delta_n) U_n x_n, U_n x_n) \\
&= d(x_n, x_{n+1}) + \delta_n d(u, U_n x_n)
\end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. By Corolally 6.1, we get $\{U_n x\}$ is a Cauchy sequence for each $x \in X$ and there exists $\lim_{n\to\infty} U_n x$. Put $Ux = \lim_{n\to\infty} U_n x$. By Lemma 6.3, U is nonexpansive and $\operatorname{Fix} U = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. Put

$$\gamma_n = d(u, P_F u)^2 - (1 - \delta_n) d(u, U_n x_n)^2.$$

We next show $\limsup_{n\to\infty} \gamma_n \le 0$. We can take a subsequence $\{\gamma_{n_i}\}$ of $\{\gamma_n\}$ such that

$$\lim_{i\to\infty} \gamma_{n_i} = \limsup_{n\to\infty} \gamma_n.$$

Further, since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \xrightarrow{\Delta} x_0 \in X$. By Lemma 6.3, we get

$$0 \le d(x_{n_{i_j}}, Ux_{n_{i_j}}) \le d(x_{n_{i_j}}, U_{n_{i_j}} x_{n_{i_j}}) + d(U_{n_{i_j}} x_{n_{i_j}}, Ux_{n_{i_j}}) \le d(x_{n_{i_j}}, U_{n_{i_j}} x_{n_{i_j}}) + \sup_{x\in C} d(U_{n_{i_j}} x, Ux)$$

and letting $j \to \infty$, we obtain $\lim_{j\to\infty} d(x_{n_{i_j}}, Ux_{n_{i_j}}) = 0$. Since U is delta-demiclosed, we have $x_0 \in \operatorname{Fix} U = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. It follows that

$$\begin{aligned}\left|\gamma_n - \left(d\left(u, P_F u\right)^2 - d(u, x_n)^2\right)\right| &= \left|d(u, x_n)^2 - d(u, U_n x_n)^2 + \delta_n d(u, U_n x_n)^2\right| \\ &\le \left|d(u, x_n)^2 - d(u, U_n x_n)^2\right| + \delta_n d(u, U_n x_n)^2 \\ &\le (d(u, x_n) + d(u, U_n x_n)) d(x_n, U_n x_n) + \delta_n d(u, U_n x_n)^2 \to 0.\end{aligned}$$

By Lemma 2.9, letting $n \to \infty$, we get

$$\begin{aligned}\limsup_{n\to\infty} \gamma_n = \lim_{i\to\infty} \gamma_{n_i} &= \lim_{j\to\infty} \gamma_{n_{i_j}} \\ &= \lim_{j\to\infty} (d(u, P_F u)^2 - d(u, x_{n_{i_j}})^2) \\ &= d(u, P_F u)^2 - \lim_{j\to\infty} d(u, x_{n_{i_j}})^2 \\ &\le d(u, P_F u)^2 - d(u, x_0)^2 \\ &\le 0.\end{aligned}$$

Then, it follows that

$$\begin{aligned}d\left(x_{n+1}, P_F u\right)^2 &\le \delta_n d\left(u, P_F u\right)^2 + (1-\delta_n) d\left(U_n x_n, P_F u\right)^2 - \delta_n (1-\delta_n) d(u, U_n x_n)^2 \\ &= (1-\delta_n) d\left(x_n, P_F u\right)^2 + \delta_n \gamma_n.\end{aligned}$$

Using Lemma 2.17, we get $\lim_{n\to\infty} d\left(x_n, P_F u\right) = 0$. Consequently, we get the desired result. □

Theorem 6.2. *Let X a complete* CAT(0) *space, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \ne \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \le n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. Let $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$ and $u \in X$. Define $\{x_n\} \subset X$ by $x_1 \in X$ and*

$$x_{n+1} = \delta_n u \oplus (1-\delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the conditions hold:

- $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;
- $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Then, $x_n \to P_F u$.

Proof. Let $p = P_F u \in F$. Then, we get

$$\begin{aligned}d(x_{n+1}, p) &\le \delta_n d(u, p) + (1-\delta_n) d(U_n x_n, p) \\ &\le \delta_n d(u, p) + (1-\delta_n) d(x_n, p) \\ &\le \max\{d(u, p), d(x_n, p)\} \le \max\{d(u, p), d(x_1, p)\}\end{aligned}$$

and hence $\{x_n\}$ and $\{U_n x_n\}$ is bounded. Further, we obtain

$$\begin{aligned} d(x_{n+1}, p)^2 &\leq \delta_n d(u, p)^2 + (1 - \delta_n) d(U_n x_n, p)^2 - \delta_n (1 - \delta_n) d(u, U_n x_n)^2 \\ &\leq (1 - \delta_{n_i}) d(x_n, p)^2 + \delta_n (d(u, p)^2 - (1 - \delta_n) d(U_n x_n, p)^2). \end{aligned}$$

Put $a_n = d(x_n, p)^2$ and $b_n = d(u, p)^2 - (1 - \delta_n) d(U_n x_n, p)$. Let $\varphi \colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function such that $\lim_{i\to\infty} \varphi(i) = \infty$. Suppose

$$\liminf_{i\to\infty} \left(a_{\varphi(i)+1} - a_{\varphi(i)} \right) \geq 0$$

and put $\varphi(i) = n_i$. Then, we get

$$\begin{aligned} 0 &\leq \liminf_{i\to\infty} (a_{n_{i+1}} - a_{n_i}) \\ &\leq \liminf_{i\to\infty} (d(x_{n_{i+1}}, p)^2 - d(x_{n_i}, p)^2) \\ &\leq \liminf_{i\to\infty} (\delta_{n_i} d(u, p)^2 + (1 - \delta_{n_i}) d(U_{n_i} x_{n_i}, p)^2 - d(x_{n_i}, p)^2) \\ &= \liminf_{i\to\infty} (\delta_{n_i} (d(u, p)^2 - d(U_{n_i} x_{n_i}, p)^2) + d(U_{n_i} x_{n_i}, p)^2 - d(x_{n_i}, p)^2) \\ &= \liminf_{i\to\infty} (d(U_{n_i} x_{n_i}, p)^2 - d(x_{n_i}, p)^2) \\ &\leq \limsup_{i\to\infty} (d(U_{n_i} x_{n_i}, p)^2 - d(x_{n_i}, p)^2) \\ &\leq 0 \end{aligned}$$

and hence

$$\lim_{i\to\infty} (d(U_{n_i} x_{n_i}, p)^2 - d(x_{n_i}, p)^2) = 0.$$

By (ii) of Theorem 5.1, we get

$$\sum_{k=1}^{n_i} \alpha_{n_i}^k d(T_k x_{n_i}, U_{n_i} x_{n_i})^2 \leq \sum_{k=1}^{n_i} \alpha_{n_i}^k d(T_k x_{n_i}, p)^2 - d(U_{n_i} x_{n_i}, p)^2 \leq d(x_{n_i}, p)^2 - d(U_{n_i} x_{n_i}, p)^2$$

and hence

$$0 \leq \alpha_{n_i}^1 d(T_1 x_{n_i}, U_{n_i} x_{n_i})^2 \leq d(x_{n_i}, p)^2 - d(U_{n_i} x_{n_i}, p)^2.$$

Since $\lim_{i\to\infty} \alpha_{n_i}^1 > 0$, letting $i \to \infty$, we have $\lim_{i\to\infty} d(T_1 x_{n_i}, U_{n_i} x_{n_i}) = 0$. Then, it follows that

$$0 \leq d(x_{n_i}, p) - d(T_1 x_{n_i}, p) \leq d(x_{n_i}, p) - (d(U_{n_i} x_{n_i}, p) - d(T_1 x_{n_i}, U_{n_i} x_{n_i})).$$

Letting $i \to \infty$, we obtain $\lim_{i\to\infty} (d(x_{n_i}, p) - d(T_1 x_{n_i}, p)) = 0$. Since T_1 is strongly quasinonexpansive, we get $\lim_{i\to\infty} d(x_{n_i}, T_1 x_{n_i}) = 0$ and thus $\lim_{i\to\infty} d(x_{n_i}, U_{n_i} x_{n_i}) = 0$. Further, it follows that

$$0 \leq |d(x_{n_i}, u) - d(U_{n_i} x_{n_i}, u)| \leq d(x_{n_i}, U_{n_i} x_{n_i})$$

and hence $\liminf_{i\to\infty} d(x_{n_i}, u) = \liminf_{i\to\infty} d(U_{n_i} x_{n_i}, u)$. Take a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $\lim_{j\to\infty} d(x_{n_{i_j}}, u) = \liminf_{i\to\infty} d(x_{n_i}, u)$ and take a subsequence $\{x_{n_{i_{j_l}}}\}$ of $\{x_{n_{i_j}}\}$ such that $x_{n_{i_{j_l}}} \xrightarrow{\Delta} x_0 \in X$. Put $w_l = n_{i_{j_l}}$. By Lemma 6.4, put $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Let C be a bounded subset of X including $\{x_n\}$. Then, we get

$$0 \leq d(x_{w_l}, U w_l) \leq d(x_{w_l}, U_{w_l} x_{w_l}) + d(U_{w_l} x_{w_l}, U x_{w_l}) \leq d(x_{w_l}, U_{w_l} x_{w_l}) + \sup_{x\in C} d(U_{w_l} x, U x_{w_l}).$$

Letting $l \to \infty$, we get $\lim_{l\to\infty} d(x_{w_l}, Ux_{w_l}) = 0$. Since U is delta-demiclosed, it follows that $x_0 \in F$. By Lemma 2.9, we get

$$\liminf_{i\to\infty} d(U_{n_i}x_{n_i}, u) = \lim_{l\to\infty} d(x_{w_l}, u) \geq d(x_0, u) \geq d(p, u).$$

Then, it follows that

$$\begin{aligned}\limsup_{i\to\infty} b_{n_i} &= \limsup_{i\to\infty}(d(u,p)^2 - (1-\delta_{n_i})d(U_{n_i}x_{n_i}, u)^2) \\ &\leq \limsup_{i\to\infty}(d(u,p)^2 - (1-\delta_{n_i})d(u,p)^2) \\ &= 0\end{aligned}$$

and hence $\limsup_{i\to\infty} b_{n_i} \leq 0$. Using Lemma 2.19, we get $\lim_{n\to\infty} a_n = 0$ and thus $x_n \to P_F u$. Consequently, we complete the proof. □

We introduce a Mann iteration using a balanced mapping of a countable family of nonexpansive mapping and prove delta-convergence to a common fixed point.

Theorem 6.3. *Let X be a complete* CAT(0) *space, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{t_n \mid n \in \mathbb{N}\} \subset [0,1[$. Define a sequence $\{x_n\}$ of X by $x_1 \in X$ and*

$$x_{n+1} = t_n x_n \oplus (1-t_n)U_n x_n$$

for all $n \in \mathbb{N}$. Suppose the following conditions:

- *$\lim_{n\to\infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$;*
- *$\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$.*

Then $x_n \xrightarrow{\Delta} x_0 \in F$.

Proof. Let $p \in F \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k$ for $n \in \mathbb{N}$. Since U_n is quasinonexpansive for $n \in \mathbb{N}$, we get

$$d(x_{n+1}, p) \leq t_n d(x_n, p) + (1-t_n)d(U_n x_n, p) \leq d(x_n, p)$$

and hence $\{d(x_n, p)\}$ is nonincreasing. Then, there exists $\lim_{n\to\infty} d(x_n, p)$. Further, we get

$$\begin{aligned}d(x_{n+1}, p)^2 &\leq t_n d(x_n, p)^2 + (1-t_n)d(U_n x_n, p)^2 - t_n(1-t_n)d(x_n, U_n x_n)^2 \\ &\leq d(x_n, p)^2 - t_n(1-t_n)d(x_n, U_n x_n)^2\end{aligned}$$

and hence

$$t_n(1-t_n)d(x_n, U_n x_n)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

Since $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$, we get $\liminf_{n\to\infty} d(x_n, U_n x_n) = 0$. We next show $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. Let C a nonempty bounded subset of X including $\{x_n\}$. By nonexpansiveness of U_n for $n \in \mathbb{N}$, we get

$$\begin{aligned}d(x_{n+1}, U_{n+1}x_{n+1}) &\leq d(x_{n+1}, U_n x_n) + d(U_n x_n, U_n x_{n+1}) + d(U_n x_{n+1}, U_{n+1}x_{n+1}) \\ &\leq d(x_{n+1}, U_n x_n) + d(x_{n+1}, x_n) + d(U_n x_{n+1}, U_{n+1}x_{n+1}) \\ &= d(x_n, U_n x_n) + d(U_n x_{n+1}, U_{n+1}x_{n+1}) \\ &\leq d(x_n, U_n x_n) + \sup_{x\in C} d(U_n x, U_{n+1}x)\end{aligned}$$

for all $n \in \mathbb{N}$. By Lemma 6.2, $\sum_{n=1}^{\infty} \sup_{x\in C} d(U_n x, U_{n+1}x) < \infty$. Since Lemma 2.18, there exists $\lim_{n\to\infty} d(x_n, U_n x_n)$ and hence $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. Since $\{U_n x\}$ is a Cauchy sequence for each

$x \in X$, there exists $\lim_{n\to\infty} U_n x$ for each $x \in X$. Put $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Then, U is nonexpansive and $\operatorname{Fix} U = F$. Take a subsequence $\{x_{n_i}\}$ of X arbitrary with $\mathrm{AC}(\{x_{n_i}\}) = \{y_0\}$. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \xrightarrow{\Delta} z_0 \in X$. Then, we get

$$d(x_{n_{i_j}}, Ux_{n_{i_j}}) \leq d(x_{n_{i_j}}, U_{n_{i_j}} x_{n_{i_j}}) + d(U_{n_{i_j}} x_{n_{i_j}}, Ux_{n_{i_j}}) \leq d(x_{n_{i_j}}, U_{n_{i_j}} x_{n_{i_j}}) + \sup_{x\in C} d(U_{n_{i_j}} x, Ux).$$

By (i) of Lemma 6.3, we get $\lim_{n\to\infty} d(x_{n_{i_j}}, Ux_{n_{i_j}}) = 0$. Then, we get

$$\begin{aligned}\limsup_{j\to\infty} d(x_{n_{i_j}}, Uz_0) &\leq \limsup_{j\to\infty} (d(x_{n_{i_j}}, Ux_{n_{i_j}}) + d(Ux_{n_{i_j}}, Uz_0)) \\ &= \limsup_{j\to\infty} d(Ux_{n_{i_j}}, Uz_0) \\ &\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, z_0)\end{aligned}$$

and hence $z_0 \in \operatorname{Fix} U = F$. Put $\mathrm{AC}(\{x_n\}) = \{x_0\}$. Then, we get

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, z_0) &= \limsup_{j\to\infty} d(x_{n_{i_j}}, z_0) \\ &\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, y_0) \\ &\leq \limsup_{i\to\infty} d(x_{n_i}, y_0) \\ &\leq \limsup_{i\to\infty} d(x_{n_i}, x_0) \\ &\leq \limsup_{n\to\infty} d(x_n, x_0) \leq \limsup_{n\to\infty} d(x_n, z_0)\end{aligned}$$

and hence $x_0 = y_0 = z_0 \in F$. Therefore, we get $x_n \xrightarrow{\Delta} x_0 \in F$ and which is the desired result. □

6.2 In a CAT(−1) space

Lemma 6.5. *Let X be a complete* CAT(−1) space, *C a nonempty bounded subset of X, T_k a quasinonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset]0, 1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$, then*

$$\sum_{n=1}^{\infty} \sup_{x\in C} d(U_{n+1}x, U_n x) < \infty.$$

Proof. Let $x \in C$ and $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. By Lemma 5.1, we get

$$\begin{aligned}&\sinh\left(\frac{d(U_n x, U_{n+1} x)}{2}\right) \\ &\leq \frac{1}{2} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| \left(\sinh \frac{d(T_k x, U_n x) + d(T_k x, U_{n+1} x)}{2} + \sinh \frac{d(T_{n+1} x, U_n x) + d(T_{n+1} x, U_{n+1} x)}{2}\right) \\ &\leq \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| \sinh 2d(x, z)\end{aligned}$$

and hence

$$d(U_nx, U_{n+1}x) \leq 2\sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| \sinh 2d(x,z) \leq 2\sinh 2M \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k|,$$

where $M = \sup_{x\in C} d(x,z)$. Since $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$, we get

$$\sum_{n=1}^{\infty} \sup_{x\in C} d(U_nx, U_{n+1}x) < \infty.$$

Consequently, we get the desired result. □

By Lemma 6.5, we can prove the following corollary easily.

Corollary 6.2. *Let X be a complete $\mathrm{CAT}(-1)$ space, T_k a quasinonexpansive mapping of X into itself with $\bigcap_{k=1}^{\infty} \mathrm{Fix}\, T_k \neq \emptyset$ and, $\{\alpha_n^k \mid n,k \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty}\sum_{k=1}^{n} \left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$, then $\sum_{n=1}^{\infty} d(U_{n+1}x, U_nx) < \infty$ and $\{U_nx\}$ is a Cauchy sequence for each $x \in X$.*

By Corollary 6.2, there exists a limit of $\{U_nx\}$. In the following lemma, we consider its properties.

Lemma 6.6. *Let X be a complete $\mathrm{CAT}(-1)$ space, C a nonempty bounded subset of X, T_k a quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ with $\bigcap_{k=1}^{\infty} \mathrm{Fix}\, T_k \neq \emptyset$, $\{\alpha_n^k \mid n,k \in \mathbb{N}, k \leq n\} \subset [0,1]$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. Suppose the following conditions hold:*

- $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$;
- $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Put $Ux = \lim_{n\to\infty} U_nx$ for each $x \in X$. Then, the following hold:

(i) $\lim_{n\to\infty} \sup_{x\in C} d(U_nx, Ux) = 0$;
(ii) $\mathrm{Fix}\, U = \bigcap_{k=1}^{\infty} \mathrm{Fix}\, T_k$;
(iii) *U is quasinonexpansive and delta-demiclosed.*

Proof. (i) Let $m,n \in \mathbb{N}$ such that $n \leq m$ and $x \in X$. Then, we get

$$\begin{aligned} d(U_mx, U_nx) &\leq d(U_mx, U_{n+1}x) + d(U_{n+1}x, U_nx) \\ &\leq d(U_mx, U_{n+2}x) + d(U_{n+2}x, U_{n+1}x) + d(U_{n+1}x, U_nx) \\ &\leq \cdots \\ &\leq \sum_{l=n}^{m-1} d(U_lx, U_{l+1}x) \leq \sum_{l=n}^{\infty} d(U_lx, U_{l+1}x) \end{aligned}$$

and hence

$$d(U_mx, U_nx) \leq \sum_{l=n}^{\infty} d(U_lx, U_{l+1}x). \tag{6.2}$$

By (6.2) and Corollary 6.2, letting $m \to \infty$, we get

$$\sup_{x\in C} d(Ux, U_nx) \leq \sum_{l=n}^{\infty} \sup_{x\in C} d(U_lx, U_{l+1}x).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} \sup_{x\in C} d(Ux, U_nx) = 0$.

(ii) Let $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k = \operatorname{Fix} U_n$. Then,

$$Uz = \lim_{n\to\infty} U_n z = \lim_{n\to\infty} z = z$$

and hence $z \in \operatorname{Fix} U$. Inversely, let $z \in \operatorname{Fix} U$ and $w \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. By Lemma 5.1, we have

$$\sum_{k=1}^{n} \alpha_n^k \cosh d(T_k z, U_n z) \cosh d(U_n z, w) \le \sum_{k=1}^{n} \alpha_n^k \cosh d(T_k z, w) \le \cosh d(z, w)$$

and hence

$$0 \le \sum_{k=1}^{n} \alpha_n^k (\cosh d(T_k z, U_n z) - 1) \le \frac{\cosh d(z, w)}{\cosh d(U_n z, w)} - 1.$$

Fix $i \in \mathbb{N}$ arbitrarily. Then, it follows that

$$0 \le \alpha_n^i (\cosh d(T_i z, U_n z) - 1) \le \sum_{k=1}^{n} \alpha_n^k (\cosh d(T_k z, U_n z) - 1) \le \frac{\cosh d(z, w)}{\cosh d(U_n z, w)} - 1.$$

Since $\lim_{n\to\infty} \alpha_n^i > 0$, letting $n \to \infty$, we get $\lim_{n\to\infty} d(T_i z, U_n z) = 0$. Then, we have

$$0 = \lim_{n\to\infty} d(T_i z, U_n z) = d(T_i z, Uz) = d(T_i z, z)$$

and hence $z \in \operatorname{Fix} T_i$. Since $i \in \mathbb{N}$ is arbitrary, we get $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$.

(iii) Let $x \in X$ and $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k$. By quasinonexpansiveness of U_n for $n \in \mathbb{N}$, we get $d(U_n x, z) \le d(x, z)$. Letting $n \to \infty$, we have $d(Ux, z) \le d(x, z)$ for $x \in X$ and $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k = \operatorname{Fix} U$. Therefore, U is quasinonexpansive. Let $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$ and $\{x_j\} \subset X$ such that $x_j \overset{\Delta}{\rightharpoonup} x_0 \in X$ and $\lim_{j\to\infty} d(x_j, Ux_j) = 0$. By Theorem 5.1, we get

$$\sum_{k=1}^{n} \alpha_n^k \cosh d(T_k x_j, U_n x_j) \cosh d(U_n x_j, z) \le \sum_{k=1}^{n} \alpha_n^k \cosh d(T_k x_j, z) \le \cosh d(x_j, z)$$

and hence

$$0 \le \sum_{k=1}^{n} \alpha_n^k \cosh d(T_k x_j, U_n x_j) - 1 = \sum_{k=1}^{n} \alpha_n^k (\cosh d(T_k x_j, U_n x_j) - 1) \le \frac{\cosh d(x_j, z)}{\cosh d(U_n x_j, z)} - 1.$$

Fix $i_0 \in \mathbb{N}$ arbitrarily. Then, we get

$$0 \le \alpha_n^{i_0} (\cosh d(T_{i_0} x_j, U_n x_j) - 1) \le \frac{\cosh d(x_j, z)}{\cosh d(U_n x_j, z)} - 1.$$

Putting $\alpha^{i_0} = \lim_{n\to\infty} \alpha_n^{i_0} > 0$ and letting $n \to \infty$, we get

$$\begin{aligned} 0 \le \alpha^{i_0} (\cosh d(T_{i_0} x_j, Ux_j) - 1) &\le \frac{\cosh d(x_j, z)}{\cosh d(Ux_j, z)} - 1 \\ &= \frac{\cosh d(x_j, z) - \cosh d(Ux_i, z)}{\cosh d(Ux_j, z)} \\ &\le 2 \sinh \frac{d(x_j, z) + d(Ux_j, z)}{2} \sinh \frac{d(x_j, z) - d(Ux_j, z)}{2} \\ &\le 2 \sinh d(x_j, z) \sinh \frac{d(x_j, Ux_j)}{2}. \end{aligned}$$

Since $\alpha^{i_0} > 0$, letting $j \to \infty$, we get $\lim_{j\to\infty} d(T_{i_0} x_j, Ux_j) = 0$ and hence $\lim_{j\to\infty} d(x_j, T_{i_0} x_j) = 0$. Since T_{i_0} is delta-demiclosed, we get $x_0 \in \operatorname{Fix} T_{i_0}$. Since i_0 is arbitrary, it follows that $x_0 \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k = \operatorname{Fix} U$. Consequently, we complete the proof. □

6.2.1 Halpern iteration

We introduce a Halpern iteration using a balanced mapping of a countable family of mappings.

Theorem 6.4. *Let X be a complete* CAT(-1) *space, T_k a strongly quasinonexpansive and delta-demiclosed mapping for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$. Let $u \in X$ and define a sequence $\{x_n\}$ by $x_1 \in X$ and*

$$x_{n+1} = \delta_n u \overset{-1}{\oplus} (1-\delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;
- $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Then, $x_n \to P_F u$.

Proof. Let $p = P_F u \in F$. Then, we get

$$\begin{aligned}
\cosh d(x_{n+1}, p) &\leq \delta_n \cosh d(u,p) + (1-\delta_n) \cosh d(U_n x_n, p) \\
&\leq \delta_n \cosh d(u,p) + (1-\delta_n) \cosh d(x_n, p) \\
&\leq \max\{\cosh d(u,p), \cosh d(x_n,p)\} \\
&\leq \max\{\cosh d(u,p), \cosh d(x_1,p)\}
\end{aligned}$$

and hence $\{x_n\}$ and $\{U_n x_n\}$ is bounded. Put

$$\begin{aligned}
a_n &= \cosh d(x_n, p) - 1, \\
b_n &= \frac{\left(1-\delta_n + \sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cosh d(u, U_n x_n) + (1-\delta_n)^2}\right)\cosh d(u,p)}{\delta_n + 2(1-\delta_n)\cosh d(u, U_n x_n)} - 1, \\
\sigma_n &= 1 - \frac{1-\delta_n}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cosh d(u, U_n x_n) + (1-\delta_n)^2}}
\end{aligned}$$

for each $n \in \mathbb{N}$. By Lemma 2.16, we get

$$a_{n+1} = \cosh d(x_{n+1}, p) - 1 \leq (1-\sigma_n) a_n + \sigma_n b_n.$$

We first show $\sum_{n=1}^{\infty} \sigma_n = \infty$. Then, it follows that

$$\begin{aligned}
\sigma_n &= 1 - \frac{1-\delta_n}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cosh d(u, U_n x_n) + (1-\delta_n)^2}} \\
&\geq 1 - \frac{1-\delta_n}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n) + (1-\delta_n)^2}} = 1 - (1-\delta_n) = \delta_n.
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \delta_n = \infty$, we get $\sum_{n=1}^{\infty} \sigma_n = \infty$. Let $\varphi \colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function such that $\lim_{i\to\infty} \varphi(i) = \infty$. Suppose

$$\liminf_{i\to\infty} \left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \geq 0$$

and put $\varphi(i) = n_i$. Then, we get

$$0 \leq \liminf_{i\to\infty} (a_{n_i+1} - a_{n_i})$$

$$
\begin{aligned}
&\leq \liminf_{i\to\infty}(\cosh d(x_{n_i+1},p) - \cosh d(x_{n_i},p)) \\
&\leq \liminf_{i\to\infty}(\delta_{n_i}\cosh d(u,p) + (1-\delta_{n_i})\cosh d(U_{n_i}x_{n_i},p) - \cosh d(x_{n_i},p)) \\
&= \liminf_{i\to\infty}(\delta_{n_i}(\cosh d(u,p) - \cosh d(U_{n_i}x_{n_i},p)) + \cosh d(U_{n_i}x_{n_i},p) - \cosh d(x_{n_i},p)) \\
&= \liminf_{i\to\infty}(\cosh d(U_{n_i}x_{n_i},p) - \cosh d(x_{n_i},p)) \\
&\leq \limsup_{i\to\infty}(\cosh d(U_{n_i}x_{n_i},p) - \cosh d(x_{n_i},p)) \leq 0
\end{aligned}
$$

and hence

$$\lim_{i\to\infty}(\cosh d(U_{n_i}x_{n_i},p) - \cosh d(x_{n_i},p)) = 0.$$

By Lemma 5.1, we get

$$\sum_{k=1}^{n_i}\alpha_{n_i}^k\cosh d(T_kx_{n_i},U_{n_i}x_{n_i})\cosh d(U_{n_i}x_{n_i},p) \leq \sum_{k=1}^{n_i}\alpha_{n_i}\cosh d(T_kx_{n_i},p) \leq \cosh d(x_{n_i},p)$$

and hence

$$0 \leq \alpha_{n_i}^1(\cosh d(T_1x_{n_i},U_{n_i}x_{n_i}) - 1) \leq \sum_{k=1}^{n_i}\alpha_{n_i}^k(\cosh d(T_kx_{n_i},U_{n_i}x_{n_i}) - 1) \leq \frac{\cosh d(x_{n_i},p)}{\cosh d(U_{n_i}x_{n_i},p)} - 1.$$

Since $\lim_{i\to\infty}\alpha_{n_i}^1 > 0$, letting $i \to \infty$, we have $\lim_{i\to\infty} d(T_1x_{n_i},U_{n_i}x_{n_i}) = 0$. Then, it follows that

$$0 \leq d(x_{n_i},p) - d(T_1x_{n_i},p) \leq d(x_{n_i},p) - (d(U_{n_i}x_{n_i},p) + d(T_1x_{n_i},U_{n_i}x_{n_i})).$$

Letting $i \to \infty$, we get $\lim_{i\to\infty}(d(x_{n_i},p) - d(T_1x_{n_i},p)) = 0$. Since T_1 is strongly quasinonexpansive, we get $\lim_{i\to\infty} d(x_{n_i},T_1x_{n_i}) = 0$. Then, we obtain that

$$d(x_{n_i},U_{n_i}x_{n_i}) \leq d(x_{n_i},T_1x_{n_i}) + d(T_1x_{n_i},U_{n_i}x_{n_i})$$

and thus $\lim_{i\to\infty} d(x_{n_i},U_{n_i}x_{n_i}) = 0$. Further, we get

$$0 \leq |d(x_{n_i},u) - d(U_{n_i}x_{n_i},u)| \leq d(x_{n_i},U_{n_i}x_{n_i})$$

and hence $\liminf_{i\to\infty} d(x_{n_i},u) = \liminf_{i\to\infty} d(U_{n_i}x_{n_i},u)$. Take a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $\lim_{j\to\infty} d(x_{n_{i_j}},u) = \liminf_{i\to\infty} d(x_{n_i},u)$. Further, take a subsequence $\{x_{n_{i_{j_l}}}\}$ of $\{x_{n_{i_j}}\}$ such that $x_{n_{i_{j_l}}} \stackrel{\Delta}{\rightharpoonup} x_0 \in X$. Put $w_l = n_{i_{j_l}}$. Then, we get $\lim_{l\to\infty} d(x_{w_l},U_{w_l}x_{w_l}) = 0$. By Corolally 6.2, $\{U_{w_l}x\}$ is a Cauchy sequence for $x \in X$ and put $Ux = \lim_{l\to\infty} U_{w_l}x$. Then, we get

$$d(x_{w_l},Ux_{w_l}) \leq d(x_{w_l},U_{w_l}x_{w_l}) + d(U_{w_l}x_{w_l},Ux_{w_l}) \leq d(x_{w_l},U_{w_l}x_{w_l}) + \sup_{x\in C} d(U_{w_l}x,Ux),$$

where C is a bounded subset of X including $\{x_n\}$. By Lemma 6.6, letting $l \to \infty$, we get $\lim_{l\to\infty} d(x_{w_l},Ux_{w_l}) = 0$. Since U is delta-demiclosed, we get $x_0 \in \operatorname{Fix} U = \bigcap_{k=1}^{\infty}\operatorname{Fix} T_k$. By Lemma 2.9, we have

$$\liminf_{i\to\infty} d(u,U_{n_i}x_{n_i}) = \lim_{i\to\infty} d(u,x_{n_i}) = \lim_{l\to\infty} d(u,x_{w_l}) \geq d(u,x_0) \geq d(u,p)$$

and hence

$$
\begin{aligned}
\limsup_{i\to\infty} b_{n_i} &= \limsup_{i\to\infty}\left(\frac{\cosh d(u,p)}{\cosh d(U_{n_i}x_{n_i},p)} - 1\right) \\
&= \frac{\cosh d(u,p)}{\liminf_{i\to\infty}\cosh d(U_{n_i}x_{n_i},p)} - 1 \leq 0.
\end{aligned}
$$

Using Lemma 2.19, we obtain $x_n \to P_F u$ and get the desired result. □

Corollary 6.3. *Let X be a complete* CAT(-1) *space, T_k a strongly quasinonexpansive and delta-demiclosed mapping for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$. Let $u \in X$ and define a sequence $\{x_n\}$ by $x_1 \in X$ and*

$$x_{n+1} = \delta_n u \oplus (1 - \delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$*;*
- $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$ and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$*.*

Then, $x_n \to P_F u$.

Proof. Let

$$\delta_n u \oplus (1 - \delta_n) U_n x_n = \gamma_n u \overset{-1}{\oplus} (1 - \gamma_n) U_n x_n$$

for each $n \in \mathbb{N}$ and $p \in F$. Since U_n is quasinonexpansive for all $n \in \mathbb{N}$, we get

$$\begin{aligned}
\cosh d(x_{n+1}, p) &\leq \delta_n \cosh d(u, p) + (1 - \delta_n) \cosh d(U_n x_n, p) \\
&\leq \delta_n \cosh d(u, p) + (1 - \delta_n) \cosh d(x_n, p) \\
&\leq \max\{\cosh d(u, p), \cosh d(x_n, p)\} \\
&\leq \max\{\cosh d(u, p), \cosh d(x_1, p)\}
\end{aligned}$$

for all $n \in \mathbb{N}$ and hence

$$d(U_n x_n, p) \leq d(x_n, p) \leq \sup_{n\in\mathbb{N}} d(x_n, p) \leq \max\{d(u, p), d(x_1, p)\} < \infty.$$

By definition of (-1)-convex combination, it follows that

$$\gamma_n = \frac{\sinh \delta_n d(u, U_n x_n)}{\sinh \delta_n d(u, U_n x_n) + \sinh(1 - \delta_n) d(u, U_n x_n)}$$

for $n \in \mathbb{N}$.Since $t/\sinh t$ is nonincreasing function and $t/\sinh t \leq 1$ for $t \geq 0$, we have

$$\begin{aligned}
\gamma_n &= \frac{\sinh \delta_n d(u, U_n x_n)}{\sinh \delta_n d(u, U_n x_n) + \sinh(1 - \delta_n) d(u, U_n x_n)} \\
&\geq \frac{\delta_n d(u, U_n x_n)}{\sinh \delta_n d(u, U_n x_n) + \sinh(1 - \delta_n) d(u, U_n x_n)} \\
&\geq \frac{\delta_n d(u, U_n x_n)}{2 \sinh d(u, U_n x_n)} \\
&\geq \frac{\delta_n}{2} \times \frac{d(u, p) + d(p, U_n x_n)}{\sinh(d(u, p) + d(p, U_n x_n))} \\
&\geq \frac{(d(u, p) + \max\{d(u, p), d(x_1, p)\})}{2 \sinh(d(u, p) + \max\{d(u, p), d(x_1, p)\})} \delta_n.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \delta_n = \infty$, we get $\sum_{n=1}^{\infty} \gamma_n = \infty$. Since $\delta_n \to 0$, we get $\gamma_n \to 0$. By Theorem 6.4, we get the desired result. □

6.3 In a CAT(1) space

Lemma 6.7. *Let X be a complete* CAT(1) *space with $\sup_{w,w' \in X} d(w, w') < \pi/2$, T_k a quasinonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset]0,1[$*

such that $\sum_{k=1}^{n} \alpha_n^k = 1$ *for* $n \in \mathbb{N}$, *and* U_n *a balanced mapping of* $\{T_k\}$ *and* $\{\alpha_n^k\}$ *for each* $n \in \mathbb{N}$. *If* $\sum_{n=1}^{\infty} \sum_{k=1}^{N} |\alpha_{n+1}^k - \alpha_n^k| < \infty$, *then*

$$\sum_{n=1}^{\infty} \sup_{x \in X} d(U_{n+1}x, U_n x) < \infty.$$

Proof. Let $x \in X$ and $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n+1} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k$. Put

$$M = \sup_{w,w' \in X} d(w, w') < \pi/2.$$

By Lemma 5.1, we get

$$\begin{aligned}
&\sin\left(\frac{d(U_{n+1}x, U_n x)}{2}\right) \\
&\leq \frac{1}{2\cos M} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| \left(\sin \frac{d(T_k x, U_n x) + d(T_k x, U_{n+1} x)}{2} + \sin \frac{d(T_{n+1} x, U_n x) + d(T_{n+1} x, U_{n+1} x)}{2}\right) \\
&\leq \frac{1}{\cos M} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| \left(\sin \frac{d(T_k x, U_n x) + d(T_k x, U_{n+1} x)}{4} + \sin \frac{d(T_{n+1} x, U_n x) + d(T_{n+1} x, U_{n+1} x)}{4}\right) \\
&\leq \frac{2 \sin d(x, z)}{\cos M} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| \\
&\leq \frac{2}{\cos M} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k|.
\end{aligned}$$

Since $t/2 \leq \sin t$ for $t \in [0, \pi/2[$, it follows that

$$\frac{d(U_n x, U_{n+1} x)}{2} \leq \frac{2}{\cos M} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k|$$

and hence

$$d(U_n x, U_{n+1} x) \leq \frac{4}{\cos M} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k|.$$

Then, we obtain

$$\sup_{x \in X} d(U_n x, U_{n+1} x) \leq \frac{4}{\cos M} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k|.$$

Since $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$, it follows that

$$\sum_{n=1}^{\infty} \sup_{x \in X} d(U_{n+1}x, U_n x) < \infty$$

and thus we get the desired result. □

By Lemma 6.7, we can prove the following corollary easily.

Corollary 6.4. *Let* X *be a complete* CAT(1) *space with* $\sup_{w,w' \in X} d(w, w') < \pi/2$, T_k *a quasinonexpansive mapping of* X *into itself for* $k \in \mathbb{N}$ *such that* $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset]0, 1[$ *with* $\sum_{k=1}^{n} \alpha_n^k = 1$ *for* $n \in \mathbb{N}$, *and* U_n *a balanced mapping of* $\{T_k\}$ *and* $\{\alpha_n^k\}$ *for each* $n \in \mathbb{N}$. *If* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$, *then,*

$$\sum_{n=1}^{\infty} d(U_{n+1}x, U_n x) < \infty$$

and $\{U_n x\}$ is a Cauchy sequence for each $x \in X$.

By Corollary 6.4, there exists a limit of $\{U_n x\}$. In the following lemma, we consider its properties.

Lemma 6.8. *Let X be a complete* CAT(1) *space with $\sup_{w,w' \in X} d(w, w') < \pi/2$, T_k a quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset]0, 1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. Suppose the following conditions hold:*

(a) $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$;
(b) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Put $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Then, the following hold:

(i) $\lim_{n\to\infty} \sup_{x\in X} d(U_n x, Ux) = 0$;
(ii) $\operatorname{Fix} U = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$;
(iii) U is quasinonexpansive and delta-demiclosed.

Proof. (i) Let $m, n \in \mathbb{N}$ such that $n \leq m$ and $x \in X$. Then, we get

$$\begin{aligned} d(U_m x, U_n x) &\leq d(U_m x, U_{n+1} x) + d(U_{n+1} x, U_n x) \\ &\leq d(U_m x, U_{n+2} x) + d(U_{n+2} x, U_{n+1} x) + d(U_{n+1} x, U_n x) \\ &\leq \cdots \\ &\leq \sum_{l=n}^{m-1} d(U_l x, U_{l+1} x) \leq \sum_{l=n}^{\infty} d(U_l x, U_{l+1} x) \end{aligned}$$

and hence

$$d(U_m x, U_n x) \leq \sum_{l=n}^{\infty} d(U_l x, U_{l+1} x).$$

Letting $m \to \infty$, we get

$$\sup_{x\in X} d(Ux, U_n x) \leq \sum_{l=n}^{\infty} \sup_{x\in X} d(U_l x, U_{l+1} x).$$

By Lemma 6.7, letting $n \to \infty$, we get $\lim_{n\to\infty} \sup_{x\in X} d(Ux, U_n x) = 0$.

(ii) Let $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k = \operatorname{Fix} U_n$. Then, we get

$$Uz = \lim_{n\to\infty} U_n z = \lim_{n\to\infty} z = z$$

and hence $z \in \operatorname{Fix} U$. Inversely, let $z \in \operatorname{Fix} U$ and $w \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. By Theorem 5.1, we have

$$\sum_{k=1}^{n} \alpha_n^k \cos d(T_k z, U_n z) \geq \frac{\sum_{k=1}^{n} \alpha_n^k \cos d(T_k z, w)}{\cos d(U_n z, w)} \geq \frac{\cos d(z, w)}{\cos d(U_n z, w)}$$

and hence

$$\sum_{k=1}^{n} \alpha_n^k (1 - \cos d(T_k z, U_n z)) \leq 1 - \frac{\cos d(z, w)}{\cos d(U_n z, w)}.$$

Fix $i \in \mathbb{N}$ arbitrarily. Then, it follows that

$$0 \leq \alpha_n^i (1 - \cos d(T_i z, U_n z)) \leq 1 - \frac{\cos d(z, w)}{\cos d(U_n z, w)}.$$

Since $\lim_{n\to\infty} \alpha_n^i > 0$, we can put $\lim_{n\to\infty} \alpha_n^i = \alpha^i \in \,]0,1]$. Letting $n \to \infty$, we get

$$0 \le \alpha^i (1 - \cos d(T_i z, z)) \le 1 - \frac{\cos d(z,w)}{\cos d(Uz,w)} = 0$$

and hence $d(T_i z, z) = 0$. Since $i \in \mathbb{N}$ is arbitrary, we get $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$.

(iii) Let $x \in X$ and $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{k=1}^{n} \operatorname{Fix} T_k$. By U_n is quasinonexpansive for $n \in \mathbb{N}$, we get

$$d(Ux, z) = \lim_{n\to\infty} d(U_n x, z) \le \lim_{n\to\infty} d(x, z) = d(x, z)$$

and thus U is quasinonexpansive. We next show U is delta-demiclosed. Let $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$, and $\{x_i\} \subset X$ with $x_i \overset{\Delta}{\rightharpoonup} x_0 \in X$ and $\lim_{i\to\infty} d(x_i, Ux_i) = 0$. By Theorem 5.1, we get

$$\sum_{k=1}^{n} \alpha_n^k \cos d(T_k x_i, U_n x_i) \ge \frac{\cos d(x_j, z)}{\cos d(U_n x_i, z)}$$

and hence

$$0 \le \sum_{k=1}^{n} \alpha_n^k (1 - \cos d(T_k x_i, U_n x_i)) \le 1 - \frac{\cos d(x_i, z)}{\cos d(U_n x_i, z)}.$$

Fix $i_0 \in \mathbb{N}$ arbitrarily. Then, it follows that

$$0 \le \alpha_n^{i_0} (1 - \cos d(T_{i_0} x_i, U_n x_i)) \le 1 - \frac{\cos d(x_i, z)}{\cos d(U_n x_i, z)}.$$

Since $\lim_{n\to\infty} \alpha_n^{i_0} > 0$, put $\lim_{n\to\infty} \alpha_n^j = \alpha^{i_0} \in \,]0,1]$. Letting $n \to \infty$, we obtain

$$\begin{aligned}
0 &\le \alpha^{i_0} (1 - \cos d(T_{i_0} x_i, Ux_i)) \\
&\le 1 - \frac{\cos d(x_i, z)}{\cos d(Ux_i, z)} \\
&= \frac{\cos d(Ux_i, z) - \cos d(x_i, z)}{\cos d(Ux_i, z)} \\
&= \frac{2}{\cos d(Ux_i, z)} \sin \frac{d(x_i, z) + d(Ux_i, z)}{2} \sin \frac{d(x_i, z) - d(Ux_i, z)}{2} \\
&\le \frac{2}{\cos d(Ux_i, z)} \sin \frac{d(x_i, z) + d(Ux_i, z)}{2} \sin \frac{d(x_i, Ux_i}{2} \\
&\le \frac{2}{\cos M} \sin \frac{d(x_i, Ux_i)}{2},
\end{aligned}$$

where $M = \sup_{w,w' \in X} d(w, w')$. Letting $i \to \infty$, we have $\lim_{i\to\infty} d(T_{i_0} x_i, Ux_i) = 0$. Since $\lim_{i\to\infty} d(x_i, Ux_i) = 0$, we get $\lim_{i\to\infty} d(x_i, T_{i_0} x_i) = 0$. By delta-demiclosedness of T_{i_0}, we obtain $x_0 \in \operatorname{Fix} T_{i_0}$. Since $i_0 \in \mathbb{N}$ is arbitrary, it follows that $x_0 \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k = \operatorname{Fix} U$ and U is delta-demiclosed. Consequently, we complete the proof. ◻

6.3.1 Halpern iteration

We introduce a Halpern iteration using a balanced mapping of a countable family of mappings.

Theorem 6.5. *Let X be a complete* CAT(1) *space with $\sup_{w,w' \in X} d(w, w') < \pi/2$, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq$*

$\emptyset$, $\{\alpha_n^k \mid n,k \in \mathbb{N}, k \le n\} \subset]0,1[$ *such that* $\sum_{k=1}^{n} \alpha_n^k = 1$ *for* $n \in \mathbb{N}$, $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$, *and* U_n *a balanced mapping of* $\{T_k\}$ *and* $\{\alpha_n^k\}$ *for each* $n \in \mathbb{N}$. *Let* $u \in X$. *Define a sequence* $\{x_n\} \subset X$ *by* $x_1 \in X$ *and*

$$x_{n+1} = \delta_n u \overset{1}{\oplus} (1-\delta_n) U_n x_n$$

for eack $n \in \mathbb{N}$. *Suppose the following conditions:*

(a) $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$;
(b) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$.

Then, $x_n \to P_F u$.

Proof. Put $p = P_F u$. Then, it follows that

$$\begin{aligned} \cos d(x_{n+1}, p) &= \cos d(\delta_n u \overset{1}{\oplus} (1-\delta_n) U_n x_n) \\ &\ge \delta_n \cos d(u,p) + (1-\delta_n) \cos d(U_n x_n, p) \\ &\ge \delta_n \cos d(u,p) + (1-\delta_n) \cos d(x_n, p) \\ &\ge \min\{\cos d(u,p), \cos d(x_n,p)\} \\ &\ge \min\{\cos d(u,p), \cos d(x_1,p)\} \end{aligned}$$

and hence

$$d(x_n, p) \le \max\{d(u,p), d(x_1,p)\} < \frac{\pi}{2}$$

for $n \in \mathbb{N}$. Put

$$\begin{aligned} a_n &= 1 - \cos d(x_n, p), \\ b_n &= 1 - \frac{\left(1 - \delta_n + \sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos d(u, U_n x_n) + (1-\delta_n)^2}\right) \cos d(u,p)}{\delta_n + 2(1-\delta_n)\cos d(u, U_n x_n)}, \\ \sigma_n &= 1 - \frac{1-\delta_n}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos d(u, U_n x_n) + (1-\delta_n)^2}} \end{aligned}$$

for all $n \in \mathbb{N}$. By Lemma 2.15, we get

$$a_{n+1} = 1 - \cos d(\delta_n u \overset{1}{\oplus} (1-\delta_n) U_n x_n, p) \le (1-\sigma_n) a_n + \sigma_n b_n$$

for all $n \in \mathbb{N}$. We first show $\sum_{n=1}^{\infty} \sigma_n = \infty$. Put $M = \sup_{w,w' \in X} d(w,w')$. Since $\delta_n^2 + (1-\delta_n)^2 \le 1/2$ and $\delta_n(1-\delta_n) \le 1/4$, we get

$$\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2} \le \sqrt{\frac{1+\cos M}{2}}.$$

Since $\delta_n \to 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $1 - \delta_n \ge 1/2$. Then, it follows that

$$\begin{aligned} \sigma_n &\ge 1 - \frac{1-\delta_n}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2}} \\ &= \frac{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2} - (1-\delta_n)}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2}} \\ &= \frac{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2} + (1-\delta_n)} \end{aligned}$$

$$\times \frac{1}{\sqrt{\delta_n^2 + 2\delta_n(1-\delta_n)\cos M + (1-\delta_n)^2} - (1-\delta_n)}$$

$$\geq \frac{\delta_n + 2(1-\delta_n)\cos M}{\left(\sqrt{\frac{1+\cos M}{2}}+1\right)\sqrt{\frac{1+\cos M}{2}}}\delta_n \geq \frac{\cos M}{\left(\sqrt{\frac{1+\cos M}{2}}+1\right)\sqrt{\frac{1+\cos M}{2}}}\delta_n$$

for all $n \geq n_0$. Since $\sum_{n=n_0}^{\infty} \delta_n = \infty$, we get $\sum_{n=1}^{\infty} \sigma_n = \infty$.

Let $\varphi\colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function such that $\lim_{i\to\infty} \varphi(i) = \infty$. Suppose

$$\liminf_{i\to\infty} \left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \geq 0$$

and put $\varphi(i) = n_i$. Then, we get

$$\begin{aligned}
0 &\leq \liminf_{i\to\infty}(a_{n_i+1} - a_{n_i}) \\
&= \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(x_{n_i+1}, p)) \\
&\leq \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - \delta_{n_i}\cos d(u, p) - (1-\delta_{n_i})\cos d(U_{n_i}x_{n_i}, p)) \\
&= \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(U_{n_i}x_{n_i}, p)) \\
&\leq \limsup_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(U_{n_i}x_{n_i}, p)) \leq 0
\end{aligned}$$

and hence $\lim_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(U_{n_i}x_{n_i}, p)) = 0$. By Theorem 5.1, we get

$$\sum_{k=1}^{n_i} \alpha_{n_i}^k \cos d(T_k x_{n_i}, U_{n_i}x_{n_i}) \cos d(U_{n_i}x_{n_i}, p) \geq \sum_{k=1}^{n_i} \alpha_{n_i} \cos d(T_k x_{n_i}, p) \geq \cos d(x_{n_i}, p)$$

and hence

$$0 \leq \alpha_{n_i}^1(1 - \cos d(T_1 x_{n_i}, U_{n_i}x_{n_i})) \leq \sum_{k=1}^{n_i} \alpha_{n_i}^k(1 - \cos d(T_k x_{n_i}, U_{n_i}x_{n_i})) \leq 1 - \frac{\cos d(x_{n_i}, p)}{\cos d(U_{n_i}x_{n_i}, p)}.$$

Since $\lim_{i\to\infty} \alpha_{n_i}^1 > 0$, letting $i \to \infty$, we have $\lim_{i\to\infty} d(T_1 x_{n_i}, U_{n_i}x_{n_i}) = 0$. Then, it follows that

$$0 \leq d(x_{n_i}, p) - d(T_1 x_{n_i}, p) \leq d(x_{n_i}, p) - (d(U_{n_i}x_{n_i}, p) + d(T_1 x_{n_i}, U_{n_i}x_{n_i})).$$

Letting $i \to \infty$, we get $\lim_{i\to\infty}(d(x_{n_i}, p) - d(T_1 x_{n_i}, p)) = 0$. Since T_1 is strongly quasinonexpansive, we get $\lim_{i\to\infty} d(x_{n_i}, T_1 x_{n_i}) = 0$. Then, we obtain that

$$d(x_{n_i}, U_{n_i}x_{n_i}) \leq d(x_{n_i}, T_1 x_{n_i}) + d(T_1 x_{n_i}, U_{n_i}x_{n_i})$$

and thus $\lim_{i\to\infty} d(x_{n_i}, U_{n_i}x_{n_i}) = 0$. Further, it follows that

$$0 \leq |d(x_{n_i}, u) - d(U_{n_i}x_{n_i}, u)| \leq d(x_{n_i}, U_{n_i})$$

and thus

$$\liminf_{i\to\infty} d(x_{n_i}, u) = \liminf_{i\to\infty} d(U_{n_i}x_{n_i}, u).$$

Take a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $\lim_{j\to\infty} d(x_{n_{i_j}}, u) = \liminf_{i\to\infty} d(x_{n_i}, u)$. Further, take a subsequence $\{x_{n_{i_{j_l}}}\}$ of $\{x_{n_{i_j}}\}$ such that $x_{n_{i_{j_l}}} \stackrel{\Delta}{\rightharpoonup} x_0 \in X$. Put $w_l = n_{i_{j_l}}$. Then, we get $\lim_{l\to\infty} d\left(x_{w_l}, U_{w_l}x_{w_l}\right) = 0$. By Corolally 6.4, $\{U_{w_l}x\}$ is a Cauchy sequence for $x \in X$ and put $Ux = \lim_{l\to\infty} U_{w_l}x$. Then, we get

$$d\left(x_{w_l}, Ux_{w_l}\right) \leq d\left(x_{w_l}, U_{w_l}x_{w_l}\right) + d\left(U_{w_l}x_{w_l}, Ux_{w_l}\right) \leq d\left(x_{w_l}, U_{w_l}x_{w_l}\right) + \sup_{x\in X} d\left(U_{w_l}x, Ux\right).$$

By Lemma 6.8, letting $l \to \infty$, we get $\lim_{l\to\infty} d(x_{w_l}, Ux_{w_l}) = 0$. Since U is delta-demiclosed, we get $x_0 \in \operatorname{Fix} U = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. By Lemma 2.9, we have

$$\liminf_{i\to\infty} d(u, U_{n_i}x_{n_i}) = \lim_{i\to\infty} d(u, x_{n_i}) = \lim_{l\to\infty} d(u, x_{w_l}) \geq d(u, x_0) \geq d(u, p)$$

and hence

$$\begin{aligned} \limsup_{i\to\infty} b_{n_i} &= \limsup_{i\to\infty} \left(1 - \frac{\cos d(u,p)}{\cos d(U_{n_i}x_{n_i}, p)}\right) \\ &= 1 - \frac{\cos d(u,p)}{\cos(\liminf_{i\to\infty} d(U_{n_i}x_{n_i}, p))} \leq 0. \end{aligned}$$

Using Lemma 2.19, we obtain $x_n \to P_F u$ and get the desired result. □

Corollary 6.5. *Let X be a complete CAT(1) space with $\sup_{w,w'\in X} d(w,w') < \pi/2$, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid n,k \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. Let $u \in X$. Define a sequence $\{x_n\} \subset X$ by $x_1 \in X$ and*

$$x_{n+1} = \delta_n u \oplus (1-\delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions:

(a) $\lim_{n\to\infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$;
(b) $\lim_{n\to\infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$.

Then, $x_n \to P_F u$.

Proof. Let

$$\delta_n u \oplus (1-\delta_n) U_n x_n = \gamma_n u \overset{1}{\oplus} (1-\gamma_n) U_n x_n$$

for each $n \in \mathbb{N}$. By definition of 1-convex combination, it follows that

$$\gamma_n = \frac{\sin \delta_n d(u, U_n x_n)}{\sin \delta_n d(u, U_n x_n) + \sin(1-\delta_n) d(u, U_n x_n)}$$

for $n \in \mathbb{N}$. Then, we get

$$\begin{aligned} \gamma_n &= \frac{\sin \dfrac{\delta_n d(u, U_n x_n)}{2} \cos \dfrac{\delta_n d(u, U_n x_n)}{2}}{\sin \dfrac{d(u, U_n x_n)}{2} \cos\left(\left(\delta_n - \dfrac{1}{2}\right) d(u, U_n x_n)\right)} \\ &\geq \frac{\delta_n \cos \dfrac{\delta_n d(u, U_n x_n)}{2}}{\cos\left(\left(\delta_n - \dfrac{1}{2}\right) d(u, U_n x_n)\right)} \geq \delta_n \cos \frac{\delta_n d(u, U_n x_n)}{2} \geq \frac{\delta_n}{\sqrt{2}} \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \delta_n = \infty$, we get $\sum_{n=1}^{\infty} \gamma_n = \infty$. Since $\delta_n \to 0$, we obtain $\gamma_n \to 0$. By Theoreom 8.8, we get the desired result. □

6.4 In a CAT(κ) space

By Lemma 6.2, 6.4, 6.5, 6.6, 6.7, and 6.8, the following lemmas hold:

Lemma 6.9. *Let X be a κ-bounded complete $\mathrm{CAT}(\kappa)$ space for $\kappa \in \mathbb{R}$, T_k a quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. Suppose $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$. Then,*

$$\sum_{n=1}^{\infty} \sup_{x \in X} d(U_n x, U_{n+1} x) < \infty$$

and $\{U_n x\}$ is a Cauchy sequence for each $x \in X$. Put $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Further, suppose $\lim_{n\to\infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$. Then, the following hold:

- $\lim_{n\to\infty} \sup_{x \in X} d(U_n x, Ux) = 0$;
- $\operatorname{Fix} U = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$;
- *U is quasinonexpansive and delta-demiclosed.*

Lemma 6.10. *Let X be a complete $\mathrm{CAT}(\kappa)$ space for $\kappa \leq 0$, C a nonempty bounded subset of X, T_k a quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. Suppose $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$. Then,*

$$\sum_{n=1}^{\infty} \sup_{x \in C} d(U_n x, U_{n+1} x) < \infty$$

and $\{U_n x\}$ is a Cauchy sequence for each $x \in X$. Put $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Further, suppose $\lim_{n\to\infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$. Then, the following hold:

- $\lim_{n\to\infty} \sup_{x \in C} d(U_n x, Ux) = 0$;
- $\operatorname{Fix} U = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$;
- *U is quasinonexpansive and delta-demiclosed.*

By Theorem 6.2, Corollary 6.3 and 6.5, the following theorems hold:

Theorem 6.6. *Let X be a κ-bounded complete $\mathrm{CAT}(\kappa)$ space for $\kappa \in \mathbb{R}$, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$. Let $u \in X$ and define a sequence $\{x_n\}$ by $x_1 \in X$ and*

$$x_{n+1} = \delta_n u \oplus (1 - \delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;
- $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Then, $x_n \to P_F u$.

Theorem 6.7. *Let X be a complete $\mathrm{CAT}(\kappa)$ space for $\kappa \leq 0$, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$. Let $u \in X$ and define a sequence $\{x_n\}$ by $x_1 \in X$ and*

$$x_{n+1} = \delta_n u \oplus (1 - \delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;
- $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Then, $x_n \to P_F u$.

Chapter 7

Projection method

In this chapter, we introduce projection methods using a balanced mapping of a countable family of mappings in a CAT(κ) space for $\kappa \in \mathbb{R}$.

7.1 Shrinking projection method

Theorem 7.1. *Let X be a κ-bounded complete CAT(κ) space for $\kappa \in \mathbb{R}$, T_k a quasinonexpansive and delta-demiclosed mapping for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. Define $\{x_n\} \subset X$ and a subset $\{C_n\}$ of X by $x_1, u \in X$, $C_1 = X$ and*

$$\begin{aligned} C_n &= \{z \in X \mid d(U_n x_n, z) \leq d(x_n, z)\} \cap C_{n-1}; \\ x_{n+1} &= P_{C_n} u \end{aligned}$$

for each $n \in \mathbb{N}$. Suppose the following conditions:

- *$\{z \in X \mid d(w, z) \leq d(w', z)\}$ is convex for $w, w' \in X$;*
- *$\lim_{n\to\infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.*

Then, $x_n \to P_F u$.

Proof. We first show $\{x_n\}$ is well-defined and $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset C_n$ for all $n \in \mathbb{N}$ by induction. If $x_1 \in X$, it is obvious that C_1 is a nonempty closed convex subset and $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset C_1$. Suppose that $x_1, x_2, \dots, x_k$ are well-defined and $C_1, C_2, \dots, C_k$ are nonempty closed convex subsets such that $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset C_i$ for $i = \{1, 2, \dots, k\}$. Since $\{z \in X \mid d(U_k x_k, z) \leq d(x_k, z)\}$ is closed and convex, C_{k+1} is closed and convex. Let $z \in \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$. Since U_k is quasinonexpansive, we have

$$d(U_k x_k, z) \leq d(x_k, z).$$

This implies that $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset C_{k+1}$. Therefore, $\{x_n\}$ is well-defined and $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \subset \bigcap_{n=1}^{\infty} C_n$ for $n \in \mathbb{N}$. By the definiton of C_n for each $n \in \mathbb{N}$, $\{C_n\}$ is a decreasing subset sequence of X and $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$. By Theorem 2.3 and 2.4, $\{x_n\}$ is convergent to $x_0 = P_{\bigcap_{n=1}^{\infty} C_n} u$. Finally, we show $x_0 \in F$. Since $x_n \to x_0$, we get $U_n x_n \to x_0$. Then, it follows that

$$d(x_n, U_n x_n) \leq d(x_n, x_0) + d(x_0, U_n x_n)$$

and hence $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. By Lemma 6.9, $\{U_n x\}$ is a Cauchy sequence for each $x \in X$, $\operatorname{Fix} U = F$, U is delta-demiclosed, and $\lim_{n\to\infty} \sup_{x \in X} d(Ux, U_n x) = 0$, where $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Then, we get

$$d(x_n, Ux_n) \leq d(x_n, U_n x_n) + d(U_n x_n, Ux_n) \leq d(x_n, U_n x_n) + \sup_{x \in X} d(Ux, U_n x_n).$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} d(x_n, Ux_n) = 0$. Since U is delta-demiclosed, we get $x_0 \in \operatorname{Fix} U = F$ and thus $x_n \to P_F u$. Then, we get the desired result. □

Theorem 7.2. *Let X be a κ-bounded complete $\mathrm{CAT}(\kappa)$ space for $\kappa \in \mathbb{R}$, T_k a quasinonexpansive and delta-demiclosed mapping for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. Define $\{x_n\} \subset X$ and a subset $\{C_n\}$ of X by $x_1 \in X$, $C_1 = X$ and*

$$C_n = \{z \in X \mid d(U_n x_n, z) \leq d(x_n, z)\} \cap C_{n-1};$$
$$x_{n+1} = P_{C_{n+1}} x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions:

- *$\{z \in X \mid d(w, z) \leq d(w', z)\}$ is convex for $w, w' \in X$;*
- *$\lim_{n\to\infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.*

Then, $x_n \overset{\Delta}{\rightharpoonup} x_0 \in F$.

Proof. We first show $\{x_n\}$ is well-defined and $F \subset C_n$ for all $n \in \mathbb{N}$ by induction. If $x_1 \in X$, it is obvious that C_1 is a nonempty closed convex subset and $F \subset C_1$. Suppose that $x_1, x_2, \dots, x_k$ are well-defined and $C_1, C_2, \dots, C_k$ are nonempty closed convex subsets such that $F \subset C_i$ for $i = \{1, 2, \dots, k\}$. Since $\{z \in X \mid d(U_k x_k, z) \leq d(x_k, z)\}$ is closed and convex, C_{k+1} is closed and convex. Let $z \in F$. Since U_k is quasinonexpansive, we have

$$d(U_k x_k, z) \leq d(x_k, z)$$

and hence $z \in C_{k+1}$. This implies that $F \subset C_{k+1}$. Therefore, $\{x_n\}$ is well-defined and $F \subset \bigcap_{n=1}^{\infty} C_n$ for $n \in \mathbb{N}$. Let $p \in F$. Since P_{C_n} is quasinonexpansice for $n \in \mathbb{N}$, we get

$$d(x_{n+1}, p) = d(P_{C_{n+1}} x_n, p) \leq d(x_n, p)$$

and thus $\{d(x_n, p)\}$ is nonincreasing. Then, there exists a limit of $\{d(x_n, p)\}$. Put

$$c_p = \lim_{n\to\infty} d(x_n, p).$$

By definition of $\{x_n\}$ and Lemma 2.11, we have

$$c_\kappa''(d(p, x_{n+1}))\phi_\kappa(x_n, x_{n+1}) \leq \phi_\kappa(x_n, p) - \phi_\kappa(x_{n+1}, p).$$

Letting $n \to \infty$, we get

$$c_\kappa''(c_p) \lim_{n\to\infty} \phi_\kappa(x_n, x_{n+1}) \leq c_p - c_p = 0$$

and hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $x_{n+1} \in C_{n+1}$, we get

$$d(U_n x_n, x_{n+1}) \leq d(x_n, x_{n+1}) \to 0$$

as $n \to \infty$. Further, it follows that

$$d(x_n, U_n x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, U_n x_n).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. By Lemma 6.9, $\{U_n x\}$ is a Cauchy sequence for each $x \in X$, $\operatorname{Fix} U = F$, U is delta-demiclosed, and $\lim_{n\to\infty} \sup_{x\in X} d(Ux, U_n x) = 0$, where $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Then, we get

$$d(x_n, Ux_n) \leq d(x_n, U_n x_n) + d(U_n x_n, Ux_n) \leq d(x_n, U_n x_n) + \sup_{x\in X} d(Ux, U_n x_n).$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} d(x_n, Ux_n) = 0$. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $\mathrm{AC}(\{x_{n_i}\}) = \{y_0\}$. Then, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} z_0$. Since

$\lim_{j\to\infty} d(x_{n_{i_j}}, Ux_{n_{i_j}}) = 0$ and U is delta-demiclosed, we get $z_0 \in \operatorname{Fix} U = F$. Put $\mathrm{AC}(\{x_n\}) = \{x_0\}$. Then, it follows that

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, z_0) &\le \lim_{j\to\infty} d(x_{n_{i_j}}, z_0)\\ &\le \limsup_{j\to\infty} d(x_{n_{i_j}}, y_0)\\ &\le \limsup_{i\to\infty} d(x_{n_i}, y_0)\\ &\le \limsup_{i\to\infty} d(x_{n_i}, x_0)\\ &\le \limsup_{n\to\infty} d(x_n, x_0) \le \limsup_{n\to\infty} d(x_n, z_0)\end{aligned}$$

and thus $x_0 = y_0 = z_0$. This implies that $x_n \xrightarrow{\Delta} x_0 \in F$ and we get the desired result. □

7.2 Nakajo–Takahashi projection method

Theorem 7.3. *Let X be a κ-bounded complete $\mathrm{CAT}(\kappa)$ space for $\kappa \in \mathbb{R}$, T_k a quasinonexpansive and delta-demiclosed mapping for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \ne \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \le n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. Define $\{x_n\} \subset X$, and subsets $\{C_n\}$ and $\{Q_n\}$ by $x_1 \in X$, $C_1 = X = Q_1$ and*

$$\begin{aligned}C_{n+1} &= \{z \in X \mid d(U_n x_n, z) \le d(x_n, z)\};\\ Q_{n+1} &= \left\{z \in X \;\middle|\; \phi_\kappa(x_n, z) + c_\kappa''(d(x_n, z))\phi_\kappa(x_1, x_n) \le \phi_\kappa(x_1, z)\right\};\\ x_{n+1} &= P_{C_{n+1}\cap Q_{n+1}} x_1\end{aligned}$$

for each $n \in \mathbb{N}$. Suppose the following conditions:

- *$\{z \in X \mid d(w, z) \le d(w', z)\}$ is convex for $w, w' \in X$;*
- *$\{z \in X \mid \phi_\kappa(w, z) + c_\kappa''(d(w, z))\phi_\kappa(w', w) \le \phi_\kappa(w', z)\}$ is convex for $w, w' \in X$;*
- *$\lim_{n\to\infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.*

Then, $x_n \to P_F x_1$.

Proof. First, we show $\{x_n\}$ is well-defined by induction. Since $x_1 \in X$ and $C_1 = Q_1 = X$, it follows that C_1 and Q_1 is nonempty closed and convex. Further, we get $F \subset C_1 \cap Q_1$. Suppose the following conditions:

- $x_1, x_2, \dots, x_k$ are well-defined;
- C_k and Q_k are nonempty, closed, and convex;
- $F \subset C_k \cap Q_k$.

By the assumptions of the theorem and continuity of a metric, it follows that C_{k+1} and Q_{k+1} are nonempty closed convex sets. Let $p \in F$. By Theorem 5.1, U_n is quasinonexpansive mapping for $n \in \mathbb{N}$. Then, it follows that $d(U_{k+1}x_{k+1}, p) \le d(x_{k+1}, p)$ and hence $p \in C_{k+1}$. This implies that $F \subset C_{k+1}$. Next, we show $F \subset Q_{k+1}$. Since $F \subset C_k \cap Q_k$, we show $C_k \cap Q_k \subset Q_{k+1}$. Let $q \in C_k \cap Q_k$ and $t \in]0,1[$. Then, we get

$$w = tq \oplus (1-t)x_k = tq \oplus (1-t)P_{C_k\cap Q_k} x_1 \in C_k \cap Q_k.$$

Put $d = d(q, x_k)$. Then, it follows that

$$\begin{aligned}\phi_\kappa(x_1, x_k) &= \phi_\kappa(x_1, P_{C_k\cap Q_k} x_1)\\ &\le \phi_\kappa(x_1, w)\end{aligned}$$

$$\leq (t)_d^\kappa \phi_\kappa(x_1, q) + (1-t)_d^\kappa \phi_\kappa(x_1, x_k) - (t)_d^\kappa \phi_\kappa(q, w) - (1-t)_d^\kappa \phi_\kappa(x_k, w)$$
$$\leq (t)_d^\kappa \phi_\kappa(x_1, q) + (1-t)_d^\kappa \phi_\kappa(x_1, x_k) - (t)_d^\kappa \phi_\kappa(q, w)$$

and hence

$$(1 - (1-t)_d^\kappa)\phi_\kappa(x_1, x_k) \leq (t)_d^\kappa \phi_\kappa(x_1, q) - (t)_d^\kappa \phi_\kappa(q, w).$$

By Lemma 2.2, dividing $(t)_d^\kappa > 0$ and letting $t \searrow 0$, we get

$$c_\kappa''(d(q, x_k))\phi_\kappa(x_1, x_k) \leq \phi_\kappa(x_1, q) - \phi_\kappa(q, x_k)$$

and thus $q \in Q_{k+1}$. This implies that $F \subset C_{k+1} \cap Q_{k+1}$. Therefore, $\{x_n\}$ is well-defined and $F \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. Since X is κ-bounded, we get $\sup_{n \in \mathbb{N}} d(x_1, x_n) < D_\kappa/2$. By the definition of Q_{n+1}, it follows that

$$d(x_1, x_n) = d(x_1, P_{Q_{n+1}} x_1) \leq d(x_1, P_{C_{n+1} \cap Q_{n+1}} x_1) = d(x_1, x_{n+1})$$

and hence $\{d(x_1, x_n)\}$ is nondecreasing. Put $c = \lim_{n\to\infty} d(x_1, x_n)$. Since $x_{n+1} \in Q_{n+1}$, we have

$$\begin{aligned}
\phi_\kappa(x_1, x_{n+1}) &\geq \phi_\kappa(x_{n+1}, x_n) + c_\kappa''(d(x_{n+1}, x_n))\phi_\kappa(x_n, x_1) \\
&= \phi_\kappa(x_{n+1}, x_n) + (1 - \kappa\phi_\kappa(x_{n+1}, x_n))\phi_\kappa(x_n, x_1) \\
&= \phi_\kappa(x_n, x_1) + (1 - \kappa\phi_\kappa(x_n, x_1))\phi_\kappa(x_n, x_{n+1}) \\
&= \phi_\kappa(x_n, x_1) + c_\kappa''(d(x_n, x_1))\phi_\kappa(x_n, x_{n+1})
\end{aligned}$$

and hence

$$c_\kappa''(d(x_n, x_1))\phi_\kappa(x_n, x_{n+1}) \leq \phi_\kappa(x_{n+1}, x_1) - \phi_\kappa(x_n, x_1).$$

Letting $n \to \infty$, we get

$$c_\kappa''(c) \lim_{n\to\infty} \phi_\kappa(x_n, x_{n+1}) \leq c - c = 0.$$

Since $c_\kappa''(c) > 0$, it follows that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $x_{n+1} \in C_{n+1}$, we get

$$d(U_n x_n, x_{n+1}) \leq d(x_n, x_{n+1})$$

and hence $\lim_{n\to\infty} d(U_n x_n, x_{n+1}) = 0$. Further, we obtain

$$d(x_n, U_n x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, U_n x_n)$$

and thus $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. By Lemma 6.9, $\{U_n x\}$ is a Cauchy sequence for each $x \in X$, Fix $U = F$, U is delta-demiclosed, and $\lim_{n\to\infty} \sup_{x \in X} d(Ux, U_n x) = 0$, where $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Then, we get

$$d(x_n, Ux_n) \leq d(x_n, U_n x_n) + d(U_n x_n, Ux_n) \leq d(x_n, U_n x_n) + \sup_{x \in X} d(Ux, U_n x_n).$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} d(x_n, Ux_n) = 0$. Take a subsequence $\{x_{n_i}\} \subset \{x_n\}$ arbitrarily. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that $x_{n_{i_j}} \xrightarrow{\Delta} x_0$. Since U is delta-demiclosed, we get $x_0 \in F$. By Lemma 2.9, we get

$$d(x_1, P_F x_1) \leq d(x_1, x_0) \leq \liminf_{j\to\infty} d(x_1, x_{n_{i_j}}) \leq \limsup_{j\to\infty} d(x_1, x_{n_{i_j}}) \leq d(x_1, P_F x_1)$$

and thus $\lim_{j\to\infty} d(x_1, x_{n_{i_j}}) = d(x_1, x_0) = d(x_1, P_F x_1)$. By Lemma 2.10, we have $x_{n_{i_j}} \to P_F x_1$. Since $\{x_{n_i}\}$ is arbitrary, we obtain $x_n \to P_F x_1$. Consequently, we complete the proof. □

Theorem 7.4. *Let X be a κ-bounded complete $\mathrm{CAT}(\kappa)$ space for $\kappa \in \mathbb{R}$, T_k a quasinonexpansive and delta-demiclosed mapping for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$. Define $\{x_n\} \subset X$, and subsets $\{C_n\}$ and $\{Q_n\}$ by $x_1 \in X$, $C_1 = X = Q_1$ and*

$$\begin{aligned}
C_{n+1} &= \{z \in X \mid d(U_n x_n, z) \leq d(x_n, z)\};\\
Q_{n+1} &= \left\{z \in X \;\middle|\; \phi_\kappa(x_n, z) + c_\kappa''(d(x_n, z))\phi_\kappa(x_{n-1}, x_n) \leq \phi_\kappa(x_{n-1}, z)\right\};\\
x_{n+1} &= P_{C_{n+1} \cap Q_{n+1}} x_n
\end{aligned}$$

for each $n \in \mathbb{N}$. Suppose the following conditions:

- *$\{z \in X \mid d(w, z) \leq d(w', z)\}$ is convex for $w, w' \in X$;*
- *$\{z \in X \mid \phi_\kappa(w, z) + c_\kappa''(d(w, z))\phi_\kappa(w', w) \leq \phi_\kappa(w', z)\}$ is convex for $w, w' \in X$;*
- *$\lim_{n\to\infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.*

Then, $x_n \overset{\Delta}{\rightharpoonup} x_0 \in F$.

Proof. First, we show $\{x_n\}$ is well-defined by induction. Since $x_1 \in X$ and $C_1 = Q_1 = X$, it follows that C_1 and Q_1 is nonempty closed and convex. Further, we get $F \subset C_1 \cap Q_1$. Suppose the following conditions:

- $x_1, x_2, \dots, x_k$ are well-defined;
- C_k and Q_k are nonempty, closed, and convex;
- $F \subset C_k \cap Q_k$.

By assumptions of the theorem and continuity of a metric, it follows that C_{k+1} and Q_{k+1} are nonempty closed convex sets. Let $p \in F$. By Theorem 5.1, U_n is a quasinonexpansive mapping for $n \in \mathbb{N}$. Then, it follows that $d(U_{k+1} x_{k+1}, p) \leq d(x_{k+1}, p)$ and hence $p \in C_{k+1}$. This implies that $F \subset C_{k+1}$. Next, we show $F \subset Q_{k+1}$. Since $F \subset C_k \cap Q_k$, we show $C_k \cap Q_k \subset Q_{k+1}$. Let $q \in C_k \cap Q_k$ and $t \in]0,1[$. Then, we get

$$w = tq \oplus (1-t)x_k = tq \oplus (1-t)P_{C_k \cap Q_k} x_{k-1} \in C_k \cap Q_k.$$

Put $d = d(q, x_k)$. Then, it follows that

$$\begin{aligned}
\phi_\kappa(x_{k-1}, x_k) &= \phi_\kappa(x_{k-1}, P_{C_k \cap Q_k} x_{k-1})\\
&\leq \phi_\kappa(x_{k-1}, w)\\
&\leq (t)_d^\kappa \phi_\kappa(x_{k-1}, q) + (1-t)_d^\kappa \phi_\kappa(x_{k-1}, x_k) - (t)_d^\kappa \phi_\kappa(q, w) - (1-t)_d^\kappa \phi_\kappa(x_k, w)\\
&\leq (t)_d^\kappa \phi_\kappa(x_{k-1}, q) + (1-t)_d^\kappa \phi_\kappa(x_{k-1}, x_k) - (t)_d^\kappa \phi_\kappa(q, w)
\end{aligned}$$

and hence

$$(1 - (1-t)_d^\kappa)\phi_\kappa(x_{k-1}, x_k) \leq (t)_d^\kappa \phi_\kappa(x_{k-1}, q) - (t)_d^\kappa \phi_\kappa(q, w).$$

By Lemma 2.2, dividing $(t)_d^\kappa > 0$ and letting $t \searrow 0$, we get

$$c_\kappa''(d(q, x_k))\phi_\kappa(x_{k-1}, x_k) \leq \phi_\kappa(x_{k-1}, q) - \phi_\kappa(q, w)$$

and thus $q \in Q_{k+1}$. This implies that $F \subset C_{k+1} \cap Q_{k+1}$. Therefore, $\{x_n\}$ is well-defined and $F \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. Let $p \in F$. Since $P_{C_n \cap Q_n}$ is quasinonexpansive for $n \in \mathbb{N}$, we get

$$d(x_{n+1}, p) = d(P_{C_n \cap Q_n} x_n, p) \leq d(x_n, p)$$

and hence $\{d(x_n, p)\}$ is nonincreasing. Then, there exists a limit of $\{d(x_n, p)\}$. Put $c_p = \lim_{n\to\infty} d(x_n, p)$. Put $\tau = tp \oplus (1-t)x_{n+1}$ and $d' = d(p, x_{n+1})$. Since $x_{n+1} = P_{C_{n+1} \cap Q_{n+1}} x_n \in C_{n+1} \cap Q_{n+1}$, we get $\tau \in C_{n+1} \cap Q_{n+1}$. Then, it follows that

$$\phi_\kappa(x_n, x_{n+1}) \leq \phi_\kappa(x_n, P_{C_{n+1} \cap Q_{n+1}} x_n)$$

$$
\begin{aligned}
&\le \phi_\kappa(x_n, \tau)\\
&\le (t)^\kappa_{d'}\phi_\kappa(x_n, p) + (1-t)^\kappa_{d'}\phi_\kappa(x_n, x_{n+1}) - (t)^\kappa_{d'}\phi_\kappa(p, \tau) - (1-t)^\kappa_{d'}(x_{n+1}, \tau)\\
&\le (t)^\kappa_{d'}\phi_\kappa(x_n, p) + (1-t)^\kappa_{d'}\phi_\kappa(x_n, x_{n+1}) - (t)^\kappa_{d'}\phi_\kappa(p, \tau)
\end{aligned}
$$

and hence

$$(1-(1-t)^\kappa_{d'})\phi_\kappa(x_n, x_{n+1}) \le (t)^\kappa_{d'}\phi_\kappa(x_n, p) - (t)^\kappa_{d'}\phi_\kappa(p, \tau).$$

By Lemma 2.2, dividing by $(t)^\kappa_d > 0$ and letting $t \searrow 0$, we get

$$c''_\kappa(d(p, x_{n+1}))\phi_\kappa(x_n, x_{n+1}) \le \phi_\kappa(x_n, p) - \phi_\kappa(p, x_{n+1}).$$

Letting $n \to \infty$, we get

$$c''_\kappa(c_p) \lim_{n\to\infty} \phi_\kappa(x_n, x_{n+1}) \le c_p - c_p = 0.$$

Since $c''_\kappa(c_p) > 0$, we get $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $x_{n+1} \in C_{n+1}$, we get

$$d(U_n x_n, x_{n+1}) \le d(x_n, x_{n+1}) \to 0$$

as $n \to \infty$. Further, it follows that

$$d(x_n, U_n x_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, U_n x_n).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. By Lemma 6.9, $\{U_n x\}$ is a Cauchy sequence for each $x \in X$, $\operatorname{Fix} U = F$, U is delta-demiclosed, and $\lim_{n\to\infty} \sup_{x\in X} d(Ux, U_n x) = 0$, where $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Then, we get

$$d(x_n, Ux_n) \le d(x_n, U_n x_n) + d(U_n x_n, Ux_n) \le d(x_n, U_n x_n) + \sup_{x\in X} d(Ux, U_n x_n).$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} d(x_n, Ux_n) = 0$. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $\operatorname{AC}(\{x_{n_i}\}) = \{y_0\}$. Then, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \xrightarrow{\Delta} z_0$. Since $\lim_{j\to\infty} d(x_{n_{i_j}}, Ux_{n_{i_j}}) = 0$ and U is delta-demiclosed, we get $z_0 \in \operatorname{Fix} U = F$. Put $\operatorname{AC}(\{x_n\}) = \{x_0\}$. Then, it follows that

$$
\begin{aligned}
\limsup_{n\to\infty} d(x_n, z_0) &\le \lim_{j\to\infty} d(x_{n_{i_j}}, z_0)\\
&\le \limsup_{j\to\infty} d(x_{n_{i_j}}, y_0)\\
&\le \limsup_{i\to\infty} d(x_{n_i}, y_0)\\
&\le \limsup_{i\to\infty} d(x_{n_i}, x_0)\\
&\le \limsup_{n\to\infty} d(x_n, x_0) \le \limsup_{n\to\infty} d(x_n, z_0)
\end{aligned}
$$

and thus $x_0 = y_0 = z_0$. This implies that $x_n \xrightarrow{\Delta} x_0 \in F$ and we get the desired result. □

7.3 Combining projection method of balanced type

Theorem 7.5. *Let X be a κ-bounded complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, T_i a quasinonexpansive mapping of X into itself satisfying condition (D) for $i \in \{1, 2, \dots, N\}$ such that $F = \bigcap_{i=1}^N \operatorname{Fix} T_i \ne \emptyset$. Let $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0, 1]$, $\{\beta^i_n \mid i \in \{1, 2, \dots, N\}, n \in \mathbb{N}\} \subset]0, 1[$ such that $\sum_{i=1}^N \beta^i_n = 1$ for $n \in \mathbb{N}$, $\{\gamma_{n,k} \mid k, n \in \mathbb{N}, k \le n\} \subset]0, 1[$ such that $\sum_{k=1}^n \gamma_{n,k} = 1$ for $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset]0, 1[$. Let $u \in X$ and define $\{x_n\}$, $\{y_n\}$ and a subset $\{C_n\}$ of X and a mapping U_n by $x_1 \in X$ and*

$$y^i_n = \alpha_n x_n \oplus (1-\alpha_n) T_i x_n \text{ for } i \in \{1, 2, \dots, N\};$$

$$
\begin{aligned}
C_n^i &= \left\{z \in X \,\middle|\, d(y_n^i, z) \le d(x_n, z)\right\} \textit{ for } i \in \{1,2,\dots,N\};\\
V_k x &= \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{N} \beta_k^i \phi_\kappa(P_{C_k^i} x, y) \textit{ for } k \le n \textit{ and } x \in X;\\
U_n x &= \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{n} \beta_n^k \phi_\kappa(V_k x, y) \textit{ for } x \in X;\\
x_{n+1} &= \delta_n u \oplus (1-\delta_n) U_n x_n
\end{aligned}
$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\{z \in X \mid d(w,z) \le d(w',z)\}$ *is convex for* $w, w' \in X$;
- $\liminf_{n\to\infty} \alpha_n < 1$;
- $\beta_n^i > 0$ *for* $i \in \{1,2,\dots,N\}$;
- $\lim_{n\to\infty} \gamma_{n,k} > 0$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\gamma_{n+1,k} - \gamma_{n,k}| < \infty$;
- $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$.

Then, $x_n \to P_F u$.

Proof. Let $p \in F$. Fix $i \in \{1,2\dots,N\}$ arbitrarily. Then, we get

$$
\begin{aligned}
\phi_\kappa(y_n^i, p) &= \phi_\kappa(\alpha_n x_n \oplus (1-\alpha_n) T_i x_n, p)\\
&\le \alpha_n \phi_\kappa(x_n, p) + (1-\alpha_n)\phi_\kappa(T_i x_n, p)\\
&\le \phi_\kappa(x_n, p)
\end{aligned}
$$

for all $n \in \mathbb{N}$ and hence $F \subset \bigcap_{n=1}^{\infty} C_n^i$. Since $i \in \{1,2,\dots,N\}$ is arbitrary, it follows that

$$
\emptyset \ne F \subset \bigcap_{i=1}^{N} \bigcap_{n=1}^{\infty} C_n^i.
$$

Since the metric projection $P_{C_k^i}$ is strongly quasinonexpansive and delta-demiclosed for $i \in \{1,2,\dots,N\}$, V_k is strongly quasinonexpansive and delta-demiclosed for $k \in \mathbb{N}$. Further, since V_k is strongly quasinonexpansive and delta-demiclosed, we get U_n is strongly quasinonexpansive and delta-demiclosed for $n \in \mathbb{N}$. Then, it follows that

$$
\operatorname{Fix} V_k = \bigcap_{i=1}^{N} \operatorname{Fix} P_{C_k^i} = \bigcap_{i=1}^{N} C_k^i
$$

and hence

$$
\bigcap_{k=1}^{\infty} \operatorname{Fix} V_k = \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{N} C_k^i \supset F \ne \emptyset
$$

Since Theorem 6.6, we get $x_n \to P_{\bigcap_{k=1}^{\infty} \bigcap_{i=1}^{N} C_k^i} u$. Put $x_0 = P_{\bigcap_{k=1}^{\infty} \bigcap_{i=1}^{N} C_k^i} u$. Since $x_0 \in \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{N} C_k^i$, we have $y_n^i \to x_0$ for all $i \in \{1,2,\dots,N\}$. Since $\liminf_{n\to\infty} \alpha_n < 1$, we can take a subsequence $\{\alpha_{n_j}\}$ of $\{\alpha_n\}$ such that $\lim_{j\to\infty} \alpha_{n_j} \in [0,1[$. Then, we have

$$
d(x_{n_j}, T_i x_j) = \frac{1}{1-\alpha_{n_j}} d(x_{n_j}, y_{n_j}^i) \le \frac{1}{1-\alpha_{n_j}} (d(x_{n_j}, x_0) + d(x_0, y_{n_j}^i))
$$

and hence $\lim_{j\to\infty} d(x_{n_j}, T_i x_{n_j}) = 0$ for all $i \in \{1,2\dots,N\}$. Since T_i satisfies condition (D) for $i \in \{1,2,\dots,N\}$, we get $x_0 \in \operatorname{Fix} T_i$. Since i is arbitrary, we obtain that $x_0 \in F$. Therefore, we get $x_0 = P_F u$ and get the desired result. □

Remark 7.1. In Theorem 7.1, 7.3 and 7.5, if $\kappa \le 0$, by lemma 6.10, we can change **a κ-bounded complete CAT(κ) space** into **a complete CAT(κ) space**.

Chapter 8

Conclusion

We introduced a resolvent for convex functions and equilibrium problems in a CAT(−1) space.

Theorem 8.1. *Let X be a complete* CAT(−1) *space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$. Let*

$$L_f x = \operatorname*{Argmin}_{y \in X} \{f(y) + \cosh d(x, y)\}$$

for each $x \in X$. Then the following properties hold:

(i) *L_f is single-valued.*
(ii) *the inequality*

$$(\cosh d(x, L_f x) + \cosh d(y, L_f y)) \cosh d(L_f x, L_f y) \leq \cosh d(L_f x, y) + \cosh d(x, L_f y)$$

holds for $x, y \in X$;
(iii) $\operatorname{Fix} L_f = \operatorname{Argmin}_X f$.

Theorem 8.2. *Let X be a complete* CAT(−1) *space having the convex hull finite property and K a nonempty closed convex subset of X. Suppose that $f \colon K \times K \to \mathbb{R}$ satisfies* Condition 3.1. *Define a set-valued mapping $L_f \colon X \to 2^K$ by*

$$L_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z, y) + \cosh d(x, y) - \cosh d(x, z)) \geq 0 \right\}$$

for all $x \in X$. Then the following hold:

(i) *$D(L_f) = X$;*
(ii) *L_f is single-valued and the inequality*

$$(\cosh d(x, L_f x) + \cosh d(y, L_f y)) \cosh d(L_f x, L_f y) \leq \cosh d(L_f x, y) + \cosh d(x, L_f y)$$

holds for $x, y \in X$.
(iii) $\operatorname{Equil} f = \operatorname{Fix} L_f$, *and thus* $\operatorname{Equil} f$ *is closed and convex.*

In the following theorems, we introduced a delta-convergence sequence with the proximal point algorithm using a resolvent for equilibrium problems in a CAT(−1) space.

Theorem 8.3. *Let X be a complete* CAT(−1) *space having the convex hull finite property, K a nonempty closed convex subset of X, $f \colon K \times K \to \mathbb{R}$ satisfying* Condition 3.1 *and $\{\lambda_n\} \subset]0, \infty[$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. For given $x_1 \in X$, define $\{x_n\}$ by*

$$x_{n+1} = L_{\lambda_n f} x_n = \left\{ z \in K \;\middle|\; \inf_{y \in K} (\lambda_n f(z, y) + \cosh d(x_n, y) - \cosh d(x_n, z)) \geq 0 \right\}$$

for all $n \in \mathbb{N}$. Then, the following hold:

(i) $\operatorname{Equil} f$ *is nonempty if and only if $\{x_n\}$ is bounded;*

(ii) *if* Equil $f \neq \emptyset$ *and* $\liminf_{n\to\infty} \lambda_n > 0$, $x_n \xrightarrow{\Delta} x_0 \in \operatorname{Equil} f$.

Next, we introduced a Halpern iteration using a balanced mapping of a countable family of mappings and proved a convergence to a common fixed point.

Theorem 8.4. *Let X be a complete* CAT(0) *space, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for all $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0,1]$. Define a sequence $\{x_n\}$ by $u, x_1 \in X$ and*

$$x_{n+1} = \delta_n u \oplus (1-\delta) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$ and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$;
- $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then, $x_n \to P_F u$.

Theorem 8.5. *Let X be a κ-bounded complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$. Let $u \in X$ and define a sequence $\{x_n\}$ by $x_1 \in X$ and*

$$x_{n+1} = \delta_n u \overset{\kappa}{\oplus} (1-\delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;
- $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$ and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Then, $x_n \to P_F u$.

Theorem 8.6. *Let X be a κ-bounded complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$. Let $u \in X$ and define a sequence $\{x_n\}$ by $x_1 \in X$ and*

$$x_{n+1} = \delta_n u \oplus (1-\delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;
- $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$ and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Then, $x_n \to P_F u$.

Theorem 8.7. *Let X be a complete* CAT(κ) *space for $\kappa \leq 0$, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$. Let $u \in X$ and define a sequence $\{x_n\}$ by $x_1 \in X$ and*

$$x_{n+1} = \delta_n u \overset{\kappa}{\oplus} (1-\delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;
- $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$ and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Then, $x_n \to P_F u$.

Theorem 8.8. *Let X be a complete* CAT(κ) *space for $\kappa \le 0$, T_k a strongly quasinonexpansive and delta-demiclosed mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \emptyset$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \le n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, U_n a balanced mapping of $\{T_k\}$ and $\{\alpha_n^k\}$ for each $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset]0,1[$. Let $u \in X$ and define a sequence $\{x_n\}$ by $x_1 \in X$ and*

$$x_{n+1} = \delta_n u \oplus (1-\delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=1}^{\infty} \delta_n = \infty$;
- $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Then, $x_n \to P_F u$.

In the following theorems, we introduced the combining projection method of balanced type in a CAT(κ) space for $\kappa \in \mathbb{R}$ to apply Thereom 8.6 and Theorem 8.8.

Theorem 8.9. *Let X be a κ-bounded complete* CAT(κ) *space for $\kappa \in \mathbb{R}$. Let T_i a quasinonexpansive of X into itself satisfying condition (D) for $i \in \{1,2,\dots,N\}$ such that $F = \bigcap_{i=1}^{N} \operatorname{Fix} T_i \neq \emptyset$, $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0,1]$, $\{\beta_n^i \mid i \in \{1,2,\dots,N\}, n \in \mathbb{N}\} \subset]0,1[$ such that $\sum_{i=1}^{N} \beta_n^i = 1$ for $n \in \mathbb{N}$, $\{\gamma_{n,k} \mid k, n \in \mathbb{N}, k \le n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \gamma_{n,k} = 1$ for $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0,1]$. Define sequences $\{x_n\}$ and $\{y_n^i\}$ of X, a sequence $\{C_n^i\}$ of subset of X, and mappings $\{U_n\}$ are defined by $u \in X$, $x_1 \in X$ and*

$$\begin{aligned}
y_n^i &= \alpha_n x_n \oplus (1-\alpha_n) T_i x_n \text{ for } i \in \{1,2,\dots,N\};\\
C_n^i &= \left\{ z \in X \,\middle|\, d(y_n^i, z) \le d(x_n, z) \right\} \text{ for } i \in \{1,2,\dots,N\};\\
V_k x &= \operatorname*{Argmin}_{y \in X} \sum_{i=1}^{N} \beta_k^i \phi_\kappa(P_{C_k^i} x, y) \text{ for } k \le n \text{ and } x \in X;\\
U_n x &= \operatorname*{Argmin}_{y \in X} \sum_{k=1}^{n} \gamma_{n,k} \phi_\kappa(V_k x, y) \text{ for } x \in X;\\
x_{n+1} &= \delta_n u \oplus (1-\delta_n) U_n x_n
\end{aligned}$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

- $\{z \in X \mid d(z,v) \le d(z,v')\}$ *is convex for all* $v, v' \in X$;
- $\liminf_{n\to\infty} \alpha_n < 1$;
- $\beta_n^i > 0$ *for* $i \in \{1,2,\dots,N\}$ *and* $n \in \mathbb{N}$;
- $\lim_{n\to\infty} \gamma_{n,k} > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\gamma_{n+1,k} - \gamma_{n,k}| < \infty$;
- $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then, $x_n \to P_F u$.

Theorem 8.10. *Let X be a complete* CAT(κ) *space for $\kappa \le 0$. Let T_i a quasinonexpansive of X into itself satisfying condition (D) for $i \in \{1,2,\dots,N\}$ such that $F = \bigcap_{i=1}^{N} \operatorname{Fix} T_i \neq \emptyset$, $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0,1]$, $\{\beta_n^i \mid i \in \{1,2,\dots,N\}, n \in \mathbb{N}\} \subset]0,1[$ such that $\sum_{i=1}^{N} \beta_n^i = 1$ for $n \in \mathbb{N}$, $\{\gamma_{n,k} \mid k, n \in \mathbb{N}, k \le n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \gamma_{n,k} = 1$ for $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0,1]$. Define sequences $\{x_n\}$ and $\{y_n^i\}$ of X, a sequence $\{C_n^i\}$ of subset of X, and mappings $\{U_n\}$ are defined by $u \in X$, $x_1 \in X$ and*

$$\begin{aligned}
y_n^i &= \alpha_n x_n \oplus (1-\alpha_n) T_i x_n \text{ for } i \in \{1,2,\dots,N\};\\
C_n^i &= \left\{ z \in X \,\middle|\, d(y_n^i, z) \le d(x_n, z) \right\} \text{ for } i \in \{1,2,\dots,N\};\\
V_k x &= \operatorname*{Argmin}_{y \in X} \sum_{i=1}^{N} \beta_k^i \phi_\kappa(P_{C_k^i} x, y) \text{ for } k \le n \text{ and } x \in X;\\
U_n x &= \operatorname*{Argmin}_{y \in X} \sum_{k=1}^{n} \gamma_{n,k} \phi_\kappa(V_k x, y) \text{ for } x \in X;
\end{aligned}$$

$$x_{n+1} = \delta_n u \oplus (1 - \delta_n) U_n x_n$$

for each $n \in \mathbb{N}$*. Suppose the following conditions hold:*

- $\{z \in X \mid d(z, v) \leq d(z, v')\}$ *is convex for all* $v, v' \in X$*;*
- $\liminf_{n\to\infty} \alpha_n < 1$*;*
- $\beta_n^i > 0$ *for* $i \in \{1, 2, \ldots, N\}$ *and* $n \in \mathbb{N}$*;*
- $\lim_{n\to\infty} \gamma_{n,k} > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\gamma_{n+1,k} - \gamma_{n,k}| < \infty$*;*
- $\lim_{n\to\infty} \delta_n = 0$*,* $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$*.*

Then, $x_n \to P_F u$*.*

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