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Approximation theorems to a solution to an equilibrium problem in complete geodesic spaces

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Contents

Chapter 1

Introduction

Let X be a nonempty set and $f\colon K \times K \to \mathbb{R}$ a function. The problem of finding a point $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$ is called an equilibrium problem. An equilibrium problem includes some important nonlinear problems such as fixed point problems, convex minimization problems, and variational inequality problems. An equilibrium problem was first studied by Blum and Oettli [1] on a Banach space. They introduced a mapping called resolvent of bifunction for an equilibrium problem. The resolvent is an important mapping that allows the equilibrium problem to be reduce to the fixed point problem. Later, Combettes and Hirstoaga introduced the resolvent in a Hilbert space, and showed many important properies of the resolvent.

Theorem 1.1 (Combettes–Hirstoaga [2])**.** *Let H be a Hilbert space, and K a nonempty, closed convex subset of H. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies the conditions (E1)–(E4).*

(E1) $f(x, x) = 0$ for all $x \in K$;
(E2) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in K$;
(E3) $f(x, \cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex for all $x \in K$;
(E4) $f(\cdot, y)\colon K \to \mathbb{R}$ is upper hemicontinuous for all $y \in K$.

Define the resolvent J_f by

$$J_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z, y) + \langle z - x, y - z \rangle) \geq 0 \right\}$$

for $x \in H$. Then J_f has the following properties:

1. *The domain of J_f is H;*
2. *J_f is single-valued and firmly nonexpansive;*
3. *the set of all fixed points of J_f coincides with* $\operatorname{Equil} f$ *and it is closed and convex.*

Further, the resolvent was generalized to the setting of CAT(0) space and CAT(1) spaces.

On the other hand, a fixed point problem is an example of an equilibrium problem, and Mann type method is a famous technique.

The following result is deduced from Reich's result [9] which was proved in the setting of uniformly convex Banach spaces.

Theorem 1.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \varnothing$. Assume that $\{\alpha_n\} \subset [0,1]$ satisfies $0 \leq \alpha_n < 1$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Let $\{x_n\}$ be a sequence of C defined by $x_1 = x \in C$, and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$$

Then $\{x_n\}$ converges weakly to an element $z \in F(T)$.

In this thesis we obtain convergence theorems of an iterative sequence to a solution of equilibrium problem on CAT(1) spaces by using the technique of approximation theorems of solution to a fixed point problem.

Chapter 2

Preliminaries

In this chapter, we define a CAT(1) space and introduce several important notions for the main results.

2.1 CAT(1) spaces

Let X be a metric space. For $x, y \in X$, a mapping $c : [0, d(x, y)] \to X$ is called a geodesic if c satisfies $c(0) = x, c(d(x, y)) = y$, and $d(c(u), c(v)) = |u - v|$ for any $u, v \in [0, d(x, y)]$. An image $[x, y]$ of c is called a geodesic segment joining x and y. X is called a uniquely π-geodesic space if for every $x, y \in X$ satisfying $d(x, y) < \pi$, there exists a unique geodesic c. In a uniquely π-geodesic space X, for all $x, y \in X$ with $d(x, y) < \pi$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1 - t)d(x, y)$, and $d(z, y) = td(x, y)$. We denote it by $tx \oplus (1 - t)y$. For $x, y, z \in X$, we define a geodesic triangle $\triangle(x, y, z)$ as the union of three segments $[x, y], [y, z]$, and $[z, x]$. For a geodesic triangle $\triangle(x, y, z)$ with $d(x, y) + d(y, z) + d(z, x) < 2\pi$, its comparison triangle $\triangle(\bar{x}, \bar{y}, \bar{z})$ in two dimensional unit sphere $\mathbb{S}^2$ is a triangle such that each corresponding edge has the same length as that of original triangle. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a comparison point of $p \in [x, y]$ if $d(x, p) = d(\bar{x}, \bar{p})$. X is called a CAT(1) space if for any $p, q \in \triangle(x, y, z)$ and their comparison point $\bar{p}, \bar{q} \in \triangle(\bar{x}, \bar{y}, \bar{z})$, the inequality

$$d(p, q) \leq d(\bar{p}, \bar{q})$$

holds for any triangle in X. We say X is admissible if for all $x, y \in X$, $d(x, y) < \pi/2$

In CAT(1) spaces, the following inequality holds;

$$\cos d(tx \oplus (1 - t)y, z) \sin d(x, y) \geq \cos d(x, z) \sin td(x, y) + \cos d(y, z) \sin(1 - t)d(x, y)$$

for each $x, y, z \in X$ and $t \in [0, 1]$.

2.2 Properties of mappings

Let X be a CAT(1) space and let $T \colon X \to X$ be a mapping. We call $x \in X$ a fixed point of T if $x = Tx$, and denote the set of all fixed point of T by $F(T)$. The closed convex hull $\operatorname{cl}\operatorname{co} C$ of subset C of X is defined as the intersection of all closed convex subsets of X including C. X has the convex full finite property if every continous mapping $T\colon \operatorname{cl}\operatorname{co} E \to \operatorname{cl}\operatorname{co} E$ has a fixed point for every finite subset E of X. A mapping $T\colon X \to X$ is said to be

- nonexpansive if
$$d(x, y) \geq d(Tx, Ty)$$
for $x, y \in X$;
- quasinonxpansive if
$$d(x, u) \geq d(Tx, u)$$
for $x \in X$ and $u \in F(T) \neq \varnothing$;
- spherically nonspreading of sum type if
$$2 \cos d(Tx, Ty) \geq \cos d(x, Ty) + \cos d(Tx, y);$$
for $x, y \in X$.

2.3 Δ-convergence

Let X be CAT(1) space and $\{x_n\} \subset X$ a sequence. An asymptotic center $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathrm{AC}(\{x_n\}) = \left\{ z \in X \;\middle|\; \inf_{x \in X} \limsup_{n \to \infty} d(x, x_n) = \limsup_{n \to \infty} d(z, x_n) \right\}.$$

A sequence $\{x_n\} \subset X$ is said to be Δ-convergent to $x_0 \in X$ if $\mathrm{AC}(\{x_{n_i}\}) = \{x_0\}$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. It is denoted by $x_n \overset{\Delta}{\rightharpoonup} x_0$.

2.4 Resolvent of equilibrium problem

Let $f\colon X \times X \to \mathbb{R}$. In this thesis, we assume the following conditions;

(E1) $f(x, x) = 0$ for all $x \in K$;
(E2) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in K$;
(E3) $f(x, \cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex for all $x \in K$;
(E4) $f(\cdot, y)\colon K \to \mathbb{R}$ is upper hemicontinuous for all $y \in K$.

Theorem 2.1 (Kimura [5]). *Let X be an admissible complete CAT(1) space having the convex hull finite property and $K \subset X$ a nonempty closed convex set. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies the conditions (E1)–(E4). Define the resolvent R_f of f by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(f(z, y) - \log \cos d(x, y) + \log \cos d(x, z) \right) \geq 0 \right\}$$

for all $x \in X$. Then the following hold:

1. *$R_f\colon X \to K$ is defined as a single-valued mapping;*
2. *R_f satisfies the following inequality for any $x, y \in X$:*

$$\frac{\cos d(x, R_f y)}{\cos d(x, R_f x)} + \frac{\cos d(y, R_f x)}{\cos d(y, R_f y)} \le 2 \cos d(R_f x, R_f y);$$

3. *$F(R_f) = \operatorname{Equil} f$ and it is closed and convex.*

Theorem 2.2 (Kimura [4]). *Let X be an admissible complete CAT(1) space having the convex hull finite property. Let K be a nonempty closed convex subset of X, and suppose that $f\colon K \times K \to \mathbb{R}$ satisfies (E1)–(E4). Let $R_f\colon X \to K$ be the resolvent of f. Then the inequality*

$$\lambda \frac{\cos d(v, R_{\lambda f} u)}{\cos d(v, R_{\mu f} v)} + \mu \frac{\cos d(u, R_{\mu f} v)}{\cos d(u, R_{\lambda f} u)} \le (\lambda + \mu) \cos d(R_{\lambda f} u, R_{\mu f} v)$$

holds for all $\lambda, \mu \in \,]0, \infty[$ and $u, v \in X$.

Chapter 3

Δ-convergence theorem for Mann iteration

In this chapter, we prove a Δ-convergence theorem for Mann type method with the resolvent of equilibrium problem. Mann type method was first studied by Mann [8] in 1953 on Hilbert spaces, and there have been many studies since then. The following theorem is a convergence theorem for Mann type method with a nonexpansive mapping on CAT(1) space.

Theorem 3.1 (He, Fang, López, and Li [3])**.** *Let X be an admissible complete CAT(1) space. Suppose that $T\colon X \to X$ is nonexpansive and $F(T) \neq \varnothing$, and let $\{t_n\} \subset]0,1[$ satisfying $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$.*

Let $\{x_n\}$ be a sequence defined by $x_1 \in X$ with $d(x_1, F(T)) < \pi/4$, and

$$x_{n+1} = t_n x_n \oplus (1-t_n)Tx_n$$

for each $n \in \mathbb{N}$. Then, $x_n \overset{\Delta}{\rightharpoonup} z_0 \in F(T)$.

The following lemma plays an important role in showing the main result.

Lemma 3.1. *Let X be an admissible complete CAT(1) space having the convex hull finite property. Let K be a nonempty closed convex subset of X, and suppose that $f\colon K \times K \to \mathbb{R}$ satisfies (E1)–(E4). Let $\{\lambda_n\} \subset]0,\infty[$ and $\{x_n\} \subset X$ be sequences satisfying $\inf_n \lambda_n > 0$, $x_n \overset{\Delta}{\rightharpoonup} x_0$ and $d(x_n, R_{\lambda_n f}x_n) \to 0$. Then, $x_0 \in \operatorname{Equil} f$.*

Proof. Let $\{n_i\} \subset \mathbb{N}$ be an increasing sequence and $z \in X$. Since

$$\begin{aligned} d(z, x_{n_i}) &\leq d(z, R_{\lambda_{n_i} f}x_{n_i}) + d(R_{\lambda_{n_i} f}x_{n_i}, x_{n_i}) \\ &\leq d(z, x_{n_i}) + 2d(R_{\lambda_{n_i} f}x_{n_i}, x_{n_i}), \end{aligned}$$

we get

$$\limsup_{i\to\infty} d(z, x_{n_i}) = \limsup_{i\to\infty} d(z, R_{\lambda_{n_i} f}x_{n_i}).$$

We first suppose that $\sup_{n\in\mathbb{N}} \lambda_n < \infty$. There exists $\{\lambda_{n_j}\} \subset \{\lambda_n\}$ such that $\lambda_{n_j} \to \lambda_0 \in [\inf_{n\in\mathbb{N}} \lambda_n, \sup_{n\in\mathbb{N}} \lambda_n]$. Fix $j \in \mathbb{N}$. From theorem 2.2,

$$\lambda_{n_j} \cos d(x_0, R_{\lambda_{n_j} f} x_{n_j}) + \cos d(x_{n_j}, R_f x_0) \le (\lambda_{n_j} + 1) \cos d(R_{\lambda_{n_j} f} x_{n_j}, R_f x_0).$$

Letting $j \to \infty$, we have

$$\begin{aligned}\lambda_0 \liminf_{j\to\infty} \cos d(x_0, R_{\lambda_{n_j} f} x_{n_j}) + \liminf_{j\to\infty} \cos d(x_{n_j}, R_f x_0)& \\ \le \liminf_{j\to\infty} (\lambda_{n_j} + 1) \cos d(R_{\lambda_{n_j} f} x_{n_j}, R_f x_0)&\end{aligned}$$

and hence

$$\begin{aligned}\lambda_0 \cos \limsup_{j\to\infty} d(x_0, R_{\lambda_{n_j} f} x_{n_j}) + \cos \limsup_{j\to\infty} d(x_{n_j}, R_f x_0)& \\ \le (\lambda_0 + 1) \cos \limsup_{j\to\infty} d(R_{\lambda_{n_j} f} x_{n_j}, R_f x_0).&\end{aligned}$$

Thus,

$$\begin{aligned}\lambda_0 \cos \limsup_{j\to\infty} d(x_0, x_{n_j}) + \cos \limsup_{j\to\infty} d(x_{n_j}, R_f x_0)& \\ \le (\lambda_0 + 1) \cos \limsup_{j\to\infty} d(x_{n_j}, R_f x_0).&\end{aligned}$$

and therefore

$$\cos \limsup_{j\to\infty} d(x_0, x_{n_j}) \le \cos \limsup_{j\to\infty} d(x_{n_j}, R_f x_0).$$

Consequently, we have

$$\limsup_{j\to\infty} d(x_0, x_{n_j}) \ge \limsup_{j\to\infty} d(x_{n_j}, R_f x_0).$$

Since $\{x_0\} = \mathrm{AC}(\{x_{n_j}\})$, we have $x_0 = R_f x_0$. It means that $x_0 \in \mathrm{Equil}\, f$.

We next suppose that $\sup_{n\in\mathbb{N}} \lambda_n = \infty$. Then, there exists $\{\lambda_{n_j}\} \subset \{\lambda_n\}$ such that $\lambda_{n_j} \to \infty$. Fix $j \in \mathbb{N}$. Then,

$$\lambda_{n_j} \cos d(x_0, R_{\lambda_{n_j} f} x_{n_j}) + \cos d(x_{n_j}, R_f x_0) \le (\lambda_{n_j} + 1) \cos d(R_{\lambda_{n_j} f} x_{n_j}, R_f x_0).$$

and thus

$$\cos d(x_0, R_{\lambda_{n_j} f} x_{n_j}) + \frac{\cos d(x_{n_j}, R_f x_0)}{\lambda_{n_j}} \le \left(1 + \frac{1}{\lambda_{n_j}}\right) \cos d(R_{\lambda_{n_j} f} x_{n_j}, R_f x_0).$$

Letting $j \to \infty$, we get

$$\liminf_{j\to\infty} \cos d(x_0, R_{\lambda_{n_j} f} x_{n_j}) \le \liminf_{j\to\infty} \cos d(R_f x_0, R_{\lambda_{n_j} f} x_{n_j})$$

and hence

$$\limsup_{j\to\infty} d(x_0, R_{\lambda_{n_j} f} x_{n_j}) \geq \limsup_{j\to\infty} d(R_f x_0, R_{\lambda_{n_j} f} x_{n_j}).$$

It implies that

$$\limsup_{j\to\infty} d(x_0, x_{n_j}) \geq \limsup_{j\to\infty} d(R_f x_0, x_{n_j}).$$

Since $\{x_0\} = \mathrm{AC}(\{x_{n_j}\})$, we have $x_0 = R_f x_0$. Therefore $x_0 \in \operatorname{Equil} f$. □

Theorem 3.2. *Let X be an admissible complete* CAT(1) *space having the convex hull finite property and $K \subset X$ a nonempty closed convex set. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies the conditions (E1)–(E4), and* $\operatorname{Equil} f \neq \varnothing$.

Let $\{\lambda_n\} \subset]0, \infty[$ and $\{\alpha_n\} \subset [0, 1[$ be sequences satisfying $\inf_n \lambda_n > 0$ and $\sup_n \alpha_n < 1$. Let $R_{\lambda_n f}\colon X \to K$ be the resolvent of $\lambda_n f$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence defined by $x_1 \in X$, and

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) R_{\lambda_n f} x_n$$

for each $n \in \mathbb{N}$. Then, $x_n \xrightarrow{\Delta} x_0 \in \operatorname{Equil} f$.

Proof. Since $\operatorname{Equil} f \neq \varnothing$, let $u \in \operatorname{Equil} f$. From Theorem 2.1,

$$\frac{\cos d(u, R_{\lambda_n f} x_n)}{\cos d(u, R_{\lambda_n f} u)} + \frac{\cos d(x_n, R_{\lambda_n f} u)}{\cos d(x_n, R_{\lambda_n f} x_n)} \leq 2 \cos d(R_{\lambda_n f} u, R_{\lambda_n f} x_n).$$

Since $u \in \operatorname{Equil} f$, we have $d(u, R_{\lambda_n f} u) = 0$ and thus

$$\cos d(u, R_{\lambda_n f} x_n) + \frac{\cos d(x_n, u)}{\cos d(x_n, R_{\lambda_n f} x_n)} \leq 2 \cos d(u, R_{\lambda_n f} x_n).$$

It implies that $\cos d(u, R_{\lambda_n f} x_n) \cos d(x_n, R_{\lambda_n f} x_n) \geq \cos d(x_n, u)$ and hence $d(u, R_{\lambda_n f} x_n) \leq d(u, x_n)$. From the parallelogram law, we get

$$\cos d(u, x_{n+1}) \geq \alpha_n \cos d(u, x_n) + (1 - \alpha_n) \cos d(u, R_{\lambda_n f} x_n) \geq \cos d(u, x_n)$$

and hence

$$d(u, x_{n+1}) \leq d(u, x_n) \leq d(u, x_1) < \frac{\pi}{2}.$$

Therefore $\{x_n\}$ is spherically bounded and $d(u, x_n) \to c \in [0, \frac{\pi}{2}[$.

Further, we have

$$\begin{aligned} \cos d(u, x_{n+1}) &\geq \alpha_n \cos d(u, x_n) + (1 - \alpha_n) \cos d(u, R_{\lambda_n f} x_n) \\ &\geq \alpha_n \cos d(u, x_n) + (1 - \alpha_n) \frac{\cos d(u, x_n)}{\cos d(R_{\lambda_n f} x_n, x_n)} \end{aligned}$$

$$= \cos d(u, x_n) + (1 - \alpha_n) \cos d(u, x_n) \left(\frac{1}{\cos d(R_{\lambda_n f} x_n, x_n)} - 1 \right),$$

which implies that

$$0 \leq (1 - \alpha_n) \left(\frac{1}{\cos d(R_{\lambda_n f} x_n, x_n)} - 1 \right) \leq \frac{\cos d(u, x_{n+1})}{\cos d(u, x_n)} - 1.$$

Since $d(u, x_n) \to c \in [0, \frac{\pi}{2}[$ and $\sup_{n \in \mathbb{N}} \alpha_n < 1$, we know $d(R_{\lambda_n f} x_n, x_n) \to 0$. Since $\{x_n\}$ is spherically bounded, any subsequence $\{x_{n_i}\}$ is spherically bounded. Let $\{x_0\} = \mathrm{AC}(\{x_n\})$ and $\{w_0\} = \mathrm{AC}(\{x_{n_i}\})$. We can take a subsequence $\{x_{n_{i_j}}\}$ such that $x_{n_{i_j}} \xrightarrow{\Delta} z_0 \in X$. From Lemma 3.1 and since $d(R_{\lambda_n f} x_{n_{i_j}}, x_{n_{i_j}}) \to 0$, we obtain $z_0 \in \operatorname{Equil} f$. Further, we have

$$\begin{aligned}
\limsup_{n \to \infty} d(x_n, z_0) &= \limsup_{j \to \infty} d(x_{n_{i_j}}, z_0) \\
&\leq \limsup_{j \to \infty} d(x_{n_{i_j}}, w_0) \\
&\leq \limsup_{i \to \infty} d(x_{n_i}, w_0) \\
&\leq \limsup_{i \to \infty} d(x_{n_i}, x_0) \\
&\leq \limsup_{n \to \infty} d(x_n, x_0) \leq \limsup_{n \to \infty} d(x_n, z_0).
\end{aligned}$$

Therefore $x_0 = w_0 = z_0$ and thus we get $\{x_0\} = \mathrm{AC}(\{x_{n_i}\})$ for all $\{x_{n_i}\} \subset \{x_n\}$. Consequently, $x_n \xrightarrow{\Delta} x_0 \in \operatorname{Equil} f$. □

Chapter 4

Δ-convergence theorem for projection method

In this chapter, we prove a theorem for projection method with the resolvent of equilibrium problem.

The following is a strong convergence theorem for a nonexpansive mapping with the scheme called a shrinking projection method on Hilbert spaces.

Theorem 4.1 (Takahashi, Takeuchi, and Kubota [10]). *Let H be a Hilbert space and $C \subset H$ a nonempty closed convex set. Let $T\colon C \to C$ be a nonexpansive mapping such that $F(T) \neq \varnothing$. Let $u \in H$ and $\{\alpha_n\} \subset [0, 1[$ be a sequence. Let $\{x_n\}$ be a sequence and $\{C_n\}$ subsets of H defined by $C_1 = C$, $x_1 \in C$ and*

$$
\begin{aligned}
y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\
C_{n+1} &= \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\
x_{n+1} &= P_{C_{n+1}} u.
\end{aligned}
$$

Then, $x_n \to P_{F(T)} u \in C$.

Further in 2023, Kimura proved the Δ-convergence theorem by modified shrinking projection method for a nonexpansive mapping in a CAT(0) space.

Theorem 4.2 ([6]). *Let X be a Hadamard space and suppose that a subset $\{z \in X | d(x, y) \leq d(y, z)\}$ is convex for $x, y \in X$. Suppose that $T\colon X \to X$ be a nonexpansive mapping and $F(T) \neq \varnothing$. Let $\{x_n\}$ be a sequence and $\{C_n\}$ be a convex subsets of X defined by $x_1 \in X$, $C_1 = X$, and*

$$
\begin{aligned}
C_{n+1} &= \{z \in X \mid d(T x_n, z) \leq d(x_n, z)\} \cap C_n, \\
x_{n+1} &= P_{C_{n+1}} x_n
\end{aligned}
$$

for each $n \in \mathbb{N}$. Then, x_n is well-defined and $x_n \xrightarrow{\Delta} x_0 \in F(T)$.

We consider the projection method for a resolvent of equilibrium problem in CAT(1) space, and show the following Δ-convergence theorem.

Theorem 4.3. *Let X be an admissible complete* $\mathrm{CAT}(1)$ *space having the convex hull finite property and $C \subset X$ a nonempty closed convex set. Suppose that $f \colon C \times C \to \mathbb{R}$ satisfies the conditions (E1)–(E4), and* $\operatorname{Equil} f \neq \varnothing$. *Let $\{\lambda_n\} \subset \,]0, \infty[$ be a sequence satisfying* $\inf_n \lambda_n > 0$. *Suppose that $\{z \in X \mid d(x, z) \leq d(y, z)\}$ is convex for all $x, y \in X$. Let $R_{\lambda_n f} \colon X \to C$ be the resolvent of $\lambda_n f$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence and C_n a convex subset of X defined by $x_1 \in C$, $C_1 = C$, and*

$$C_{n+1} = \{z \in C \mid d(R_{\lambda_n f} x_n, z) \leq d(x_n, z)\} \cap C_n,$$
$$x_{n+1} = P_{C_{n+1}} x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ is well-defined and $x_n \overset{\Delta}{\rightharpoonup} x_0 \in$ $\operatorname{Equil} f$.

Proof. We first show by induction that $F(R_{\lambda_n f} x) \subset C_n$ and x_n is well-defined for all $n \in \mathbb{N}$. $F(R_{\lambda_1 f}) \subset C_1$ is obvious. Suppose $F(R_{\lambda_k f}) \subset C_k$. Let $x \in C$ and $z \in F(R_{\lambda_k f})$. Then

$$\frac{\cos d(x, R_{\lambda_k f} z)}{\cos d(x, R_{\lambda_k f} x)} + \frac{\cos d(z, R_{\lambda_k f} x)}{\cos d(z, R_{\lambda_k f} z)} \leq 2 \cos d(R_{\lambda_k f} x, R_{\lambda_k f} z).$$

Since $z \in F(R_{\lambda_k f})$, we have $\cos d(R_{\lambda_k f} x, z) \cos d(x, R_{\lambda_k f} x) \geq \cos d(x, z)$ and thus $d(x, R_{\lambda_k f} x) \leq d(x, z)$. This implies that $z \in C_{k+1}$ and hence $F(R_{\lambda_{k+1} f}) \subset C_{k+1} \neq \varnothing$. Therefore $F(R_{\lambda_n f}) \subset C_n \neq \varnothing$ for all $n \in \mathbb{N}$. Further, we know that C_n is closed and convex for all $n \in \mathbb{N}$ and thus $\{x_n\}$ is well-defined.

Next, we show that $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \operatorname{Equil} f$. Let $u \in F(R_{\lambda_n f})$. Since $\{P_{C_n}\}$ is quasi-nonexpansive, $d(x_{n+1}, u) = d(P_{C_n+1}, u) \leq d(x_n, u)$ and thus $\{d(x_n, u)\}$ converges to some $c \in [0, \pi/2[$. From the definition of CAT(1) space, we have

$$\begin{aligned}
&\cos d(x_{n+1}, x_n) \sin d(u, x_{n+1}) \\
&= \cos d(P_{C_{n+1}} x_n, x_n) \sin d(u, P_{C_{n+1}} x_n) \\
&\geq \cos d(tu \oplus (1-t) P_{C_{n+1}} x_n, x_n) \sin d(u, P_{C_{n+1}} x_n) \\
&\geq \cos d(u, x_n) \sin t d(u, P_{C_{n+1}} x_n) + \cos d(P_{C_{n+1}} x_n, x_n) \sin (1-t) d(u, P_{C_{n+1}} x_n) \\
&= \cos d(u, x_n) \sin t d(u, x_{n+1}) + \cos d(x_{n+1}, x_n) \sin (1-t) d(u, x_{n+1})
\end{aligned}$$

and hence

$$(\sin d(u, x_{n+1}) - \sin (1-t) d(u, x_{n+1})) \cos d(x_{n+1}, x_n) \geq \cos d(u, x_{n+1}) \sin t d(u, x_{n+1}).$$

Using sum to product formulas, we have

$$\begin{aligned}
&(\sin d(u, x_{n+1}) - \sin (1-t) d(u, x_{n+1})) \cos d(x_{n+1}, x_n) \\
&\qquad = 2 \cos \frac{(2-t) d(u, x_{n+1})}{2} \sin \frac{t d(u, x_{n+1})}{2} \cos \frac{t}{2} d(x_n, x_{n+1}).
\end{aligned}$$

From half angle formulas, we get

$$\begin{aligned}
&2\cos\frac{(2-t)d(u,x_{n+1})}{2}\sin\frac{td(u,x_{n+1})}{2}\cos d(x_n,x_{n+1})\\
&\geq \cos d(u,x_n)\sin d(u,x_{n+1})\\
&= \cos d(u,x_n)2\sin\frac{t}{2}d(u,x_{n+1})\cos\frac{t}{2}d(u,x_{n+1}).
\end{aligned}$$

This implies that

$$\cos\frac{(2-t)d(u,x_{n+1})}{2}\cos d(x_n,x_{n+1}) \geq \cos d(u,x_n)\cos\frac{t}{2}d(u,x_{n+1}),$$

and letting $t \to 0$, we get

$$\cos d(u,x_{n+1})\cos d(x_n,x_{n+1}) \geq \cos d(u,x_n).$$

Since $d(x_n,u) \to c$, wa have

$$1 \geq \cos d(x_{n+1},x_n) \geq \frac{\cos d(u,x_n)}{\cos d(u,x_{n+1})} \to \frac{\cos c}{\cos c} = 1$$

as $n \to \infty$, and thus $d(x_{n+1},x_n) \to 0$. Since $x_{n+1} \in C_{n+1}$ and $d(x_{n+1},x_n) \to 0$, we have

$$\begin{aligned}
0 \leq d(R_{\lambda_n f}x_n,x_n) &\leq d(R_{\lambda_n f}x_n,x_{n+1}) + d(x_{n+1},x_n)\\
&\leq d(x_n,x_{n+1}) + d(x_{n+1}+x_n) \to 0
\end{aligned}$$

and hence

$$d(R_{\lambda_n f}x_n,x_n) \to 0.$$

Since $\{x_n\}$ is spherically bounded, every subsequence $\{x_{n_i}\}$ of $\{x_n\}$ is also spherically bounded. Let $\{x_0\} = \mathrm{AC}(\{x_n\})$ and $\{w_0\} = \mathrm{AC}(\{x_{n_i}\})$. We can take a subsequence $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that Δ-convergence to some $z_0 \in C$. Since $d(R_{\lambda_n f}x_n,x_n) \to 0$, by Lemma 3.1 we get $z_0 \in \operatorname{Equil} f$. From the definition of the asymptotic center, we get

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n,z_0) &= \limsup_{j\to\infty} d(x_{n_{i_j}},z_0)\\
&\leq \limsup_{j\to\infty} d(x_{n_{i_j}},w_0)\\
&\leq \limsup_{i\to\infty} d(x_{n_i},w_0)\\
&\leq \limsup_{i\to\infty} d(x_{n_i},x_0)\\
&\leq \limsup_{n\to\infty} d(x_n,x_0) \leq \limsup_{n\to\infty} d(x_n,z_0).
\end{aligned}$$

Therefore $x_0 = w_0 = z_0$ and thus we get $\{x_0\} = \mathrm{AC}(\{x_{n_i}\})$ for all $\{x_{n_i}\} \subset \{x_n\}$. Consequently, $x_n \xrightarrow{\Delta} x_0 \in \operatorname{Equil} f$. □

Further, we show that a projection onto subset of CAT(1) space strong convergence to the same point as sequence for metric projection.

Theorem 4.4. *Let X be an admissible complete* CAT(1) *space having the convex hull finite property and $C \subset X$ a nonempty closed convex set. Suppose that $f\colon C \times C \to \mathbb{R}$ satisfies the conditions (E1)–(E4), and* $\operatorname{Equil} f \neq \varnothing$*. Let $\{\lambda_n\} \subset]0, \infty[$ be a sequence satisfying $\inf_n \lambda_n > 0$. Suppose that $\{z \in X \mid d(x, z) \le d(y, z)\}$ is convex for all $x, y \in C$. Let $R_{\lambda_n f}\colon X \to C$ be the resolvent of $\lambda_n f$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence and C_n be a convex subset of X defined by $x_1 \in C$, $C_1 = C$, and*

$$\begin{aligned} C_{n+1} &= \{z \in C \mid d(R_{\lambda_n f} x_n, z) \le d(x_n, z)\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x_n \end{aligned}$$

for each $n \in \mathbb{N}$. Let $D \subset X$ be closed and convex such that $\operatorname{Equil} f \subset D \subset \bigcap_{n=1}^{\infty} C_n$*. Then, $P_D x_n \to x_0$, where x_0 is the Δ-limit of $\{x_n\}$.*

Proof. By theorem 4.3, we have that $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \operatorname{Equil} f$. Using this fact, we show $P_D x_n \to x_0$. Since

$$d(P_D x_{n+1}, x_{n+1}) \le d(P_D x_n, x_{n+1}) = d(P_D x_n, P_{C_{n+1}} x_n) \le d(P_D x_n, x_n),$$

$\{\cos d(P_D x_n, x_n)\}$ is a Cauchy sequence. Hence, there exists $\{\alpha_n\} \subset \mathbb{R}$ such that $\alpha_n \to 0$ and $\cos d(P_D x_m, x_m) - \cos d(P_D x_n, x_n) \le \alpha_n$ if $m \ge n$. We have

$$\frac{\alpha_n}{\cos d(P_D x_m, x_m)} \ge \frac{\cos d(P_D x_m, x_m)}{\cos d(P_D x_m, x_m)} - \frac{\cos d(P_D x_n, x_n)}{\cos d(P_D x_m, x_m)},$$

and thus

$$\frac{\cos d(P_D x_n, x_n)}{\cos d(P_D x_m, x_m)} \ge 1 - \frac{\alpha_n}{\cos d(P_D x_m, x_m)} \ge 1 - \frac{\alpha_n}{\cos d(P_D x_1, x_1)}.$$

Further, since $\cos d(P_D x_m, x_m) \cos d(P_D x_m, P_D x_n) \ge \cos d(P_D x_n, x_m)$, we have

$$\begin{aligned} \cos d(P_D x_m, P_D x_n) &\ge \frac{\cos d(P_D x_n, x_m)}{\cos d(P_D x_m, x_m)} \\ &\ge \frac{\cos d(P_t D x_n, x_n)}{\cos d(P_D x_m, x_m)} \ge 1 - \frac{\alpha_n}{\cos d(P_D x_1, x_1)}. \end{aligned}$$

Hence, $\{P_D x_n\}$ is a Cauchy sequence and we get $\{P_D x_n\}$ converges to some $y_0 \in C$. Since $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \operatorname{Equil} f$, we have

$$\begin{aligned} \limsup_{n\to\infty} d(y_0, x_n) &\le \limsup_{n\to\infty} (d(y_0, P_D x_n) + d(P_D x_n, x_n)) \\ &= \limsup_{n\to\infty} d(P_D x_n, x_n) \le \limsup_{n\to\infty} d(x_0, x_n). \end{aligned}$$

Therefore, $y_0 = x_0$ and we get $P_D x_n \to x_0 \in \operatorname{Equil} f$. ∎

Chapter 5

Conclusion

In this thesis, we considered an equilibrium problem defined on an admissible complete CAT(1) space and obtain two different types of approximation theorems by using the resolvent operator for an equilibrium problem.

The first result is a Mann type Δ-convergence theorem as follows;

Theorem 3.3. *Let X be an admissible complete* CAT(1) *space having the convex hull finite property and $K \subset X$ a nonempty closed convex set. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies the conditions (E1)–(E4), and* $\operatorname{Equil} f \neq \varnothing$.

Let $\{\lambda_n\} \subset \,]0, \infty[$ and $\{\alpha_n\} \subset [0, 1[$ be sequences satisfying $\inf_n \lambda_n > 0$ and $\sup_n \alpha_n < 1$. Let $R_{\lambda_n f}\colon X \to K$ be the resolvent of $\lambda_n f$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence defined by $x_1 \in X$, and

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) R_{\lambda_n f} x_n$$

for each $n \in \mathbb{N}$. Then, $x_n \stackrel{\Delta}{\rightharpoonup} x_0 \in \operatorname{Equil} f$.

In the second result, we used a modified shrinking projection method, which was recently proposed by Kimura, and obtain an iterative sequence which is Δ-convergent to the solution to an equilibrium problem.

Theorem 4.3. *Let X be an admissible complete* CAT(1) *space having the convex hull finite property and $C \subset X$ a nonempty closed convex set. Suppose that $f\colon C \times C \to \mathbb{R}$ satisfies the conditions (E1)–(E4), and* $\operatorname{Equil} f \neq \varnothing$. *Let $\{\lambda_n\} \subset \,]0, \infty[$ be a sequence satisfying $\inf_n \lambda_n > 0$. Suppose that $\{z \in X \mid d(x, z) \leq d(y, z)\}$ is convex for all $x, y \in X$. Let $R_{\lambda_n f}\colon X \to C$ be the resolvent of $\lambda_n f$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence and C_n a convex subset of X defined by $x_1 \in C$, $C_1 = C$, and*

$$\begin{aligned} C_{n+1} &= \{z \in C \mid d(R_{\lambda_n f} x_n, z) \leq d(x_n, z)\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x_n \end{aligned}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ is well-defined and $x_n \stackrel{\Delta}{\rightharpoonup} x_0 \in \operatorname{Equil} f$.

Further, in this scheme, we found a strongly convergent sequence to the same point as the original iterative sequence by using metric projections onto a subset having an appropriate property.

Theorem 4.4. *Let X be an admissible complete* $\mathrm{CAT}(1)$ *space having the convex hull finite property and $C \subset X$ a nonempty closed convex set. Suppose that $f\colon C \times C \to \mathbb{R}$ satisfies the conditions (E1)–(E4), and* $\operatorname{Equil} f \neq \varnothing$. *Let $\{\lambda_n\} \subset \left]0, \infty\right[$ be a sequence satisfying* $\inf_n \lambda_n > 0$. *Suppose that $\{z \in X \mid d(x, z) \leq d(y, z)\}$ is convex for all $x, y \in C$. Let $R_{\lambda_n f}\colon X \to C$ be the resolvent of $\lambda_n f$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence and C_n be a convex subset of X defined by $x_1 \in C$, $C_1 = C$, and*

$$C_{n+1} = \{z \in C \mid d(R_{\lambda_n f} x_n, z) \leq d(x_n, z)\} \cap C_n,$$
$$x_{n+1} = P_{C_{n+1}} x_n$$

for each $n \in \mathbb{N}$. Let $D \subset X$ be closed and convex such that $\operatorname{Equil} f \subset D \subset \bigcap_{n=1}^{\infty} C_n$. *Then, $P_D x_n \to x_0$, where x_0 is the Δ-limit of $\{x_n\}$.*

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