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東邦大学審査学位論文(博士)

Crystal Bases and K-hives

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Abstract. In this thesis, we study the theory of A_{n-1}-crystal bases and K-hives. This thesis has three themes. The first theme is a combinatorial realization of crystal bases of highest weight modules over the quantized enveloping algebra of type A by K-hives. This contains the determination of a crystal structure on a set of K-hives using two approaches. One approach is obtained by considering an embedding of a set of K-hives determined by a dominant weight into a tensor product of sets of K-hives determined by fundamental weights. The other approach is obtained by considering a combinatorial description of the crystal structure. The second theme is a combinatorial tensor product decomposition map of crystal bases in terms of K-hives. This map is described using the notion of path operators on K-hives, and then the decomposition map can be computed graphically. The third theme is a set of algorithms for computing the crystal structure on a set of K-hives and their implementation as a Python package. Additionally, we show some examples of performing this package.

Contents

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1. Introduction

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. The quantized enveloping algebra $U_q(\mathfrak{g})$ is the q-analogue of $U(\mathfrak{g})$, which is introduced in the study of the quantum Yang-Baxter equation in [2, 4] When $q = 1$, $U_q(\mathfrak{g})$ is the same as $U(\mathfrak{g})$. When $q = 0$, the representation of $U_q(\mathfrak{g})$ can be studied from combinatorics by crystal bases.

Crystal bases are special bases of modules over $U_q(\mathfrak{g})$ at $q = 0$ developed in [6, 7, 5]. These bases have nice properties and give a combinatorial tool for studying the representation theory of $U_q(\mathfrak{g})$. For example, computing the action of $U_q(\mathfrak{g})$ on the tensor product of the modules is laborious. However, it can be simply computed at $q = 0$ using the crystal basis. Moreover, some crystal bases have combinatorial realizations: let $\mathfrak{g}$ be a simple Lie algebra of type A_{n-1}. For a dominant weight λ of A_{n-1}, let $V(\lambda)$ be the highest weight module of the highest weight λ over $U_q(\mathfrak{g})$. Let Y be the Young tableau corresponding to λ. Let $B(Y)$ be the set of semistandard tableaux of shape Y. Then, $B(Y)$ is isomorphic to $B(\lambda)$. This means that the crystal base of the highest weight module of the highest weight λ of type A_{n-1} is realized by semistandard tableaux of shape Y [8]. Furthermore, the decomposition of a tensor product of crystal bases of highest weight modules of type A_{n-1} is given using the realization by Young tableaux. This decomposition rule is obtained by determining that an element of a tensor product is a highest weight element in terms of Young tableaux [14].

In this thesis, we study the theory of A_{n-1}-crystal bases and K-hives. This thesis has three themes.

The first theme is a combinatorial realization of crystal bases of highest weight modules of type A by K-hives [19, 17, 18]. A K-hive is the labeling of the vertices of an equilateral triangular graph introduced in [12, 13]. K-hives have correspondence with semistandard Young tableaux or Gelfand-Tsetlin patterns and have applications, for example, to (Stretched) Kostka coefficients [16, 10] (also, see [11, 24]). Then, the crystal structure on K-hives introduced in this thesis is based on the construction of the crystal structure on $B(Y)$ [8]: For a dominant weight λ of type A_{n-1}, let $\mathbb{H}(\lambda)$ be the set of K-hives determined by λ. Let Λ_k $(k = 1, 2, \dots, n-1)$ be the fundamental weights of A_{n-1}. First, we consider the crystal structure on $\mathbb{H}(\Lambda_k)$. Then, we construct the embedding $\mathbb{H}(\lambda)$ to $\bigotimes_k \mathbb{H}(\Lambda_k)$. Then the crystal structure on $\mathbb{H}(\lambda)$ is defined so that the embedding is a crystal morphism. Further, we can show that $\mathbb{H}(\lambda)$ is isomorphic to $B(\lambda)$. Also, we give a direct combinatorial description of the crystal structure, which enables us to define the crystal structure on $\mathbb{H}(\lambda)$ directly.

The second theme is the combinatorial tensor product decomposition map of crystal bases by K-hives [20, 21]. This is an application of the realization by K-hives. We first define maps from $\mathbb{H}(\lambda)$ to $\mathbb{H}(\mu)$ and then define a map from $\mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$ to $\mathbb{H}(\nu) \otimes \mathbb{H}(\xi)$ by combining these maps. Then, the decomposition map is constructed as a sequence of the maps. We introduce the notion of path operators which can be graphically computed.

From the fact that the maps from $\mathbb{H}(\lambda)$ to $\mathbb{H}(\mu)$ are path operators, the decomposition map can be graphically computed.

The third theme is a set of algorithms for computing the crystal structure on K-hives, and the implementation of these algorithms [15]. We also give some examples of executing these algorithms. The implementation is provided as a Python package named *khive-crystal*. The source code is available in [22].

This thesis is organized as follows. In Section 2, we review basic notions and notation of quantized enveloping algebras, crystals, and K-hives. Sections 3, 4, and 5 concern the first, second, and third themes explained above, respectively. Finally, Section 6 gives some concluding remarks.

2. Preliminaries

In this section, we review basic notions and notation. In 2.1, we review the definition of quantized enveloping algebras and related notions. In 2.2, we review the definitions of crystals, the tensor product of crystals, crystal graphs, and morphisms between crystals. In 2.3, we review K-hives and define some notations.

2.1. Quantized Enveloping Algebras.

In this subsection, we review the definition of quantized enveloping algebras of type A, see [3] for more details.

Let $\mathfrak{sl}_n$ be the Lie algebra of type A_{n-1} over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$ consisting of traceless diagonal matrices. Let $I = \{1, 2, \ldots, n-1\}$ be an index set. Let $A = (a_{ij})_{i,j \in I}$ be the Cartan matrix of type A_{n-1}. For $m \in \mathbb{Z}_{>0}$, let $[m] = \{1, 2, \ldots, m\}$. For $i \in [n]$, define a linear map $\epsilon_i \colon \mathfrak{h} \to \mathbb{C}$ by $\epsilon_i(h) = c_i$, where $h = \operatorname{diag}(c_j \mid j \in [n]) \in \mathfrak{h}$. For $i \in I$, set $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Let $\Pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ be simple roots and $\Pi^\vee = \{h_i\}_{i \in I} \subset \mathfrak{h}$ be simple coroots. Let Δ be the root system of $\mathfrak{sl}_n$. Set $\Delta^+ = \Delta \cap \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ and $\Delta^- = \Delta - \Delta^+$. For all $i \in I$, let $\Lambda_i = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_i \in \mathfrak{h}^*$ be an i-th fundamental weight. Set $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$, $P^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i$, and $P^\vee = \bigoplus_{i \in I} \mathbb{Z} h_i$. We call P the weight lattice, P^+ the set of dominant integral weights, and $P^\vee$ the dual weight lattice, respectively. Using this notation, the Cartan datum for $\mathfrak{sl}_n$ is defined as $(A, \Pi, \Pi^\vee, P, P^\vee)$.

Let q be an indeterminate. Let $U_q(\mathfrak{sl}_n)$ be the quantized enveloping algebra over $\mathbb{Q}(q)$ associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. Let $V(\lambda)$ be the irreducible highest weight module of weight $\lambda \in P^+$ with the highest weight vector v_λ over $U_q(\mathfrak{sl}_n)$.

2.2. Crystals.

In this subsection, we review the notion of crystals, see [3, 6, 7] for more details.

Definition 2.1. A **crystal** associated with Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is a set B together with the maps $\operatorname{wt} \colon B \to P$, $e_i, f_i \colon B \to B \cup \{0\}$, and $\varepsilon_i, \varphi_i \colon B \to \mathbb{Z} \cup \{-\infty\}$ ($i \in I$) satisfying the following properties.

(1) $\varphi_i(b) = \varepsilon_i(b) + \operatorname{wt}(b)(h_i)$ for $i \in I$,
(2) $\operatorname{wt}(e_i b) = \operatorname{wt}(b) + \alpha_i$ if $e_i b \in B$,
(3) $\operatorname{wt}(f_i b) = \operatorname{wt}(b) - \alpha_i$ if $e_i b \in B$,
(4) $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1$, $\varphi_i(e_i b) = \varphi_i(b) + 1$ if $e_i b \in B$,
(5) $\varepsilon_i(f_i b) = \varepsilon_i(b) + 1$, $\varphi_i(f_i b) = \varphi_i(b) - 1$ if $f_i b \in B$,
(6) $f_i b = b'$ if and only if $b = e_i b'$ for $b, b' \in B$, $i \in I$,
(7) if $\varphi_i(b) = -\infty$, then $e_i b = f_i b = 0$.

Since $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Cartan datum of type A_{n-1}, a crystal associated with $(A, \Pi, \Pi^\vee, P, P^\vee)$ is also called a $U_q(\mathfrak{sl}_n)$-crystal.

A $U_q(\mathfrak{sl}_n)$-crystal can be thought of as a colored and oriented graph in the following manner.

Definition 2.2. Let B be a $U_q(\mathfrak{sl}_n)$-crystal. A **crystal graph** of B is an I-colored oriented graph whose vertices are elements of B and the arrows are written as $b \xrightarrow{i} b'$ when $f_i b = b'$ for $i \in I$ and $b, b' \in B$.

The tensor product of crystals is defined as follows.

Definition 2.3. Let B_1 and B_2 be crystals. The tensor product $B_1 \otimes B_2$ of B_1 and B_2 is defined to be the set $B_1 \times B_2$ whose crystal structure is defined by

(1) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,
(2) $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}(b_1)(h_i))$,
(3) $\varphi(b_1 \otimes b_2) = \max(\varphi(b_2), \varphi(b_1) + \mathrm{wt}(b_2)(h_i))$,
(4) $e_i(b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2 & \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i b_2 & \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$
(5) $f_i(b_1 \otimes b_2) = \begin{cases} f_i b_1 \otimes b_2 & \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i b_2 & \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$

In general, we have the following proposition([8, Proposition 2.1.1]).

Proposition 2.4. For $j \in \{1, \ldots, N\}$, let B_j be a $U_q(\mathfrak{sl}_n)$-crystal. Fix $i \in I$. Take $b_j \in B_j$ $(j = 1, \ldots, N)$, and we set

$$a_k = \sum_{1 \leq j < k} (\varphi_i(b_j) - \varepsilon_i(b_{j+1})) \quad 1 \leq k \leq N.$$

In particular, we set $a_1 = 0$. Then we have

(1) $\varepsilon_i(b_1 \otimes \cdots \otimes b_N) = \max\left\{\sum_{1 \leq j \leq k} \varepsilon_i(b_j) - \sum_{1 \leq j < k} \varphi_i(b_j) \mid 1 \leq k \leq N\right\}$,
(2) $\varphi_i(b_1 \otimes \cdots \otimes b_N) = \max\left\{\varphi_i(b_N) + \sum_{k \leq j < N} (\varphi_i(b_j) - \varepsilon_i(b_{j+1})) \mid 1 \leq k \leq N\right\}$,
(3) If k is the largest element such that $a_k = \min\{a_j \mid 1 \leq j \leq N\}$ then, we have
$$f_i(b_1 \otimes \cdots \otimes b_N) = b_1 \otimes \cdots \otimes b_{k-1} \otimes f_i b_k \otimes b_{k+1} \otimes \cdots \otimes b_N,$$
(4) If k is the smallest element such that $a_k = \min\{a_j \mid 1 \leq j \leq N\}$ then, we have
$$e_i(b_1 \otimes \cdots \otimes b_N) = b_1 \otimes \cdots \otimes b_{k-1} \otimes e_i b_k \otimes b_{k+1} \otimes \cdots \otimes b_N.$$

Isomorphism of crystals is defined to be a bijection preserving the crystal structure. Later we will also construct a crystal embedding as defined in the following.

Definition 2.5. Let B_1, B_2 be $U_q(\mathfrak{sl}_n)$-crystals. A **crystal morphism** $\Psi\colon B_1 \to B_2$ is a map $\Psi\colon B_1 \cup \{0\} \to B_2 \cup \{0\}$ satisfying

(1) $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b), \varepsilon_i(\Psi(b)) = \varepsilon_i(b), \varphi_i(\Psi(b)) = \varphi_i(b)$ if $b \in B_1, \Psi(b) \in B_2$,
(2) $f_i \Psi(b) = \Psi(f_i b), e_i \Psi(b) = \Psi(e_i b)$ if $\Psi(b), \Psi(e_i b), \Psi(f_i b) \in B_2$ for $b \in B_1$,
(3) $\Psi(0) = 0$.

A morphism $\Psi\colon B_1 \to B_2$ is called an **embedding** if Ψ induces an injection $B_1 \cup \{0\} \to B_2 \cup \{0\}$. A morphism $\Psi\colon B_1 \to B_2$ is called an **isomorphism** if Ψ induces a bijection $B_1 \cup \{0\} \to B_2 \cup \{0\}$. We write $B_1 \cong B_2$ if there exists an isomorphism $\Psi\colon B_1 \to B_2$.

The irreducible highest weight module $V(\lambda)$ of weight $\lambda \in P^+$ with the highest weight vector v_λ has the crystal basis $(L(\lambda), B(\lambda))$. In particular, $B(\lambda)$ is a $U_q(\mathfrak{sl}_n)$-crystal with a highest weight element b_λ of weight λ.

2.3. **K-hives.** Hives are introduced by T. Tao and A. Knutson [12, 13] as the labeling of the vertices of an equilateral triangular graph. There are three forms of hives, one of which, the upright gradient representation, is used in this paper. See [24] for more details. In this paper, we use K-hives, which are a special kind of hives introduced in [9].

Let $m, n \in \mathbb{Z}_{\geq 0}$ and $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n) \in \mathbb{Z}^n_{\geq 0}$. $\tilde{\mu}$ is called a **composition** of m if $\tilde{\mu}_1 + \cdots + \tilde{\mu}_n = m$. A composition $\tilde{\lambda}$ is called a **partition** of m if $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_n \geq 0$. If $\tilde{\lambda}$ is a partition of m such that $\tilde{\lambda}_i = k$ for $1 \leq i \leq j \leq n$ and $\tilde{\lambda}_i = 0$ for $j < i \leq n$, then we write $\tilde{\lambda}$ as (k^j). In particular, we simply write (0^n) as 0 if there is no fear of confusion. Also, $\ell(\tilde{\lambda})$ denotes the length of $\tilde{\lambda}$.

For $\lambda \in P^+$, there exists a partition $\tilde{\lambda}$ such that $\tilde{\lambda}_1\epsilon_1 + \tilde{\lambda}_2\epsilon_2 + \cdots + \tilde{\lambda}_n\epsilon_n = \lambda$. Similarly, for $\mu \in P$, there exists a composition $\tilde{\mu}$ such that $\tilde{\mu}_1\epsilon_1 + \tilde{\mu}_2\epsilon_2 + \cdots + \tilde{\mu}_n\epsilon_n = \mu$. Note that a composition $(\tilde{\mu}_1 + k, \ldots, \tilde{\mu}_n + k)$ also represents $\mu \in P$ since $\epsilon_1 + \cdots + \epsilon_n = 0$.

Let $\xi \in P$ be a weight of $V(\lambda)$. Then ξ is written as $\lambda - \sum_{i\in I} k_i\alpha_i \in P$ $(k_i \in \mathbb{Z})$. For ξ, there exists a composition $\tilde{\xi}$ such that $\tilde{\xi}_1\epsilon_1 + \tilde{\xi}_2\epsilon_2 + \cdots + \tilde{\xi}_n\epsilon_n = \xi$ and $\sum_{k=1}^n \tilde{\xi}_k = \sum_{k=1}^n \tilde{\lambda}_k$.

Let $\lambda, \mu \in P^+$. Let $\tilde{\lambda}$ (resp. $\tilde{\mu}$) be a partition which represents λ (resp. μ). Suppose $V(\lambda) \otimes V(\mu) \cong \bigoplus_\nu V(\nu)$. Then, for each $\nu \in P^+$ appearing on the right-hand side above, we can take a partition $\tilde{\nu}$ such that $\tilde{\nu}_1\epsilon_1 + \tilde{\nu}_2\epsilon_2 + \cdots + \tilde{\nu}_n\epsilon_n = \nu$ and $\sum_{k=1}^n \tilde{\nu}_k = \sum_{k=1}^n \tilde{\lambda}_k + \sum_{k=1}^n \tilde{\mu}_k$.

In the following, a partition (resp. composition) $\tilde{\lambda}$ representing a dominant weight (resp. an integral weight) λ is also denoted by λ by abuse of notation.

Definition 2.6. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$. Let $(U_{ij})_{1\leq i<j\leq n} \in \mathbb{Z}^{n(n-1)/2}$. An **integer hive** of size n in upright gradient representation ([24]) is a tuple $(\alpha, \beta, \gamma, (U_{ij})_{1\leq i<j\leq n})$ that satisfies

$$\beta_k = (\gamma_k + \sum_{i=1}^{k-1} U_{ik}) + (\alpha_k - \sum_{j=k+1}^{n} U_{kj}). \tag{2.1}$$

Remark 2.7. In [12, 13, 24], the term hive refers to a hive with additional inequality conditions called the rhombus inequalities. We rather follow the terminology of [9, 10, 11].

An integer hive in upright gradient representation is illustrated as the labeling of an equilateral triangular graph with boundary edge labels $(\alpha_i)_i$, $(\beta_i)_i$, $(\gamma_i)_i$, and upright gradients $(U_{ij})_{i<j}$ as shown in FIGURE 1.

In the following, for $i \in [n]$, set

$$U_{ii} = \beta_i - \sum_{k=1}^{i-1} U_{ki} \tag{2.2}$$

and $U_{ij} = 0$ if $i > j$ or $j > n$ or $i < 1$. Also, for simplicity, we will write $(U_{ij})_{1\leq i<j\leq n}$ as $(U_{ij})_{i<j}$.

In this paper, we consider a kind of integer hive called a K-hive.

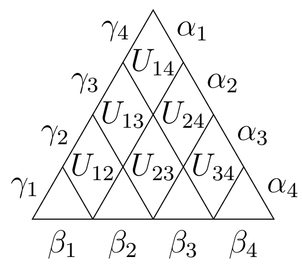


Figure 1. integer hive of size 4

Definition 2.8. Let $m,n \in \mathbb{Z}_{\geq 0}$. Let $\alpha,\beta,\gamma \in \mathbb{Z}^n_{\geq 0}$. For $1 \leq i < j \leq n$, set $L_{ij} = \sum_{k=1}^{j-1} U_{ik} - \sum_{k=1}^{j} U_{i+1,k}$. Then an integer hive in upright gradient representation $H = (\alpha,\beta,\gamma,(U_{ij})_{i<j})$ is called a **K-hive** if the following conditions are satisfied

(1) α is a partition of m,
(2) β is a composition of m,
(3) $\gamma = (0^n)$,
(4) $U_{ij} \geq 0$ for $1 \leq i < j \leq n$,
(5) $L_{ij} \geq 0$ for $1 \leq i < j \leq n$,
(6) $\beta_i \geq \sum_{k=1}^{i-1} U_{ki}$ for $i \in [n]$.

For a partition α of m and a composition of β, let

$$\mathcal{H}^{(n)}(\alpha,\beta,0) = \{H = (\alpha,\beta,0,(U_{ij})_{i<j}) \mid H \text{ is a K-hive}\}.$$

Set

$$\mathbb{H}(\alpha) = \bigcup_{\beta} \mathcal{H}^{(n)}(\alpha,\beta,0),$$

where the union runs through all compositions of m. We sometimes call an element of $\mathbb{H}(\alpha)$ an α**-K-hive**.

Remark 2.9. For $H = (\alpha,\beta,0,(U_{ij})_{i<j}) \in \mathcal{H}^{(n)}(\alpha,\beta,0)$, we have

$$\begin{aligned}\sum_{k=1}^{n} \beta_k &= \sum_{k=1}^{n}(\sum_{i=1}^{k-1} U_{ik} + \alpha_k - \sum_{j=k+1}^{n} U_{kj}) \\ &= \sum_{k=1}^{n} \alpha_k.\end{aligned}$$

Thus, if $\sum_{i=1}^n \alpha_i \neq \sum_{i=1}^n \beta_i$, we have $\mathcal{H}^{(n)}(\alpha,\beta,0) = \varnothing$.

Remark 2.10. Let $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n)$ be a partition of $m \in \mathbb{Z}_{\geq 0}$. Let $l \in \mathbb{Z}_{\geq 0}$. Set $\alpha' = (\alpha_i + l)_i$. We know that α and α' represent the same dominant weight. We also have that $\mathbb{H}(\alpha) \cong \mathbb{H}(\alpha')$ as a set. The bijection from $\mathbb{H}(\alpha)$ to $\mathbb{H}(\alpha')$ is given by the map which maps $(\alpha,\beta,0,(U_{ij})_{i<j})$ to $(\alpha',\beta',0,(U'_{ij})_{i<j})$, where $\beta' = (\beta_i + l)_i$ and $(V_{ij})_{i<j} = (U_{ij})_{i<j}$. Note that $V_{ii} = U_{ii} + l$ holds for $i = 1,2,\dots,n-1$.

Remark 2.11. Let $H \in \mathcal{H}^{(n)}(\alpha,\beta,0) \subset \mathbb{H}(\alpha)$. In this case, we have $U_{ii} = \alpha_i - \sum_{j=i+1}^{n} U_{ij}$ by Definition 2.6 (2.1)(2.2). Also, we have $U_{ij} = 0$ for $j \in [n]$ if $\alpha_i = 0$ since $U_{kl} \geq 0$ for $1 \leq k \leq l \leq n$.

Example 2.12. Let $n = 4$, $\lambda = (3, 2, 1, 0)$ and $\mu = (2, 3, 1, 0)$. We have an example of $H \in \mathcal{H}^{(4)}(\lambda, \mu, 0) \subset \mathbb{H}(\lambda)$ as shown in FIGURE 2.

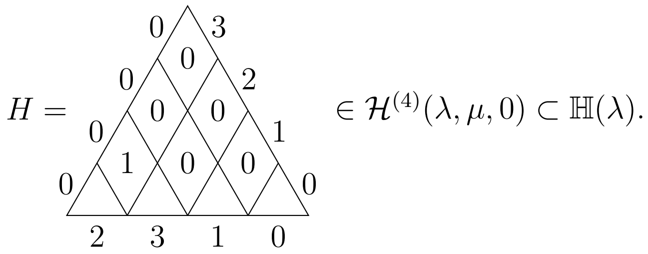


FIGURE 2. An example of a K-hive

Remark 2.13. Let $\lambda \in P^+$ and let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let T be a Young tableau of shape λ and weight μ, and let U_{ij} be the number of j in the i-th row of T. Then the map that sends H to T is a bijection from $\mathbb{H}(\lambda)$ to the set of semistandard tableaux of shape λ (cf. [10]).

Remark 2.14. Let $\tilde{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of $m \in \mathbb{Z}_{\geq 0}$. Let $\tilde{\lambda}' = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of $m \in \mathbb{Z}_{\geq 0}$.

3. Crystal Structure on K-hives

In this section, we introduce a crystal structure on the set of K-hives and show that the crystal structure is isomorphic to the crystal basis of a highest weight module. In 3.1, the crystal structure on $\mathbb{H}(\Lambda_k)$ is given. In 3.2, for an arbitrary dominant weight λ an embedding of $\mathbb{H}(\lambda)$ into a tensor product of crystals of the form $\mathbb{H}(\Lambda_k)$ is defined. Then, the crystal structure on $\mathbb{H}(\lambda)$ is introduced such that the embedding is a crystal morphism. In 3.3, a direct combinatorial description of the crystal structure on $\mathbb{H}(\lambda)$ is given. The main reference is [19].

3.1. Crystal Structure on $\mathbb{H}(\Lambda_\nu)$.

We will start with the case where a weight is a fundamental weight. Since the ν-th fundamental can be viewed as the partition (1^ν), the upper right boundary edge labels of $H \in \mathbb{H}(\Lambda_\nu)$ are 1 or 0.

Lemma 3.1. Let $\nu \in I$ and $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$.

(1) For all $i \in \{1, 2, \ldots, \nu\}$, there exists a unique $j \in \{i, i+1, \ldots, n\}$ such that $U_{ij} = 1$.
(2) Fix $j \in I$. If there exists $i, i' \in \{1, 2, \ldots, j\}$ such that $U_{ij}, U_{i'j} > 0$, then $i = i'$ holds.

Proof. (1) Set $\lambda = \Lambda_\nu$. Let $i \in \{1, \dots, \nu\}$. By Definition 2.6 (2.1), we have

$$\sum_{l=1}^{i-1} U_{li} + \lambda_i - \sum_{l=i+1}^{n} U_{il} = \mu_i$$
$$\left(\mu_i - \sum_{l=1}^{i-1} U_{li}\right) + \sum_{l=i+1}^{n} U_{il} = \lambda_i$$
$$\sum_{l=i}^{n} U_{il} = 1.$$

Thus there exists a unique $j \in \{i, i+1, \dots, n\}$ such that $U_{ij} = 1$ since $H \in \mathbb{H}(\lambda)$.

(2) Set $\lambda = \Lambda_\nu$. Fix $j \in I$. Suppose that there exists $i, i' \in \{1, 2, \dots, j\}$ such that $U_{ij}, U_{i'j} > 0$. Assume $i \neq i'$. From Definition 2.6 (2.1) and $i, i' \in \{1, 2, \dots, j\}$, we have

$$\mu_j = \sum_{k=1}^{j-1} U_{kj} + \lambda_j - \sum_{k=j+1}^{n} U_{jk}$$
$$= \sum_{k=1}^{j} U_{kj} \geq 2.$$

On the other hand, it follows from Definition 2.6 (2.1) and Lemma 3.1 that $\mu_k \in \{0, 1\}$. This is a contradiction, and hence we have $i = i'$. □

Definition 3.2. Let $\nu \in I$. The maps $\mathrm{wt}\colon \mathbb{H}(\Lambda_\nu) \to P$, $e_i, f_i\colon \mathbb{H}(\Lambda_\nu) \to \mathbb{H}(\Lambda_\nu) \cup \{0\}$ and $\varepsilon_i, \varphi_i\colon \mathbb{H}(\Lambda_\nu) \to \mathbb{Z}_{\geq 0}$ $(i \in I)$ are defined in the following manner. Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$.

(1) $\mathrm{wt}(H) \coloneqq \sum_{k=1}^{n-1} (\mu_k - \mu_{k+1}) \Lambda_k \in P$,
(2) $\varepsilon_i(H) \coloneqq \max(\mu_{i+1} - \mu_i, 0)$,
(3) $\varphi_i(H) \coloneqq \max(\mu_i - \mu_{i+1}, 0)$,
(4) Set $\mu' = \sum_{k=1}^{n} \mu'_k \epsilon_k \in P$, where $\mu'_i = \mu_i + 1$, $\mu'_{i+1} = \mu_{i+1} - 1$, and $\mu'_k = \mu_k$ for $k \neq i, i+1$. Set $U'_{k_0,i} = U_{k_0,i} + 1$, $U'_{k_0,i+1} = U_{k_0,i+1} - 1$ if there exists $k_0 \in \{1, 2, \dots, i+1\}$ such that $U_{k_0,i+1} > 0$. Set $U'_{kl} = U_{kl}$ if $k \neq k_0$ and $l \neq i, i+1$. Then, for $i \in I$, $e_i\colon \mathbb{H}(\Lambda_\nu) \to \mathbb{H}(\Lambda_\nu) \cup \{0\}$ is defined as follows:
$$e_i H = \begin{cases} (\Lambda_\nu, \mu', 0, (U'_{kl})_{k<l}) & \varepsilon_i(H) > 0, \\ 0 & \varepsilon_i(H) = 0, \end{cases}$$
(5) Set $\mu' = \sum_{k=1}^{n} \mu'_k \epsilon_k \in P$, where $\mu'_i = \mu_i - 1$, $\mu'_{i+1} = \mu_{i+1} + 1$, and $\mu'_k = \mu_k$ for $k \neq i, i+1$. Set $U'_{k_0,i} = U_{k_0,i} - 1$, $U'_{k_0,i+1} = U_{k_0,i+1} + 1$ if there exists $k_0 \in \{1, 2, \dots, i\}$ such that $U_{k_0,i} > 0$. Set $U'_{kl} = U_{kl}$ if $k \neq k_0$ and $l \neq i, i+1$. $f_i\colon \mathbb{H}(\Lambda_\nu) \to \mathbb{H}(\Lambda_\nu) \cup \{0\} (i \in I)$ is defined as follows:
$$f_i H = \begin{cases} (\Lambda_\nu, \mu', 0, (U'_{kl})_{k<l}) & \varphi_i(H) > 0, \\ 0 & \varphi_i(H) = 0. \end{cases}$$

Proposition 3.3. Let $H \in \mathbb{H}(\Lambda_\nu)$. Suppose $f_i H$, $e_i H \neq 0$. Then we have $f_i H, e_i H \in \mathbb{H}(\Lambda_\nu)$.

Proof. Let $\nu \in I$. Set $\lambda = \Lambda_\nu$. Let $H = (\lambda, \mu, 0, (U_{kl})_{k<l}) \in \mathbb{H}(\lambda)$. We show that $f_i H, e_i H \in \mathbb{H}(\lambda)$ if $f_i H, e_i H \neq 0$. Assume that $f_i H \neq 0$ and $f_i H = (\lambda, \mu', 0, (U'_{kl})_{k<l})$.

First, we show that f_iH is an integer hive. By Definition 3.2, we have $\mu'_i = \mu_i - 1$, $\mu'_{i+1} = \mu_{i+1}+1$ and $\mu'_k = \mu_k$ for $k \neq i, i+1$. Since $f_iH \neq 0$, there exists a $k_0 \in \{1, 2, \ldots, i\}$ such that $U_{k_0,i} > 0$. Then we have $U'_{k_0,i} = U_{k_0,i} - 1$, $U'_{k_0,i+1} = U_{k_0,i+1} + 1$ and $U'_{kl} = U_{kl}$ for $k \neq k_0$ and $l \neq i, i+1$. Thus, by Lemma 3.1, we have

$$\begin{aligned}
\sum_{k=1}^{j-1} U'_{kj} + \left(\lambda_j - \sum_{k=j+1}^{n} U'_{jk}\right) &= \sum_{k=1}^{j} U'_{kj} \\
&= \begin{cases} \sum_{k\neq k_0} U_{ki} + (U_{k_0,i} - 1) & j = i, \\ \sum_{k\neq k_0} U_{k,i+1} + (U_{k_0,i+1} + 1) & j = i+1, \\ \sum_{k=1}^{j} U_{kj} & \text{else} \end{cases} \\
&= \begin{cases} \sum_{k=1}^{j} U_{ki} - 1 & j = i, \\ \sum_{k=1}^{j} U_{k,i+1} + 1 & j = i+1, \\ \sum_{k=1}^{j} U_{kj} & \text{else} \end{cases} \\
&= \mu'_j.
\end{aligned}$$

Thus Definition 2.6 (2.1) holds. Then we have that f_iH is an integer hive.

Next, we show that f_iH is a K-hive. It then suffices to show that f_iH satisfies the conditions from (1) to (6) in Definition 2.8. By Definition 3.2, (1), (2), (3) and (4) hold.

Set $L'_{kl} = \sum_{m=1}^{l-1} U'_{km} - \sum_{m=1}^{l} U'_{k+1,m}$. If $k \neq k_0$ and $l \neq i+1$, then $L'_{kl} \geq 0$ is obvious. By Lemma 3.1, k_0 is the unique element in $\{1, 2, \ldots, i\}$ such that $U_{k_0,i} > 0$, then the following holds.

$$\begin{aligned}
L'_{k_0,i} &= \sum_{m=1}^{i-1} U'_{k_0,m} - \sum_{m=1}^{i} U'_{k_0+1,m} \\
&= \sum_{m=1}^{i-1} U_{k_0,m} - \sum_{m=1}^{i} U_{k_0+1,m} \\
&= -\sum_{m=1}^{i} U_{k_0+1,m} \geq 0.
\end{aligned}$$

Then $U_{k_0+1,m} = 0$ holds for $m = 1, 2, \ldots, i$. Also, we have $U_{k_0+1,i+1} = 0$ since $\mu_{i+1} = 0$. Thus, the following holds.

$$\begin{aligned}
L'_{k_0,i+1} &= \sum_{m=1}^{i} U'_{k_0,m} - \sum_{m=1}^{i+1} U'_{k_0+1,m} \\
&= (U_{k_0,i} - 1) - \sum_{m=1}^{i+1} U_{k_0+1,m} \\
&= U_{k_0,i} - 1 \geq 0.
\end{aligned}$$

Therefore we have (5).

If $k \neq i, i+1$, then $\mu'_k \geq \sum_{l=1}^{k-1} U'_{lk}$ is obvious since $H \in \mathbb{H}(\Lambda_\nu)$. For $k = i+1$, we have the following.

$$\begin{aligned}\mu'_{i+1} - \sum_{l=1}^{i} U'_{l,i+1} &= (\mu_{i+1}+1) - \left(\sum_{l \neq k_0} U_{l,i+1} + (U_{k_0,i+1}+1)\right) \\ &= \mu_{i+1} - \sum_{l=1}^{i} U_{l,i+1} \geq 0.\end{aligned}$$

For $k = i$, there are two cases: $k_0 < i$ and $k_0 = i$. If $k_0 < i$, then we have

$$\begin{aligned}\mu'_i - \sum_{l=1}^{i-1} U'_{li} &= (\mu_i - 1) - (\sum_{l=1}^{i-1} U_{li} - 1) \\ &= \mu_i - \sum_{l=1}^{i-1} U_{li} \geq 0.\end{aligned}$$

If $k_0 = i$, then we have

$$\begin{aligned}\mu'_i - \sum_{l=1}^{i-1} U'_{li} &= (\mu_i - 1) - \sum_{l=1}^{i-1} U_{li} \\ &= \mu_i - 1 \geq 0.\end{aligned}$$

Note that we have $\mu_i > 0$ since $f_i H \neq 0$. Thus, (6) holds. Therefore, $f_i H \in \mathbb{H}(\lambda)$ holds. $e_i H \in \mathbb{H}(\lambda)$ is proved in a similar manner. □

Remark 3.4. It follows from Definition 2.6 (2.1) that $\mu_i \in \{0, 1\}$ for all $i \in [n]$ since Λ_ν corresponds to (1^k). Thus, we have $\varphi_i(H), \varepsilon_i(H) \in \{0, 1\}$. Moreover, the following holds.

$$\varphi_i(H) = \begin{cases} 1 & f_i H \neq 0, \\ 0 & f_i H = 0. \end{cases}$$

$$\varepsilon_i(H) = \begin{cases} 1 & e_i H \neq 0, \\ 0 & e_i H = 0. \end{cases}$$

Proposition 3.5. Let $\nu \in I$. Then $\mathbb{H}(\Lambda_\nu)$ is a $U_q(\mathfrak{sl}_n)$-crystal together with the maps wt, e_i, f_i, φ_i, ε_i in Definition 3.2.

Proof. It suffices to show that the maps satisfy the conditions from (1) to (7) in Definition 2.1. By Definition 3.2, the condition (7) is obvious. Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$.

(1) By Definition 3.2, we have

$$\begin{aligned}\mathrm{wt}(H)(h_i) &= \mu_i - \mu_{i+1} \\ &= \varphi_i(H) - \varepsilon_i(H).\end{aligned}$$

(3) Suppose $f_i H \in \mathbb{H}(\Lambda_\nu)$. By the definition of f_i, we have

$$\begin{aligned}\mathrm{wt}(f_i H) &= \sum_{k \neq i, i+1} \mu_k \epsilon_k + (\mu_i - 1)\epsilon_i + (\mu_{i+1}+1)\epsilon_{i+1} \\ &= \mu - (\epsilon_i - \epsilon_{i+1}) \\ &= \mathrm{wt}(H) - \alpha_i.\end{aligned}$$

We can then prove (2) in a similar manner to (3).

(5) Suppose $f_iH \in \mathbb{H}(\Lambda_\nu)$. In this case, $\varphi_i(H) = 1$ and $\varepsilon_i(H) = 0$ hold. Then we have $\varphi_i(f_iH) = \varphi_i(H) - 1$ since

$$\varphi_i(f_iH) = max\{(\mu_i - 1) - (\mu_{i+1} + 1), 0\} = 0.$$

We also have $\varepsilon_i(f_iH) = \varepsilon_i(H) + 1$ since

$$\varepsilon_i(f_iH) = max\{(\mu_{i+1} + 1) - (\mu_i - 1), 0\} = 1.$$

We can then prove (4) in a similar manner to (5).

(6) Suppose $f_iH \in \mathbb{H}(\Lambda_\nu)$. Assume $f_iH = (\Lambda_\nu, \mu^{(1)}, 0, (U^{(1)}_{ij})_{i<j})$. Since $f_iH \in \mathbb{H}(\Lambda_\nu)$, we have $\mu_i - \mu_{i+1} > 0$, and hence $\mu^{(1)}_{i+1} - \mu^{(1)}_i > 0$ holds by the definition of f_i. Then we have $e_i(f_iH) \in \mathbb{H}(\Lambda_\nu)$. Assume $e_i(f_iH) = (\Lambda_\nu, \mu^{(2)}, 0, (U^{(2)}_{ij})_{i<j})$. By Definition 3.2, we have

$$\begin{aligned}\mu^{(2)} &= \sum_{k\neq i,i+1} \mu^{(1)}_k \epsilon_k + (\mu^{(1)}_i + 1) + (\mu^{(1)}_{i+1} - 1) \\ &= \mu.\end{aligned}$$

Since $f_iH \in \mathbb{H}(\Lambda_\nu)$, there exists a unique $k_0 \in \{1, 2, \ldots, i\}$ such that $U_{k_0,i}, U^{(1)}_{k_0,i+1} > 0$. Then the following holds.

$$\begin{aligned}U^{(2)}_{kl} &= \begin{cases} U^{(1)}_{kl} + 1 & k = k_0, l = i, \\ U^{(1)}_{kl} - 1 & k = k_0, l = i+1, \\ U^{(1)}_{kl} & else \end{cases} \\ &= \begin{cases} (U_{kl} - 1) + 1 & k = k_0, l = i, \\ (U_{kl} + 1) - 1 & k = k_0, l = i+1, \\ U_{kl} & else \end{cases} \\ &= U_{kl}.\end{aligned}$$

Then we have $e_i(f_iH) = H$.

Suppose $e_iH \in \mathbb{H}(\Lambda_\nu)$. Assume $e_iH = (\Lambda_\nu, \mu^{(1)}, 0, (U^{(1)}_{ij})_{i<j})$. Since $e_iH \in \mathbb{H}(\Lambda_\nu)$, we have $\mu_{i+1} - \mu_i > 0$, and hence $\mu^{(1)}_i - \mu^{(1)}_{i+1} > 0$ holds. Thus, we have $f_i(e_iH) \in \mathbb{H}(\Lambda_\nu)$. Assume $f_i(e_iH) = (\Lambda_\nu, \mu^{(2)}, 0, (U^{(2)}_{ij})_{i<j})$. By Definition 3.2, we have

$$\begin{aligned}\mu^{(2)} &= \sum_{k\neq i,i+1} \mu^{(1)}_k \epsilon_k + (\mu^{(1)}_i - 1) + (\mu^{(1)}_{i+1} + 1) \\ &= \mu.\end{aligned}$$

Since $e_iH \in \mathbb{H}(\Lambda_\nu)$, there exists a $k_0 \in I\{1, 2, \ldots, i+1\}$ such that $U_{k_0,i+1}, U^{(1)}_{k_0,i} > 0$. Then the following holds.

$$
\begin{aligned}
U^{(2)}_{kl} &= \begin{cases} U^{(1)}_{kl} - 1 & k = k_0, l = i, \\ U^{(1)}_{kl} + 1 & k = k_0, l = i+1, \\ U^{(1)}_{kl} & else, \end{cases} \\
&= \begin{cases} (U_{kl} + 1) - 1 & k = k_0, l = i, \\ (U_{kl} - 1) + 1 & k = k_0, l = i+1, \\ U_{kl} & else, \end{cases} \\
&= U_{kl}.
\end{aligned}
$$

Then we have $f_i(e_iH) = H$. □

Example 3.6. The action of f_i on the $U_q(\mathfrak{sl}_4)$-crystal $\mathbb{H}(\Lambda_3)$ is computed as follows. Let $H = (\Lambda_3, \Lambda_3, (U_{kl})_{k<l}) \in \mathbb{H}(\Lambda_3)$, where $U_{kl} = 0$ for $1 \le k < l \le 4$.

It follows that $f_1H = f_2H = 0$ from $\varphi_1(H) = \varphi_2(H) = 0$ since $\mu_1 - \mu_2 = \mu_2 - \mu_3 = 0$. Since $\varphi_3(H) = \mu_3 - \mu_4 = 1$ and $U_{33} = 1$, f_3H is as shown in FIGURE 3.

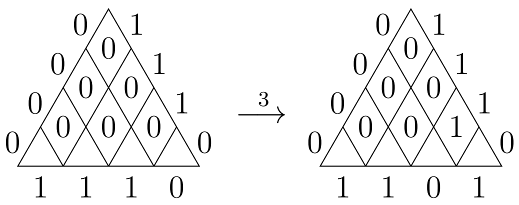


FIGURE 3. Action of f_3 on the $U_q(\mathfrak{sl}_4)$-crystal $\mathbb{H}(\Lambda_3)$

In the following, we investigate the crystal structure on $\mathbb{H}(\Lambda_\nu)$ defined by Definition 3.2. We will show that $\mathbb{H}(\Lambda_\nu)$ is isomorphic to $B(\Lambda_\nu)$ with these results, see Proposition 3.34.

Lemma 3.7. Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$. Suppose that there exists $i_0, j_0, i_1, j_1 \in [n]$ such that $U_{i_0,j_0}, U_{i_1,j_1} > 0$. Then $i_1 > i_0$ if and only if $j_1 > j_0$.

Proof. Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$. Suppose that there exists $i_0, j_0, i_1, j_1 \in [n]$ such that $U_{i_0,j_0}, U_{i_1,j_1} > 0$.

Assume $i_1 > i_0$ and $i_1 = i_0 + l$ for some $l \in \mathbb{Z}$. By Lemma 3.1 and $H \in \mathbb{H}(\Lambda_\nu)$,

$$
\begin{aligned}
\sum_{k=0}^{l-1} L_{i_0+k, j_0+k} &= \sum_{k=1}^{j_0-1} U_{i_0,k} - \sum_{k=1}^{j_0+l-1} U_{i_0+l,k} \\
&= -\sum_{k=1}^{j_0+l-1} U_{i_0+l,k} \ge 0.
\end{aligned}
$$

Then, we have $U_{i_1k} = 0$ for $k = 1, 2, \ldots, j_0 + l - 1$, especially $U_{i_1k} = 0$ if $k \le j_0$. Thus, $j_1 > j_0$ holds.

Assume $j_1 > j_0$. Suppose that $i_0 \geq i_1$ and $i_0 = i_1 + l$ for some $l \in \mathbb{Z}$. By Lemma 3.1 and $H \in \mathbb{H}(\Lambda_\nu)$,

$$\sum_{k=0}^{l-1} L_{i_1+k,j_1+k} = \sum_{k=1}^{j_1-1} U_{i_1k} - \sum_{k=1}^{j_1+l-1} U_{i_1+l,k} = -\sum_{k=1}^{j_1+l-1} U_{i_1+l,k} \geq 0$$

Then, we have $U_{i_0k} = 0$ for $k = 1, 2, \ldots, j_1 + l - 1$, especially $U_{i_0k} = 0$ if $k < j_1$, however, this is a contradiction for $j_1 > j_0$. Thus, $i_1 > i_0$ holds. □

Proposition 3.8. Let $H, H' \in \mathbb{H}(\Lambda_\nu)$. If $\mathrm{wt}(H) = \mathrm{wt}(H')$, then $H = H'$ holds.

Proof. Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$. Set $\lambda = \Lambda_\nu$. For $s = 1, 2, \ldots, \nu$, there exists a unique $j_s \in [n]$ such that $U_{sj_s} = 1$ by Lemma 3.1. By Lemma 3.1 and (2.2), $\mu_k = 1$ if $k = j_s$ for some $s = 1, 2, \ldots, \nu$, otherwise $\mu_k = 0$. By Lemma 3.7, we have $j_1 < j_2 < \cdots < j_\nu$. Thus, (s, j_s) is uniquely determined by λ and μ. Therefore, if $\mathrm{wt}(H) = \mathrm{wt}(H')$, then $H = H'$ holds for $H, H' \in \mathbb{H}(\Lambda_\nu)$. □

By the proof of Proposition 3.8, we have the following.

Corollary 3.9. Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$. For $s = 1, 2, \ldots, \nu$, let $j_s \in [n]$ such that $\mu_{j_s} = 1$. Assume $j_1 < j_2 < \cdots < j_\nu$. Then,

$$U_{ij} = \begin{cases} 1 & \text{if } (i, j) = (s, j_s), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.10. For $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$, set $\Omega(H) = (\Lambda_\nu, \xi, 0, (V_{ij})_{i<j})$, where $\xi_i = \mu_{n+1-i}$ ($i \in [n]$) and $V_{ij} = U_{\nu+1-i,n+1-j}$ ($1 \leq i < j \leq n$). Then, $\Omega(H) \in \mathbb{H}(\Lambda_\nu)$.

Proof. Set $\lambda = \Lambda_\nu$. For $s = 1, 2, \ldots, \nu$, we can take $j_s \in [n]$ such that $\mu_{j_s} = 1$ since $H \in \mathbb{H}(\Lambda_\nu)$. We may assume $j_1 < j_2 < \cdots < j_\nu$ by retaking j_s if necessary. By Corollary 3.9,

$$U_{ij} = \begin{cases} 1 & \text{if } (i, j) = (s, j_s) \text{ for some } s \in \{1, 2, \ldots, \nu\}, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of Ω, $\xi_k = 1$ if $k = n + 1 - j_s$, otherwise $\xi_k = 0$. Also, we have

$$V_{ij} = U_{\nu+1-i,n+1-j} = \begin{cases} 1 & \text{if } (i, j) = (\nu + 1 - s, n + 1 - j_s), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\xi \in P$ and $\sum_{i=1}^{n} \xi_i = \nu$, we can take $H' \in \mathbb{H}(\Lambda_\nu)$ such that $\mathrm{wt}(H') = \xi$. By Corollary 3.9, $\Omega(H) = H'$ holds, and hence $\Omega(H) \in \mathbb{H}(\Lambda_\nu)$ holds. □

Definition 3.11. The map $\Omega\colon \mathbb{H}(\Lambda_\nu) \cup \{0\} \to \mathbb{H}(\Lambda_\nu) \cup \{0\}$ is defined by H maps to $\Omega(H)$ for $H \in \mathbb{H}(\Lambda_\nu)$ and $\Omega(0) = 0$.

Proposition 3.12. The map $\Omega\colon \mathbb{H}(\Lambda_\nu) \cup \{0\} \to \mathbb{H}(\Lambda_\nu) \cup \{0\}$ is an involution.

Proof. Let $H \in \mathbb{H}(\Lambda_\nu)$. By Definition 3.11, we have $\Omega(\Omega(H)) = H$. Also, we have $\Omega(0) = 0$. Then, Ω is a surjection. Let $H, K \in \mathbb{H}(\Lambda_\nu) \cup \{0\}$. Assume $\Omega(H) = \Omega(K)$. By Definition 3.11, we have $H = \Omega(\Omega(H)) = \Omega(\Omega(K)) = K$. Then Ω is an injection. Thus, Ω is a bijection, especially Ω is an involution. □

Proposition 3.13. $\Omega\colon \mathbb{H}(\Lambda_\nu) \to \mathbb{H}(\Lambda_\nu)$ has the following properties. For $H \in \mathbb{H}(\Lambda_\nu)$ and $i \in I$,

(1) $\mathrm{wt}(\Omega(H)) = w_0 \mathrm{wt}(H)$,
(2) $\varphi_i(\Omega(H)) = \varepsilon_{n-i}(H)$,
(3) $\varepsilon_i(\Omega(H)) = \varphi_{n-i}(H)$,
(4) $f_i(\Omega(H)) = \Omega(e_{n-i}(H))$,
(5) $e_i(\Omega(H)) = \Omega(f_{n-i}(H))$,

where w_0 denotes the longest element in the Weyl group of type A_{n-1}.

Proof. Let $H = (\Lambda_\nu, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_\nu)$. Let w_0 be the longest element in the Weyl group of type A_{n-1}. By Definition 3.11, we have

$$\begin{aligned}\mathrm{wt}(\Omega(H)) &= \sum_{k=1}^{n} \mu_{n+1-k}\epsilon_k = \sum_{k=1}^{n} \mu_k \epsilon_{n+1-k} \\ &= \sum_{k=1}^{n} \mu_k w_0(\epsilon_k) = w_0 \mathrm{wt}(H),\end{aligned}$$

hence (1) holds.

By Definition 3.11, we have

$$\begin{aligned}\varphi_i(\Omega(H)) &= \max\{\mu_{n+1-i} - \mu_{n-i}, 0\} \\ &= \varepsilon_{n-1}(H).\end{aligned}$$

Then (2) holds. Also, we have

$$\begin{aligned}\varepsilon_i(\Omega(H)) &= \max\{\mu_{n-i} - \mu_{n+1-i}, 0\} \\ &= \varphi_{n-1}(H).\end{aligned}$$

Then (3) holds.

From (2), (4) is obvious if $f_i\Omega(H) = 0$. Suppose $f_i\Omega(H) \neq 0$. Set $\xi = \mathrm{wt}(f_i\Omega(H))$ and $o = \mathrm{wt}(\Omega(e_{n-i}(H)))$. By Definitions 3.2 and 3.11, for $k = 1, 2, \ldots, n$,

$$\begin{aligned}\xi_k &= \begin{cases} \mu_{n+1-k} - 1 & \text{if } k = i, \\ \mu_{n+1-k} + 1 & \text{if } k = i+1, \\ \mu_{n+1-k} & \text{otherwise} \end{cases} \\ &= o_k.\end{aligned}$$

By Proposition 3.8, (4) holds.

From (3), (5) is obvious if $e_i\Omega(H) = 0$. Suppose $e_i\Omega(H) \neq 0$. Set $\xi = \mathrm{wt}(e_i\Omega(H))$ and $o = \mathrm{wt}(\Omega(f_{n-i}(H)))$. By Definitions 3.2 and 3.11, for $k = 1, 2, \ldots, n$,

$$\begin{aligned}\xi_k &= \begin{cases} \mu_{n+1-k} + 1 & \text{if } k = i, \\ \mu_{n+1-k} - 1 & \text{if } k = i+1, \\ \mu_{n+1-k} & \text{otherwise} \end{cases} \\ &= o_k.\end{aligned}$$

By Proposition 3.8, (5) holds. □

3.2. **Crystal Structure on $\mathbb{H}(\lambda)$.** In this subsection, we determine a crystal structure on $\mathbb{H}(\lambda)$ for $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. This structure is induced by an embedding of $\mathbb{H}(\lambda)$ into a tensor product of crystals of the form $\mathbb{H}(\Lambda_\nu)$ with $\nu \in I$. We will then prove that $\mathbb{H}(\lambda) \cong B(\lambda)$.

We start with constructing an embedding such that $\mathbb{H}(\lambda)$ is split into a tensor product of sets of K-hives.

Definition 3.14. Let $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. Set $N = \sum_{i\in I} m_i$. Let $l_N = \max\{i \in I \mid m_i \neq 0\}$. For $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, $H_N = (\Lambda_{l_N}, \mu^{(N)}, 0, (U_{ij}^{(N)})_{i<j})$ is defined by

$$U_{ij}^{(N)} = \begin{cases} 1 & \text{if } j = \min\{j \in [n] \mid U_{ij} > 0\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_k^{(N)} = \begin{cases} 1 & \text{if there exists } j \in [n] \text{ such that } U_{kj}^{(N)} > 0, \\ 0 & \text{otherwise} \end{cases}$$

For H and H_N, $H^{(N-1)} = (\lambda^{(N-1)}, \xi^{(N-1)}, 0, (V_{ij}^{(N-1)})_{i<j})$ is defined by $\lambda^{(N-1)} = \lambda - \Lambda_{l_N}$, $\xi^{(N-1)} = \mu - \mu^{(N)}$, and $V_{ij}^{(N-1)} = U_{ij} - U_{ij}^{(N)}$ $(1 \leq i < j \leq n)$.

Lemma 3.15. Let $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. Set $N = \sum_{i\in I} m_i$. Let $H \in \mathbb{H}(\lambda)$. Let H_N and $H^{(N-1)}$ in Definition 3.14. Then, $H_N \in \mathbb{H}(\Lambda_{l_N})$ and $H^{(N-1)} \in \mathbb{H}(\lambda^{(N-1)})$ hold.

Proof. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let $H_N = (\Lambda_{l_N}, \mu^{(N)}, 0, (U_{ij}^{(N)})_{i<j})$ and $H^{(N-1)} = (\lambda^{(N-1)}, \xi^{(N-1)}, 0, (V_{ij}^{(N-1)})_{i<j})$ in Definition 3.14.

For $s = 1, 2, \ldots, l_N$, we can take $j_s \in [n]$ such that $\mu_{j_s}^{(N)} = 1$. We may assume $j_1 < j_2 < \cdots < j_{l_N}$ by retaking j_s if necessary. By Definition 3.14,

$$U_{ij}^{(N)} = \begin{cases} 1 & \text{if } (i,j) = (s, j_s) \text{ for some } s \in \{1, 2, \ldots, l_N\}, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 3.8 and Corollary 3.9, we have $H_N \in \mathbb{H}(\Lambda_{l_N})$.

By Definition 3.14 and (2.2), then we have

$$\begin{aligned} \xi_i^{(N-1)} &= \mu_i - \mu_i^{(N)} \\ &= \sum_{k=1}^{i} U_{ki} - \sum_{k=1}^{i} U_{ki}^{(N)} \\ &= \sum_{k=1}^{i-1} V_{ki}^{(N-1)} + \left(\lambda_i^{(N-1)} - \sum_{k=i+1}^{n} V_{ik}^{(N-1)}\right) \end{aligned}$$

for $i = 1, 2, \ldots, n$. Then $H^{(N-1)}$ is an integer hive.

By Definition 3.14, Definition 2.8 (1), (2), (3), and (4) holds for $H^{(N-1)}$. Set $L_{ij}^{(N)} = \sum_{k=1}^{j-1} U_{ik}^{(N)} - \sum_{k=1}^{j} U_{i+1,k}^{(N)}$ and $L_{ij}^{(N-1)} = \sum_{k=1}^{j-1} V_{ik}^{(N-1)} - \sum_{k=1}^{j} V_{i+1,k}^{(N-1)}$ for $1 \leq i < j \leq n$.

Then we know that

$$\begin{aligned} L_{ij} &= \sum_{k=1}^{j-1} U_{ik} - \sum_{k=1}^{j} U_{i+1,k} \\ &= \sum_{k=1}^{j-1} (U_{ik}^{(N)} + V_{ik}^{(N-1)}) - \sum_{k=1}^{j} (U_{i+1,k}^{(N)} + V_{i+1,k}^{(N-1)}) \\ &= L_{ij}^{(N)} + L_{ij}^{(N-1)} \geq 0. \end{aligned}$$

By Definition 3.14, $L_{ij}^{(N-1)} = L_{ij} - L_{ij}^{(N)} \geq 0$.

By Definition 3.14, we have

$$\begin{aligned} \mu_j - \sum_{k=1}^{j-1} U_{kj} &= (\mu_j^{(N)} + \xi_j^{(N-1)}) - \sum_{k=1}^{j-1} (U_{kj}^{(N)} + V_{kj}^{(N-1)}) \\ &= \mu_j^{(N)} - \sum_{k=1}^{j-1} U_{kj}^{(N)} + \xi_j^{(N-1)} - \sum_{k=1}^{j-1} V_{kj}^{(N-1)}. \end{aligned}$$

By Definition 3.14, $\mu_j \geq \mu_j^{(N)}$ for $j = 1, 2, \ldots, n$ and $U_{ij} \geq U_{ij}^{(N)}$ for $1 \leq i < j \leq n$ hold, and hence $\xi_j^{(N-1)} - \sum_{k=1}^{j-1} V_{kj}^{(N-1)} \geq 0$. Therefore, $H^{(N-1)} \in \mathbb{H}(\lambda^{(N-1)})$. □

Definition 3.16. Let $\lambda = \sum_{i \in I} m_i \Lambda_i \in P^+$. Set $N = \sum_{i \in I} m_i$. For each $H \in \mathbb{H}(\lambda)$, take $H_N \in \mathbb{H}(\Lambda_{l_N})$ and $H^{(N-1)} \in \mathbb{H}(\lambda^{(N-1)})$ as in Definition 3.14. Then define the map $\Psi_\lambda \colon \mathbb{H}(\lambda) \to \mathbb{H}(\lambda^{(N-1)}) \times \mathbb{H}(\Lambda_{l_N})$ by $\Psi_\lambda(H) = H^{(N-1)} \times H_N$.

Lemma 3.17. The map Ψ_λ is an injection.

Proof. Let $H, K \in \mathbb{H}(\lambda)$. Let $\Psi_\lambda(H) = H^{(N-1)} \times H_N$ and $\Psi_\lambda(K) = K^{(N-1)} \times K_N$ where $H^{(N-1)}, K^{(N-1)} \in \mathbb{H}(\lambda^{(N-1)})$ and $H_N, K_N \in \mathbb{H}(\Lambda_{l_N})$. Suppose that $H^{(N-1)} \times H_N = K^{(N-1)} \times K_N$. Then we have $H^{(N-1)} = K^{(N-1)}$ and $H_N = K_N$. By the construction of Ψ_λ, we have $H = K$. □

Example 3.18. Let $n = 4$, $\lambda = (3, 2, 1, 0)$ and $\mu = (2, 3, 1, 0)$. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathcal{H}^{(4)}(\lambda, \mu, 0)$ be the diagram on the left in FIGURE 4. Then $\Psi_\lambda(H) = H^{(2)} \otimes H_3$ is as shown in FIGURE 4. Note that $U_{11} = 2 \geq 0$.

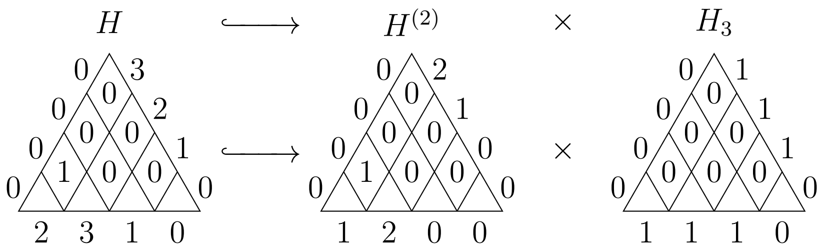


FIGURE 4. Action of Ψ_λ on $\mathbb{H}(\lambda)$

By applying Lemma 3.17 repeatedly, we obtain the following.

Proposition 3.19. Let $\lambda = \sum_{i \in I} m_i \Lambda_i \in P^+$. Then there exists an injection

$$\Psi \colon \mathbb{H}(\lambda) \to \bigotimes_{i \in I} \mathbb{H}(\Lambda_i)^{\otimes m_i}.$$

Proof. Let $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. Set $N = \sum_{i\in I} m_i$. By Lemma 3.17, there exists an injection $\Psi_\lambda \colon \mathbb{H}(\lambda) \to \mathbb{H}(\lambda^{(N-1)}) \times \mathbb{H}(\Lambda_{l_N})$. Then we apply Lemma 3.17 for $\lambda^{(N-1)}$, and by repeating this argument, we get a map $\Psi \colon \mathbb{H}(\lambda) \to \Pi_{i\in I}\mathbb{H}(\Lambda_i)^{m_i}$. Since each $\mathbb{H}(\Lambda_i)$ is a $U_q(\mathfrak{sl}_n)$-crystal, $\Pi_{i\in I}\mathbb{H}(\Lambda_i)^{m_i}$ has $U_q(\mathfrak{sl}_n)$-crystal structure by Definition 2.3. Thus we can write $\Pi_{i\in I}\mathbb{H}(\Lambda_i)^{m_i}$ as $\bigotimes_{i\in I} \mathbb{H}(\Lambda_i)^{\otimes m_i}$. Since Ψ_λ is an injection, Ψ is an injection. $\square$

Example 3.20. Let $n = 4$, $\lambda = (3, 2, 1, 0)$ and $\mu = (2, 3, 1, 0)$. Let $H \in \mathcal{H}^{(4)}(\lambda, \mu, 0)$ be the diagram on the left of FIGURE 5. Then $\Psi(H) = H_1 \otimes H_2 \otimes H_3$ is as shown in FIGURE 5.

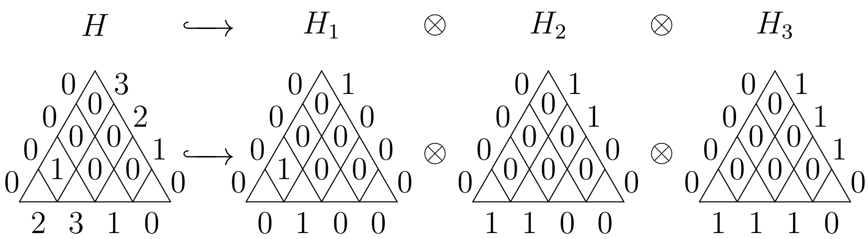


FIGURE 5. Action of Ψ on $\mathbb{H}(\lambda)$

By the construction of Ψ, we have the following.

Lemma 3.21. Let $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. Set $N = \sum_{i\in I} m_i$. Let $H \in \mathbb{H}(\lambda)$. Let $\Psi(H) = H_1 \otimes \cdots \otimes H_N$, where $H_k = (\Lambda_{l_k}, \mu^{(k)}, 0, (U_{ij}^{(k)})_{i<j})$ $(k = 1, \ldots, N)$. For $k \in \{1, \ldots, N\}$ and $i \in [n]$, if there exists $j \in [n]$ such that $U_{i,j}^{(k)} > 0$, then set $j_{i,k}$ to its j, otherwise set $j_{i,k}$ to 0. Suppose that $j_{i,k} > 0$ for some $k \in \{1, \ldots, N\}$ and $i \in [n]$. Then we have $j_{i,k'} \geq j_{i,k}$ if $k \geq k'$.

Proof. Set $H^{(N)} = H$ and $\lambda^{(N)} = \lambda$. By Definition 3.14, for $m = 1, 2, \ldots, N$ there exists $H_m \in \mathbb{H}(\Lambda_{l_m})$ and $H^{(m-1)} \in \mathbb{H}(\lambda^{(m-1)})$ such that

$$\Psi_{\lambda^{(m)}}(H^{(m)}) = H^{(m-1)} \otimes H_m.$$

For $m = 1, 2, \ldots, N$, let $H^{(m)} = (\lambda^{(m)}, \xi^{(m)}, 0, (V_{ij}^{(m)})_{i<j})$. Fix $k \in \{1, 2, \ldots, N\}$. It follows from the definition of Ψ and Ψ_λ $(\lambda \in P^+)$ that

$$V_{ij}^{(k)} = U_{ij}^{(1)} + \cdots + U_{ij}^{(k)} \quad (1 \leq i < j \leq n).$$

Then, by the definition of $\Psi_{\lambda^{(k)}}$,

$$U_{ij}^{(k)} = \begin{cases} 1 & j = \min\{j \in [n] \mid V_{ij}^{(k)} > 0\}, \\ 0 & \text{else.} \end{cases}$$

This means that for $1 \leq k' \leq k \leq N$

$$\begin{aligned} j_{i,k} &= \min\{j \in [n] \mid U_{ij}^{(1)} + \cdots + U_{ij}^{(k)} > 0\} \\ &\leq \min\{j \in [n] \mid U_{ij}^{(1)} + \cdots + U_{ij}^{(k')} > 0\} \\ &= j_{i,k'}. \end{aligned}$$

$\square$

Remark 3.22. It follows from Lemma 3.21 that

$$j_{i,k} = \min\{j \in [n] \mid U_{ij}^{(l)} > 0, l = 1, \ldots, k\}$$
$$= \max\{j \in [n] \mid U_{ij}^{(l)} > 0, l = k, \ldots, N\}.$$

Proposition 3.23. Let $\lambda = \sum_{i\in I} m_i\Lambda_i = \sum_{i\in I} \lambda_i\epsilon_i \in P^+$. Set $N = \sum_{i\in I} m_i$. Then,

(3.3)

$$\Psi(\mathbb{H}(\lambda)) = \{H_1 \otimes \cdots \otimes H_N \in \bigotimes_{k=1}^{N} \mathbb{H}(\Lambda_{l_k}) \mid j_{i,\lambda_{N+1-i}} \geq j_{i,\lambda_{N+1-i}+1} \geq \cdots \geq j_{i,N} \text{ for all } i \in I\},$$

where $j_{i,k}\,(i \in I, k \in \{1, \ldots, N\})$ is defined in Lemma 3.21.

Proof. Let $\lambda = \sum_{i\in I} m_i\Lambda_i = \sum_{i\in I} \lambda_i\epsilon_i \in P^+$. Set $\mathcal{F}$ to the right set of (3.3).

First, we show $\Psi(\mathbb{H}(\lambda)) \subset \mathcal{F}$. Let $H = H_1 \otimes \cdots \otimes H_N \in \Psi(\mathbb{H}(\lambda))$, where $H_k \in \mathbb{H}(\Lambda_{l_k})$ for $k = 1, 2, \ldots, N$. We know $\lambda_i = m_i + m_{i+1} + \cdots + m_{n-1}$ for $i \in I$. Then by the construction of Ψ, $\Lambda_{l_{\lambda_{N+1-i}}} = \Lambda_{N+1-i}$. By Lemma 3.1, $j_{i,\lambda_{N+1-i}} > 0$ holds. By Lemma 3.21, $j_{i,\lambda_{N+1-i}} \geq j_{i,\lambda_{N+1-i}+1} \geq \cdots \geq j_{i,N}$ holds. Thus, $H \in F$ holds.

Next, we show $\mathcal{F} \subset \mathbb{H}(\lambda)$. Let $H = H_1 \otimes \cdots \otimes H_N \in \bigotimes_{k=1}^{N} \mathbb{H}(\Lambda_{l_k})$, where $H_k = (\Lambda_{l_k}, \mu^{(k)}, 0, (U_{ij}^{(k)})_{i<j})$ for $k = 1, 2, \ldots, N$. Let $\tilde{H} = (\tilde{\lambda}, \tilde{\mu}, 0, (\tilde{U}_{ij})_{i<j})$, where $\tilde{\lambda} = \sum_{k=1}^{N} \Lambda_{l_k}$, $\tilde{\mu} = \sum_{k=1}^{N} \mu^{(k)}$, and $\tilde{U}_{ij} = \sum_{k=1}^{N} U_{ij}^{(k)}\,(1 \leq i < j \leq n)$. Then we can check $\tilde{H} \in \mathbb{H}(\lambda)$ as follows. For $i \in I$,

$$\begin{aligned}
\tilde{\mu}_i &= \sum_{k=1}^{N} \mu_i^{(k)} \\
&= \sum_{k=1}^{N} \left(\sum_{l=1}^{i-1} U_{li}^{(k)} + \left((\Lambda_{l_k})_i - \sum_{l=i+1}^{n} U_{il}^{(k)} \right) \right) \\
&= \sum_{l=1}^{i-1} \tilde{U}_{li}^{(k)} + \left(\tilde{\lambda}_i^{(k)} - \sum_{l=i+1}^{n} \tilde{U}_{il}^{(k)} \right).
\end{aligned}$$

Then $\tilde{H}$ is an integer hive. $\tilde{\lambda} \in P^+$, $\tilde{\mu} \in P$, $\sum_{i\in I} \tilde{\lambda}_i = \sum_{i\in I} \tilde{\mu}_i$, and $\tilde{U}_{ij} \geq 0\,(1 \leq i < j \leq n)$ immediately hold from the definition of $\tilde{H}$ and $H_k \in \mathbb{H}(\Lambda_{l_k})$. For $1 \leq i < j \leq n$,

$$\begin{aligned}
\tilde{L}_{ij} &= \sum_{k=1}^{j-1} \tilde{U}_{ik} - \sum_{k=1}^{j} \tilde{U}_{i+1,k} \\
&= \sum_{k=1}^{j-1} \sum_{l=1}^{N} U_{ik}^{(l)} - \sum_{k=1}^{j} \sum_{l=1}^{N} U_{i+1,k}^{(l)} \\
&= \sum_{l=1}^{N} L_{ij}^{(l)} \geq 0.
\end{aligned}$$

Also, for $i \in I$,

$$\begin{aligned}
\tilde{\mu}_i - \sum_{k=1}^{i-1} \tilde{U}_{ki} &= \sum_{l=1}^{N} \mu_i^{(l)} - \sum_{k=1}^{i-1} \sum_{l=1}^{N} U_{ki}^{(l)} \\
&= \sum_{l=1}^{N} (\mu_i^{(l)} - \sum_{k=1}^{i-1} U_{ki}^{(l)}) \geq 0.
\end{aligned}$$

By the choice of H, $\tilde{\lambda} = \lambda$. Then $\tilde{H} \in \mathbb{H}(\lambda)$.

We may assume $\Psi(\tilde{H}) = \tilde{H}_1 \otimes \cdots \otimes \tilde{H}_N$, where $\tilde{H}_k = (\Lambda_{l_k}, \tilde{\mu}^{(k)}, 0, (\tilde{U}^{(k)}_{ij})_{i<j})$ for $k = 1, 2, \ldots, N$. We show $\tilde{H}_k = H_k$ for $k = 1, \ldots, N$ by induction on k. Set $\tilde{H}^{(N)} = \tilde{H}$ and $\lambda^{(N)} = \lambda$. By Definition 3.14, we know $\Psi_{\lambda^{(k)}}(\tilde{H}^{(k)}) = \tilde{H}^{(k-1)} \otimes \tilde{H}_k$, where $\tilde{H}^{(k)} = (\lambda^{(k)}, \tilde{\mu}^{(k)}, 0, (V^{(k)}_{ij})_{i<j})$ for $k = 1, 2, \ldots, N$. By Definition 3.14 and $H \in \mathcal{F}$,

$$\begin{aligned}\tilde{U}^{(N)}_{ij} &= \begin{cases} 1 & \text{if } j = \min\{j \in [n] \mid U^{(1)}_{ij} + \cdots + U^{(N)}_{ij} > 0\}, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } j = j_{i,N}, \\ 0 & \text{otherwise,} \end{cases} \\ &= U^{(N)}_{ij}.\end{aligned}$$

By Definition 3.14, $\tilde{\mu}^{(N)} = \mu^{(N)}$, namely $\tilde{H}_N = H_N$ holds. Assume that $\tilde{H}_s = H_s$ for $s = k+1, k+2, \ldots, N$. By Definition 3.14, $H \in \mathcal{F}$, and the induction hypothesis,

$$\begin{aligned}\tilde{U}^{(k)}_{ij} &= \begin{cases} 1 & \text{if } j = \min\{j \in [n] \mid U^{(1)}_{ij} + \cdots + U^{(k)}_{ij} > 0\}, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } j = j_{i,k}, \\ 0 & \text{otherwise,} \end{cases} \\ &= U^{(k)}_{ij}.\end{aligned}$$

By Definition 3.14, $\tilde{\mu}^{(k)} = \mu^{(k)}$, namely $\tilde{H}_k = H_k$ holds. Thus, $H \in \Psi(\mathbb{H}(\lambda))$. □

Remark 3.24. For $H \in \mathbb{H}(\lambda)\,(\lambda \in P^+)$, let $\Psi(H) = H_1 \otimes \cdots \otimes H_N$, where $H_k = (\Lambda_{l_k}, \mu^{(k)}, 0, (U^{(k)}_{ij})_{i<j}) \in \mathbb{H}(\Lambda_k)$ for $k = 1, 2, \ldots, N$. For $i \in [n]$ and $k \in \{1, 2, \ldots, N\}$, let $j_{i,k}$ be as in Lemma 3.21. Then, for each $k = 1, 2, \ldots, N$, we have

$$j_{1,k} < j_{2,k} < \cdots < j_{l_k,k} \tag{3.4}$$

from Lemma 3.7.

Proposition 3.25. Let $\lambda \in P$. $\Psi(\mathbb{H}(\lambda)) \cup \{0\}$ is stable under the action of e_i and f_i for $i \in I$.

Proof. We show that $f_i(\Psi(\mathbb{H}(\lambda)) \cup \{0\}) \subset \Psi(\mathbb{H}(\lambda)) \cup \{0\}$. Let $H = H_1 \otimes \cdots \otimes H_N \in \Psi(\mathbb{H}(\lambda))$, where $H_k = (\Lambda_{l_k}, \mu^{(k)}, 0, (U^{(k)}_{ij})_{i<j})$. Assume $f_iH = H_1 \otimes \cdots \otimes f_iH_{k_0} \otimes \cdots \otimes H_N$. If $f_iH = 0$, the statement is obvious.

Suppose $f_iH \neq 0$. Let $f_iH_{k_0} = (\Lambda_{l_{k_0}}, \tilde{\mu}^{(k_0)}, 0, (\tilde{U}^{(k_0)}_{ij})_{i<j})$. For $i \in I$, if there exists $j \in [n]$ such that $\tilde{U}^{(k_0)}_{ij} > 0$, then set $\tilde{j}_{i,k_0}$ to its j, otherwise set $\tilde{j}_{i,k_0}$ to 0. For H_{k_0} and i, let k_0 in Definition 3.2 (5) be written as k_{f_iH}. Then we know $j_{k_{f_iH},k_0} = i$. By Definition 3.2, we have $\tilde{j}_{k_{f_iH},k_0} = i+1$ and $\tilde{j}_{k,k_0} = j_{k,k_0}$ if $k \neq k_{f_iH}$. By Proposition 3.23, to show that $f_iH \in \Psi(H)$, it suffices to check that $j_{k_{f_iH},k_0-1} \geq \tilde{j}_{k_{f_iH},k_0} = i+1$. Note that we have $j_{k_{f_iH},k_0-1} \geq j_{k_{f_iH},k_0} = i$ since $H \in \Psi(\mathbb{H}(\lambda))$. It also follows that $\varphi_i(H_{k_0-1}) = 0$ since $\varphi_i(H_{k_0-1}) - \varepsilon_i(H_{k_0}) \leq 0$ holds from Proposition 2.4.

Suppose $j_{k_{f_iH},k_0-1} = i$. Then, $\mu^{(k_0-1)}_i = \mu^{(k_0-1)}_{i+1} = 1$ follows from Remark 3.4 and $\varphi_i(H_{k_0-1}) = 0$. By Lemma 3.7, $j_{k_{f_iH}+1,k_0-1} = i+1$ and $j_{k_{f_iH}+1,k_0} > i$ holds. Since $f_iH^{(k_0)} \neq 0$, we know $\mu^{(k_0)}_{i+1} = 0$ by Remark 3.4. Then, we have $j_{k_{f_iH}+1,k_0} > i+1$ from

(2.2). Now, we have $j_{k_{f_iH}+1,k_0-1} = i+1 < j_{k_{f_iH}+1,k_0}$, however this is a contradiction for $H \in \Psi(\mathbb{H}(\lambda))$. Thus, $j_{k_{f_iH},k_0-1} \geq i+1$ holds.

Similarly, $e_i(\Psi(\mathbb{H}(\lambda)) \cup \{0\}) \subset \Psi(\mathbb{H}(\lambda)) \cup \{0\}$ is can be shown. □

Definition 3.26. Let $\lambda = \sum_{i \in I} m_i \Lambda_i \in P^+$. The crystal structure on $\mathbb{H}(\lambda)$ is defined so that Ψ is a morphism of crystals.

Remark 3.27. In this subsection, we take $\lambda \in P^+$ as $\lambda = \sum_{i\in I} m_i\Lambda_i$. Assume that $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$, then we have $\lambda_n = 0$. For $l \in \mathbb{Z}_{\geq 0}$, let $\lambda' = \sum_{i=1}^n \lambda'_i \epsilon_i$, where $\lambda'_i = (\lambda_i + l)$ for $i = 1, 2, \dots, n$. Then $\lambda'_n > 0$ holds. The construction of the crystal structure on $\mathbb{H}(\lambda)$ can be applied for λ'. Note that $\lambda = \lambda'$ holds since $\epsilon_1 + \cdots + \epsilon_n = 0$.

Also, we know that partitions $\tilde{\lambda} = (\lambda_i)_i$ and $\tilde{\lambda}' = (\lambda'_i)_i$ represent λ. By Remark 2.10, we know that $\mathbb{H}(\tilde{\lambda}) \cong \mathbb{H}(\tilde{\lambda}')$ as a set. We also have that the bijection in Remark 2.10 preserves the crystal structure, and hence $\mathbb{H}(\tilde{\lambda}) \cong \mathbb{H}(\tilde{\lambda}')$ holds as a crystal.

In the rest of this subsection, we prove that $\mathbb{H}(\lambda) \cong B(\lambda)$.

Definition 3.28. Let $\lambda \in P^+$. Then define $H_\lambda \in \mathbb{H}(\lambda)$ by $H_\lambda = (\lambda, \lambda, 0, (0)_{i<j})$.

Remark 3.29. Let $\lambda \in P^+$. Let $H_\lambda = (\lambda, \lambda, 0, (0)_{i<j}) \in \mathbb{H}(\lambda)$. For $i = 1, 2, \dots, \ell(\lambda)$, we have

$$U_{ii} = \lambda_i - \sum_{k=1}^{i-1} U_{ki} = \lambda_i > 0.$$

Remark 3.30. Let $\lambda \in P^+$. Let $H = (\lambda, \lambda, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. By Definition 2.6, we have $\lambda_1 = \lambda_1 - \sum_{k=2}^n U_{1k}$. This means that $U_{1k} = 0$ for $2 \leq k \leq n$ since $U_{ij} \geq 0$. Repetition of this argument yields $U_{ij} = 0$ for $1 \leq i < j \leq n$. Thus, $H = H_\lambda$ holds, and hence we have that H_λ is the unique element $H \in \mathbb{H}(\lambda)$ such that $\mathrm{wt}(H) = \lambda$.

Lemma 3.31. Let $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. Let $H_\lambda = (\lambda, \lambda, 0, (0)_{i<j}) \in \mathbb{H}(\lambda)$. Then $e_i H_\lambda = 0$ for all $i \in I$.

Proof. Let $N = \sum_{i\in I} m_i$. Let $\Psi(H_\lambda) = H_1 \otimes \cdots \otimes H_N$. By Proposition 2.4, there exists ν such that

$$e_i H = H_1 \otimes \cdots \otimes e_i H_\nu \otimes \cdots \otimes H_N.$$

Assume $H_\nu = (\Lambda_{l_\nu}, \mu^{(\nu)}, 0, (U_{ij}^{(\nu)})_{i<j})$. By the construction of Ψ and the definition of H_λ, $\mu^{(\nu)}$ is a partition, and hence $e_i H_\nu = 0$ for all $\nu \in 1, 2 \dots, N$. Thus, $e_i H_\lambda = 0$ holds for all $i \in I$. □

Lemma 3.32. Let $\lambda = \sum_{i\in I} m_i\Lambda_i \in P^+$. Then we have

$$\mathbb{H}(\lambda) = \{f_{i_1} \dots f_{i_k} H_\lambda \mid k \geq 0, i_1, \dots, i_k \in I\}.$$

Therefore $\mathbb{H}(\lambda)$ is connected.

Proof. It suffices to show that if $H \in \mathbb{H}(\lambda)$ such that $e_j H = 0$ for all $j \in I$, then $H = H_\lambda$. Let $H \in \mathbb{H}(\lambda)$. Set $N = \sum_{i\in I} m_i$. Let $\Psi(H) = H_1 \otimes \cdots \otimes H_N$ and $H_m = (\Lambda_{l_m}, \mu^{(m)}, 0, (U_{kl}^{(m)})_{k<l}) \in \mathbb{H}(\Lambda_{l_m})$.

Suppose that $e_j H = 0$ for all $j \in I$. We show that for all $k \in \{1, 2, \dots, N\}$ there exists a $\nu \in I$ such that $H_k = H_{\Lambda_\nu}$. Assume that there exists a $k \in \{1, 2, \dots, N\}$ such that $H_k \neq H_{\Lambda_\nu}$ for all $\nu \in I$, and let k_0 be the smallest such k. Then we have $\mu^{(k_0)}$ is a composition. Then we can take an $i \in I$ such that $\mu_{i+1}^{(k_0)} = 1$ and $\mu_i^{(k_0)} = 0$, and hence there exists a $j_0 \in \{i+1, i+2, \ldots, n\}$ such that $U_{i,j_0}^{(k_0)} > 0$. By Remark 3.22, we have $j_0 = \min\{l \in I \cup \{n\} \mid U_{i,l}^{(k)} > 0, k = 1, 2, \ldots, k_0\}$. Thus, for $k = 1, 2, \ldots, k_0 - 1$, if $U_{il}^{(k)} > 0$ then $l \geq i+1$ holds. However, it follows from $H_k = H_{\Lambda_k}$ that $U_{il}^{(k)} = 0$ for $l \geq i+1$ if $k = 1, 2, \ldots, k_0 - 1$. This is a contradiction. Therefore, for all $k \in \{1, 2, \ldots, N\}$ there exists a $\nu \in I$ such that $H_k = H_{\Lambda_\nu}$. Then, by the construction of Ψ, $H = H_\lambda$ holds. □

From Lemma 3.31 and Lemma 3.32, we have the following.

Lemma 3.33. Let $\lambda \in P^+$. Then H_λ is the highest weight element of weight λ in $\mathbb{H}(\lambda)$.

Proposition 3.34. Let $k \in I$. There is an isomorphism from $\mathbb{H}(\Lambda_k)$ to $B(\Lambda_k)$.

Proof. Let $k \in I$. From [23][1, Theorem 4.13], it suffices to show that

(1) If $e_i(H) = 0$, then $\varepsilon_i(H) = 0$ for $H \in \mathbb{H}(\Lambda_k)$, $i \in I$,
(2) If $f_i(H) = 0$, then $\varphi_i(H) = 0$ for $H \in \mathbb{H}(\Lambda_k)$, $i \in I$,
(3) When $i, j \in I$ and $i \neq j$, if $H, K \in \mathbb{H}(\Lambda_k)$ and $K = e_i H$, then $\varepsilon_j(K)$ equals $\varepsilon_j(H)$ or $\varepsilon_j(H) + 1$. The second case where $\varepsilon_j(K) = \varepsilon_j(H) + 1$ is possible only if α_i and α_j are not orthogonal roots,
(4) When $i, j \in I$ and $i \neq j$, if $H, K \in \mathbb{H}(\Lambda_k)$ and $K = f_i H$, then $\varphi_j(K)$ equals $\varphi_j(H)$ or $\varphi_j(H) + 1$. The second case where $\varphi_j(K) = \varphi_j(H) + 1$ is possible only if α_i and α_j are not orthogonal roots,
(5) Assume that $i, j \in I$ and $i \neq j$. If $H \in \mathbb{H}(\Lambda_k)$ with $\varepsilon_i(H) > 0$ and $\varepsilon_j(e_i H) = \varepsilon_j(H) > 0$, then $e_i e_j H = e_j e_i H$ and $\varphi_i(e_j H) = \varphi_i(H)$,
(6) Assume that $i, j \in I$ and $i \neq j$. If $H \in \mathbb{H}(\Lambda_k)$ with $\varphi_i(H) > 0$ and $\varphi_j(f_i H) = \varphi_j(H) > 0$, then $f_i f_j H = f_j f_i H$ and $\varepsilon_i(f_j H) = \varepsilon_i(H)$,
(7) Assume that $i, j \in I$ and $i \neq j$. If $H \in \mathbb{H}(\Lambda_k)$ with $\varepsilon_j(e_i H) = \varepsilon_j(H) + 1 > 1$ and $\varepsilon_i(e_j H) = \varepsilon_i(H) + 1 > 1$, then $e_i e_j^2 e_i H = e_j e_i^2 e_j H \neq 0$, $\varphi_i(e_j H) = \varphi_i(e_j^2 e_i H)$ and $\varphi_j(e_i H) = \varphi_j(e_i^2 e_j H)$,
(8) Assume that $i, j \in I$ and $i \neq j$. If $H \in \mathbb{H}(\Lambda_k)$ with $\varphi_j(f_i H) = \varphi_j(H) + 1 > 1$ and $\varphi_i(f_j H) = \varphi_i(H) + 1 > 1$, then $f_i f_j^2 f_i H = f_j f_i^2 f_j H \neq 0$, $\varepsilon_i(f_j H) = \varepsilon_i(f_j^2 f_i H)$ and $\varepsilon_j(f_i H) = \varepsilon_j(f_i^2 f_j H)$.

by Remark 3.4, Lemmas 3.32 and 3.33. By Remark 3.4, (1) and (2) hold. Also, again by Remark 3.4, we know that there is no $i \in I$ such that $\varepsilon_i(H) > 1$ (resp. $\varphi_i(H) > 1$), so (7) (resp. (8)) is true.

Let $i, j \in I$ with $i \neq j$. Let $H, K \in \mathbb{H}(\Lambda_k)$. Assume $K = e_i H$. By Definition 3.2, $\varepsilon_j(K) = \varepsilon_j(H)$ is obvious if $j \neq i-1, i+1$. Let $H = (\Lambda_k, \mu, 0, (U_{ij})_{i<j})$ and $K = (\Lambda_k, \xi, 0, (V_{ij})_{i<j})$. We know $\varepsilon_i(H) = 1$ from $K \neq 0$ and Remark 3.4, especially $\mu_{i+1} = 1$ and $\mu_i = 0$. By Definition 3.2, if $\mu_{i-1} = 0$, then $\varepsilon_{i-1}(K) = \varepsilon_{i-1}(H) + 1$, otherwise $\varepsilon_{i-1}(K) = \varepsilon_{i-1}(H)$. Also, if $\mu_{i+2} = 1$, then $\varepsilon_{i+1}(K) = \varepsilon_{i+1}(H) + 1$, otherwise $\varepsilon_{i+1}(K) = \varepsilon_{i+1}(H)$. Then (3) holds.

Let $i, j \in I$ with $i \neq j$. Let $H \in \mathbb{H}(\Lambda_k)$. Assume that $\varepsilon_i(H) > 0$ and $\varepsilon_j(e_i H) = \varepsilon_j(H) > 0$. By Definition 3.2, $\mathrm{wt}(e_i e_j H) = \mathrm{wt}(e_j e_i H)$ holds. Then, $e_i e_j H = e_j e_i H$ holds by Proposition 3.8. By assumption and (3), we can assume $j \neq i-1, i+1$. Then, we have $\varphi_i(e_j H) = \varphi_i(H)$ by Definition 3.2. Thus, (5) is satisfied.

By Propositions 3.12, 3.13, and (5), (6) immediately holds. □

Theorem 3.35. Let $\lambda \in P^+$. Then we have a crystal isomorphism $\Phi\colon \mathbb{H}(\lambda) \to B(\lambda)$ such that $\Phi(H_\lambda) = b_\lambda$.

Proof. By Proposition 3.34 and Proposition 3.19, if $\mathbb{H}(\lambda)$ has the highest weight vector of the highest weight λ, then we have $\mathbb{H}(\lambda) \cong B(\lambda)$. By Lemma 3.32 and Lemma 3.31, they hold. Then we have a crystal isomorphism $\Phi\colon \mathbb{H}(\lambda) \to B(\lambda)$ such that $\Phi(H_\lambda) = b_\lambda$. □

3.3. **Direct Combinatorial Description of Crystal Structure on $\mathbb{H}(\lambda)$.** In this subsection, we describe the crystal structure of $\mathbb{H}(\lambda)$ directly. More specifically, we give an explicit formula for computing the maps wt, e_i, f_i, φ_i, ε_i $(i \in I)$ for $\mathbb{H}(\lambda)$.

Theorem 3.36. Let $\lambda = \sum_{i\in I} m_i\Lambda_i$. For $H \in \mathbb{H}(\lambda)$, the maps wt, f_j, e_j, φ_j, ε_j $(j \in I)$ are computed as follows. Fix $j \in I$.

(1) $\mathrm{wt}(H) = \sum_{i\in I}(\mu_i - \mu_{i+1})\Lambda_i$.
(2) For $k \in \{1, 2, \ldots, j\}$, set $\varphi_j^{(k)}(H) = \max\{\varphi_j^{(k-1)}(H) + U_{k,j} - U_{k+1,j+1}, 0\}$. Note that we regard $\varphi_j^{(0)}$ as 0. Then we have $\varphi_j(H) = \varphi_j^{(j)}(H)$.
(3) For $k \in \{1, 2, \ldots, j+1\}$, set $\varepsilon_j^{(k)}(H) = \max\{\varepsilon_j^{(k-1)}(H) + U_{j+2-k,j+1} - U_{j+1-k,j}, 0\}$. Note that we regard $\varepsilon_j^{(0)}$ as 0. Then we have $\varepsilon_j(H) = \varepsilon_j^{(j+1)}(H)$.
(4) If $\varphi_j(H) = 0$ then $f_jH = 0$. If $\varphi_j(H) \neq 0$, let

$$k' = \min\{k \in [n] \mid \forall l \geq k,\ \varphi_j^{(l)}(H) > 0\}.$$

Then we have $f_jH = (\lambda, \mu', 0, (U'_{kl})_{k<l})$ where

$$\mu' = \sum_{k\neq j,j+1} \mu_k\epsilon_k + (\mu_j - 1)\epsilon_j + (\mu_{j+1} + 1)\epsilon_{j+1},$$

$$U'_{kl} = \begin{cases} U_{kl} - 1 & \text{if } k = k',\ l = j, \\ U_{kl} + 1 & \text{if } k = k',\ l = j+1, \\ U_{kl} & \text{else.} \end{cases}$$

(5) If $\varepsilon_j(H) = 0$ then $e_jH = 0$. If $\varepsilon_j(H) \neq 0$, let

$$k' = \min\{k \in [n] \mid \forall l \geq k,\ \varepsilon_j^{(l)}(H) > 0\}.$$

Then we have $e_jH = (\lambda, \mu', 0, (U'_{kl})_{k<l})$ where

$$\mu' = \sum_{k\neq j,j+1} \mu_k\epsilon_k + (\mu_j + 1)\epsilon_j + (\mu_{j+1} - 1)\epsilon_{j+1},$$

$$U'_{kl} = \begin{cases} U_{kl} + 1 & \text{if } k = j+2-k',\ l = j, \\ U_{kl} - 1 & \text{if } k = j+2-k',\ l = j+1, \\ U_{kl} & \text{else.} \end{cases}$$

Proof. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let $N = \sum_{i\in I} m_i$. Assume that $\Psi(H) = H_1 \otimes \cdots \otimes H_N$ and $H_k = (\Lambda_{l_k}, \mu^{(k)}, 0, (U_{ij}^{(k)})_{i<j})$ for $k = 1, 2, \ldots, N$. Fix $j \in I$. By Proposition 2.4, we have

$$\varphi_j(H) = \max\left\{\varphi_j(H_N) + \sum_{k=\nu}^{N-1} (\varphi_j(H_\nu) - \varepsilon_j(H_{\nu+1})) \mid 1 \leq \nu \leq N\right\}.$$

Suppose that ν_0 is the largest integer such that

$$\varphi_j(H) = \varphi_j(H_N) + \sum_{k=\nu_0}^{N-1} \left(\varphi_j(H_k) - \varepsilon_j(H_{k+1})\right).$$

Since $\varphi_j(H_k) = \max\{\mu_j^{(k)} - \mu_{j+1}^{(k)}, 0\}$ and $\varepsilon_j(H_k) = \max\{\mu_{j+1}^{(k)} - \mu_j^{(k)}, 0\}$, we have

$$\begin{aligned}
\varphi_j(H) &= \varphi_j(H_N) + \sum_{k=\nu_0}^{N-1} \left(\varphi_j(H_k) - \varepsilon_j(H_{k+1})\right) \\
&= \varphi_j(H_{\nu_0}) + \sum_{k=\nu_0+1}^{N} \left(\varphi_j(H_k) - \varepsilon_j(H_k)\right) \\
&= \max\{\mu_j^{(\nu_0)} - \mu_{j+1}^{(\nu_0)}, 0\} + \sum_{k=\nu_0+1}^{N} \left(\mu_j^{(k)} - \mu_{j+1}^{(k)}\right).
\end{aligned}$$

Note that, by the choice of ν_0, we have $\varphi_j(H_{\nu_0}) = \mu_j^{(\nu_0)} - \mu_{j+1}^{(\nu_0)} > 0$ if $\varphi_j(H) > 0$.

If $\varphi_j^{(j)}(H) > 0$, then we can take $k' = \min\{k \in [n] \mid \forall l \geq k,\ \varphi_j^{(l)}(H) > 0\}$. Then we have

$$\varphi_j^{(j)}(H) = \sum_{k=k'}^{j} (U_{kj} - U_{k+1,j+1}) > 0$$

since $\varphi_j^{(k'-1)}(H) = 0$ by the choice of k'.

(1) By Definition 2.3, we have $\mathrm{wt}(H) = \sum_{k=1}^{N} \mathrm{wt}(H_k) = \sum_{i \in I} (\mu_i - \mu_{i+1})\Lambda_i$.

(2) First, we consider the case where $\varphi_j(H) = 0$. By the choice of ν_0, we have $\nu_0 = N$. Then we have that $\mu_j^{(N)} - \mu_{j+1}^{(N)} \leq 0$ and $\sum_{k=\nu}^{N} (\mu_j^{(k)} - \mu_j^{(k)}) \leq 0$ for $\nu = 1, 2. \ldots, N$. By the above discussion, if $\varphi_j^{(j)}(H) > 0$, then there exists k' such that $\sum_{k=k'}^{j} (U_{kj} - U_{k+1,j+1}) > 0$. Then, to show that $\varphi_j^{(j)}(H) = 0$, it suffices to show that

$$\sum_{k=\nu}^{j} (U_{kj} - U_{k+1,j+1}) \leq 0 \quad \text{for all } \nu \in \{1, 2, \ldots, j\}.$$

For $\nu \in \{1, \ldots, N\}$ and $k \in [n]$, if there exists $l \in [n]$ such that $U_{k,l}^{(\nu)} > 0$, then set $j_{k,\nu}$ to its j, otherwise set $j_{k,\nu}$ to 0. Fix $k' \in \{1, 2, \ldots, j\}$. For k' and j, set

$$\begin{aligned}
\mathcal{N}_{k',j} &= \left\{k \in \{1, \ldots, N\} \mid U_{lj}^{(k)} > 0, l \geq k'\right\}, \\
\mathcal{N}'_{k',j+1} &= \left\{k \in \{1, \ldots, N\} \mid U_{l,j+1}^{(k)} > 0, l > k'\right\}.
\end{aligned}$$

If $\mathcal{N}_{k',j} \cup \mathcal{N}'_{k',j+1} = \varnothing$, then $\sum_{k=k'}^{j} (U_{kj} - U_{k+1,j+1}) = 0$ holds since $U_{kl} = U_{kl}^{(1)} + \cdots + U_{kl}^{(N)}$. Suppose that $\mathcal{N}_{k',j} \cup \mathcal{N}'_{k',j+1} \neq \varnothing$. Let $\nu_1 = \min(\mathcal{N}_{k',j} \cup \mathcal{N}'_{k',j+1})$. Suppose that there exists $k_1 \in \{k', k'+1, \ldots, j\}$ such that $j_{k_1,\nu_1} = j$. By the choice of ν_1, $j_{k,\nu} \neq j, j+1$ for $k' \leq k$ and $\nu < \nu_1$. Also, by Remark 3.24 and Lemma 3.21, $j_{k,\nu} \neq j, j+1$ for $k < k_1$ and

$\nu_1 \leq \nu$. Then we have

$$\begin{aligned}\sum_{k=k'}^{j}(U_{kj}-U_{k+1,j+1}) &= \sum_{k=k'}^{j}\sum_{\nu=1}^{N}U_{kj}^{(\nu)} - \sum_{k=k'}^{j}\sum_{\nu=1}^{N}U_{k+1,j+1}^{(\nu)} \\ &= \sum_{k=k'}^{j}\sum_{\nu=\nu_1}^{N}U_{kj}^{(\nu)} - \sum_{k=k'}^{j}\sum_{\nu=\nu_1}^{N}U_{k+1,j+1}^{(\nu)} \\ &= \sum_{k=1}^{j}\sum_{\nu=\nu_1}^{N}U_{kj}^{(\nu)} - \sum_{k=1}^{j}\sum_{\nu=\nu_1}^{N}U_{k+1,j+1}^{(\nu)} \\ &= \sum_{\nu=\nu_1}^{N}\left(\mu_j{}^{(\nu)} - \mu_{j+1}^{(\nu)}\right) \leq 0.\end{aligned}$$

Suppose that there exists $k_1 \in \{k'+1,\dots,j\}$ such that $j_{k_1,\nu_1} = j+1$. By the choice of ν_1, $j_{k,\nu} \neq j, j+1$ for $k' \leq k$ and $\nu < \nu_1$. Also, by Remark 3.24 and Lemma 3.21, $j_{k,\nu} \neq j$ for $k < k_1$ and $\nu_1 < \nu$, and $j_{k,\nu} \neq j+1$ for $k < k_1$ and $\nu_1 \leq \nu$. From $j_{k_1,\nu_1} = j+1$, $k' < k_1$, and Remark 3.24, if there exists k such that $j_{k,\nu_1} = j$, then $k = k_1 - 1 \geq k'$. Then $j_{k,\nu_1} \neq j$ for $k < k'$. Then we similarly have the following.

$$\sum_{k=k'}^{j}(U_{kj}-U_{k+1,j+1}) = \sum_{\nu=\nu_1}^{N}\left(\mu_j{}^{(\nu)} - \mu_{j+1}^{(\nu)}\right) \leq 0.$$

Next, we consider the case where $\varphi_j(H) \neq 0$. Suppose that there exists $\nu' \in \{1,2,\dots,N\}$ such that $\varphi_j(H_{\nu'}) > 0$ and let $j_{k',\nu'} = j$ for some $k' \in \{1,2,\dots,j\}$. By Remark 3.24 and Lemma 3.21, $j_{k,\nu} \neq j, j+1$ for $k < k'$ and $\nu' \leq \nu$. We know that $j = j_{k',\nu'} < j_{k'+1,\nu'}$ from Remark 3.24, especially $j+1 < j_{k'+1,\nu'}$ holds since we have $\mu_{j+1}^{(\nu')} = 0$ from $\varphi_j(H_{\nu'}) > 0$. Then, $j_{k,\nu} \neq j, j+1$ for $k' < k$ and $\nu \leq \nu'$. Suppose that $j_{k',\nu} \neq j$ for $\nu = 1,2,\dots,\nu'-1$. Then we have

$$\begin{aligned}\sum_{\nu=\nu'}^{N}\left(\mu_j^{(\nu)} - \mu_{j+1}^{(\nu)}\right) &= \sum_{\nu=\nu'}^{N}\sum_{k=1}^{j}U_{kj}^{(\nu)} - \sum_{\nu=\nu'}^{N}\sum_{k=1}^{j+1}U_{k,j+1}^{(\nu)} \\ &= \sum_{\nu=\nu'}^{N}\sum_{k=k'}^{j}U_{kj}^{(\nu)} - \sum_{\nu=\nu'}^{N}\sum_{k=k'+1}^{j+1}U_{k,j+1}^{(\nu)} \\ &= \sum_{\nu=1}^{N}\sum_{k=k'}^{j}U_{kj}^{(\nu)} - \sum_{\nu=1}^{N}\sum_{k=k'+1}^{j+1}U_{k,j+1}^{(\nu)} \\ &= \sum_{k=k'}^{j}(U_{kj}-U_{k+1,j+1}).\end{aligned}$$

Since $\varphi_j(H) \neq 0$, then we have $\varphi_j(H_{\nu_0}) > 0$. Then there exists $k_0 \in \{1,2,\dots,j\}$ such that $j_{k_0,\nu_0} = j$. By the choice of ν_0, for $\nu = 1,2,\dots,\nu_0-1$

$$\begin{aligned}\sum_{k=\nu}^{\nu_0-1}(\varphi_j(H_k)-\varepsilon_j(H_{k+1})) &= \varphi_j(H_\nu) - \varepsilon_j(H_{\nu_0}) + \sum_{k=\nu+1}^{\nu_0-1}(\varphi_j(H_k)-\varepsilon_j(H_k)) \\ &= \varphi_j(H_\nu) + \sum_{k=\nu+1}^{\nu_0-1}\left(\mu_j^{(k)} - \mu_{j+1}^{(k)}\right) \leq 0.\end{aligned} \tag{3.5}$$

Note that $\varepsilon_j(H_{\nu_0}) = 0$ holds from $\varphi_j(H_{\nu_0}) > 0$ and Definition 3.2. If there exists $\nu_0' \in \{1,2,\dots,\nu_0-1\}$ such that $j_{k_0,\nu_0'} = j$, then we have that $j_{k,\nu} \neq j, j+1$ for $k < k_0$ and

$\nu_0' \leq \nu$ by Remark 3.24 and Lemma 3.21. Also, we know that $j_{k,\nu} \neq j, j+1$ for $k_0 < k$ and $\nu \leq \nu_0$. Then, we have that $\varphi_j(H_{\nu_0'}) > 0$ and $\mu_{j+1}^{(\nu)} = 0$ for $\nu < \nu_0$. Moreover,

$$\varphi_j(H_{\nu_0'}) + \sum_{k=\nu_0'+1}^{\nu_0-1} \left(\mu_j^{(k)} - \mu_{j+1}^{(k)}\right) = \varphi_j(H_{\nu_0'}) + \sum_{k=\nu_0'+1}^{\nu_0-1} \mu_j^{(k)} > 0.$$

This is a contradiction for (3.5), and hence we have $j_{k_0,\nu} \neq j$ for all $\nu = 1, 2, \dots, \nu_0 - 1$. Then, by the discussion above and $\varphi_j(H) > 0$,

$$\sum_{k=\nu_0}^{N} \left(\mu_j^{(k)} - \mu_{j+1}^{(k)}\right) = \sum_{k=k_0}^{j} (U_{kj} - U_{k+1,j+1}) > 0.$$

To show that $\sum_{k=k_0}^{m}(U_{kj} - U_{k+1,j+1}) > 0\,(m = k_0, k_0+1, \dots, j)$, suppose that there exists $m \in \{k_0, k_0+1, \dots, j\}$ such that $\sum_{k=k_0}^{m}(U_{kj} - U_{k+1,j+1}) \leq 0$. Let m_0 be the largest among such m. In this case, since we have

$$\sum_{k=k_0}^{j} (U_{kj} - U_{k+1,j+1}) = \sum_{k=m_0+1}^{j} (U_{kj} - U_{k+1,j+1}) + \sum_{k=k_0}^{m_0} (U_{kj} - U_{k+1,j+1}) > 0,$$

then $\sum_{k=m_0+1}^{j}(U_{kj} - U_{k+1,j+1}) > 0$ holds, especially $U_{m_0+1,j} - U_{m_0+2,j+1} > 0$. Then we can take $\nu' \in \{\nu_0+1, \nu_0+2, \dots, N\}$ such that $j_{m_0+1,\nu'} = j$ since we have that $j_{k,\nu} \neq j$ for $k_0 < k$ and $\nu < \nu_0$. Let ξ_0 be the smallest $\nu \in \{\nu_0+1, \dots, N\}$ such that $j_{m_0+1,\nu} = j$. We know that $j = j_{m_0+1,\xi_0} < j_{m_0+2,\xi_0}$, especially $j+1 < j_{m_0+2,\xi_0}$ holds since $U_{m_0+2,j+1} = 0$. Then $\varphi_j(H_{\xi_0}) > 0$ holds. By the choice of ξ_0, $j_{m_0+1,\nu} \neq j$ for $\nu = 1, 2, \dots, \xi_0 - 1$. Then by the above discussion,

$$\sum_{k=\xi_0}^{N} \left(\mu_j^{(k)} - \mu_{j+1}^{(k)}\right) = \sum_{k=m_0+1}^{j} (U_{kj} - U_{k+1,j+1}).$$

Recall that $j_{k,\nu} \neq j, j+1$ for $k < m_0 + 1$ and $\xi_0 \leq \nu$, or $k_0 < k$ and $\nu \leq \nu_0$, or $k < k_0$ and $\nu_0 \leq \nu$. In addition, we know $j_{k_0,\nu} \neq j$ for $k < m_0 + 1$ and $\nu < \nu_0$. Thus,

$$\begin{aligned}
\sum_{k=k_0}^{m_0} (U_{kj} - U_{k+1,j+1}) &= \sum_{\nu=1}^{N}\sum_{k=k_0}^{m_0} U_{kj}^{(\nu)} - \sum_{\nu=1}^{N}\sum_{k=k_0}^{m_0} U_{k+1,j+1}^{(\nu)} \\
&= \sum_{\nu=\nu_0}^{\xi_0-1}\sum_{k=k_0}^{m_0} U_{kj}^{(\nu)} - \sum_{\nu=\nu_0}^{\xi_0-1}\sum_{k=k_0}^{m_0} U_{k+1,j+1}^{(\nu)} \\
&= \sum_{\nu=\nu_0}^{\xi_0-1}\sum_{k=1}^{j} U_{kj}^{(\nu)} - \sum_{\nu=\nu_0}^{\xi_0-1}\sum_{k=1}^{j} U_{k+1,j+1}^{(\nu)} \\
&= \sum_{\nu=\nu_0}^{\xi_0-1} (\mu_j^{(\nu)} - \mu_{j+1}^{(\nu)})
\end{aligned}$$

holds. Then, since we have

$$\begin{aligned}
\sum_{k=k_0}^{j} (U_{kj} - U_{k+1,j+1}) &= \sum_{k=m_0+1}^{j} (U_{kj} - U_{k+1,j+1}) + \sum_{k=k_0}^{m_0} (U_{kj} - U_{k+1,j+1}) \\
&= \sum_{k=\xi_0}^{N} \left(\mu_j^{(k)} - \mu_{j+1}^{(k)}\right) + \sum_{k=\nu_0}^{\xi_0-1} \left(\mu_j^{(k)} - \mu_{j+1}^{(k)}\right) > 0,
\end{aligned}$$

then $\sum_{\nu=\xi_0}^{N}(\mu_j^{(\nu)}-\mu_{j+1}^{(\nu)})>0$. However, this is a contradiction to the choice of ν_0. Thus, we have $\sum_{k=k_0}^{m}(U_{kj}-U_{k+1,j+1})>0$ for all $m=k_0,k_0+1,\dots,j$. Also, from (3.5) we have

$$\sum_{k=1}^{\nu_0-1}(\mu_j^{(k)}-\mu_{j+1}^{(k)})\leq 0.$$

Then,

$$k_0=\min\{k\in[n]\mid \forall l\geq k,\ \varphi_j^{(l)}(H)>0\}$$

holds. Therefore we have

$$\varphi_j(H)=\varphi_j^{(j)}(H).$$

(3) is proved in a similar way.

(4) By the above discussion and Proposition 2.4, we have

$$f_jH=H_1\otimes\cdots\otimes f_jH_{\nu_0}\otimes\cdots\otimes H_N.$$

Suppose $\varphi_j(H)=0$. By the choice of ν_0, we have $\nu_0=N$. Then $\varphi_j(H)=\varphi_j(H_N)=0$ holds. By Remark 3.4, we have $f_jH=0$.

Suppose $\varphi_j(H)\neq 0$. By the above discussion, we know $U_{k_0,j}^{(\nu_0)}>0$. Then, by Definition 3.2, $f_iH_{\nu_0}=(\Lambda_{\nu_0},\tilde{\mu}^{(\nu_0)},0,(\tilde{U}_{kl}^{(\nu_0)})_{k<l})\in\mathbb{H}(\Lambda_{\nu_0})$, where

$$\tilde{U}_{kl}^{(\nu_0)}=\begin{cases}U_{kl}^{(\nu_0)}-1 & \text{if } (k,l)=(k_0,j),\\ U_{kl}^{(\nu_0)}+1 & \text{if } (k,l)=(k_0,j+1),\\ U_{kl}^{(\nu_0)} & \text{otherwise,}\end{cases}$$

$$\tilde{\mu}_k^{(\nu_0)}=\begin{cases}\mu_k^{(\nu_0)}-1 & \text{if } k=j,\\ \mu_k^{(\nu_0)}+1 & \text{if } k=j+1,\\ \mu_k^{(\nu_0)} & \text{otherwise.}\end{cases}$$

Thus the statement holds by the construction of Ψ. We can then prove (5) in a similar manner. □

Example 3.37. Let $n=4$, $\lambda=\mu=\Lambda_1+\Lambda_3$. The action of f_3 on the $U_q(\mathfrak{sl}_4)$-crystal $\mathbb{H}(\Lambda_1+\Lambda_3)$ is computed as follows. Let $H=(\Lambda_1+\Lambda_3,\mu,(U_{kl})_{k<l})\in\mathbb{H}(\Lambda_1+\Lambda_3)$, where $U_{kl}=0$ for $1\leq k<l\leq 4$. Then we have

$$\begin{aligned}\varphi_3^{(1)}&=\max(U_{13}-U_{24},0)=0,\\ \varphi_3^{(2)}&=\max(\varphi_3^{(1)}+U_{23}-U_{34},0)=0,\\ \varphi_3^{(3)}&=\max(\varphi_3^{(2)}+U_{33}-U_{44},0)=1.\end{aligned}$$

Thus, we have $\varphi_3(H)=1$ and $k'=3$, and so f_3H is as shown in FIGURE 6.

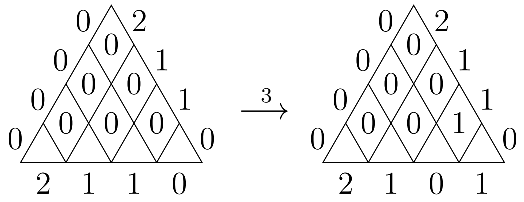
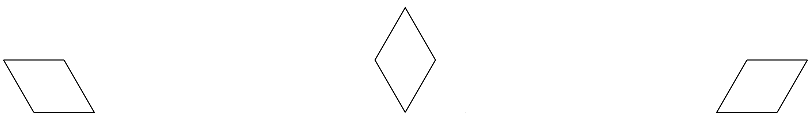


FIGURE 6. Action of f_3 on the $U_q(\mathfrak{sl}_4)$-crystal $\mathbb{H}(\Lambda_1 + \Lambda_3)$

Proposition 3.38 below shows how other parameters built into H can help expressing the results in Theorem 3.36. A hive H is actually a collection of edge labels of all elementary triangles in the hive graph (see [24, on the right of (2.3)]), satisfying some compatibility conditions, and each (elementary) rhombus, consisting of two adjacent elementary triangles, determines its gradient as the difference between the labels of its parallel edges, U_{ij} is the gradient of an upright rhombus as shown in FIGURE 1, and L_{ij}, although expressible in terms of the U_{ij}'s, is the gradient of a left-leaning rhombus, having the orientation shown on the left of FIGURE 7. Each right-leaning rhombus, whose orientation is shown on the right of FIGURE 7, also gives a gradient R_{ij}, expressible in terms of the U_{ij}'s as in Proposition 3.38. Note that the R_{ij} are not assumed to be non-negative, as opposed to the case of LR-hives. Although the rhombi that would correspond to R_{ij} with $i = j$ lie outside the hive triangle, it is convenient to include such parameters.



(A) left-leaning (B) upright (C) right-leaning

FIGURE 7. Rhombus

Proposition 3.38. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Set $R_{ij} = \sum_{k=1}^{i-1} U_{k,j-1} - \sum_{k=1}^{i} U_{kj}$ for $1 \leq i \leq j \leq n$. Fix $j \in I$.

(1) For $i = 1, 2, \ldots, j+1$, set $\tilde{R}_{i,j+1} = R_{j+1,j+1} - R_{i,j+1}$. If there is no $i \in \{1, 2, \ldots, j+1\}$ such that $\tilde{R}_{i,j+1} > 0$, then $\varphi_j(H) = 0$ holds. If there exists $i \in \{1, 2, \ldots, j+1\}$ such that $\tilde{R}_{i,j+1} > 0$, let i_0 be the maximum element such that $\tilde{R}_{i_0,j+1} = \max\{\tilde{R}_{i,j+1} \mid i \in \{1, 2, \ldots, j+1\}\}$. Then, $\varphi_j(H) = \tilde{R}_{i_0,j+1}$ and

$$i_0 = \min\{k \in [n] \mid \forall l \geq k, \varphi_j^{(l)}(H) > 0\}.$$

(2) If there is no $i \in \{1, 2, \ldots, j+1\}$ such that $R_{i,j+1} < 0$, then $\varepsilon_j(H) = 0$ holds. If there exists $i \in \{1, 2, \ldots, j+1\}$ such that $R_{i,j+1} < 0$, let i_0 be the minimum element such that $R_{i_0,j+1} = \min\{R_{i,j+1} \mid i \in \{1, 2, \ldots, j+1\}\}$. Then, $\varepsilon_j(H) = -R_{i_0,j+1}$ and

$$i_0 = \min\{k \in [n] \mid \forall l \geq k, \varepsilon_j^{(l)}(H) > 0\}.$$

Proof. (1) Assume that there is no $i \in \{1,2,\dots,j+1\}$ such that $\tilde{R}_{i,j+1} > 0$. Then we have

$$\sum_{k=i}^{j} U_{k,j} - \sum_{k=i+1}^{j+1} U_{k,j+1} \leq 0$$

for $i \in \{1,2,\dots,j\}$. This means that $\varphi_j^{(k)}(H) = 0$ for $k \in \{1,2,\dots,j\}$, especially $\varphi_j(H) = \varphi_j^{(j)}(H) = 0$.

Assume that there is $i \in \{1,2,\dots,j+1\}$ such that $\tilde{R}_{i,j+1} > 0$. Let i_0 be the maximum element such that $\tilde{R}_{i_0,j+1} = \max\{\tilde{R}_{i,j+1} \mid i \in \{1,2,\dots,j+1\}\}$. For $i \in \{1,2,\dots,j+1\}$ and $l \in \{1,2,\dots,j+1-i\}$, we have

$$\tilde{R}_{i,j+1} - \tilde{R}_{i+l,j+1} = \sum_{k=i}^{i+l-1} U_{k,j} - \sum_{k=i+1}^{i+l} U_{k,j+1}.$$

By the choice of i_0, for $l = 1,2,\dots,j+1-i_0$, $\tilde{R}_{i_0,j+1} - \tilde{R}_{i_0+l,j+1} > 0$ holds, and hence $\varphi_j^{(k)}(H) > 0$ for $k = i_0, i_0+1,\dots,j+1$. Also, by the choice of i_0 again, for $l = 1,2,\dots,i_0-1$, $\tilde{R}_{i_0-l,j+1} - \tilde{R}_{i_0,j+1} \leq 0$, and hence $\varphi_j^{(i_0-1)}(H) = 0$. Thus, $i_0 = \min\{k \in [n] \mid \forall l \geq k, \varphi_j^{(l)}(H) > 0\}$.

(2) Assume that there is no $i \in \{1,2,\dots,j+1\}$ such that $R_{i,j+1} < 0$. Then we have

$$\sum_{k=1}^{i-1} U_{k,j} - \sum_{k=1}^{i} U_{k,j+1} \geq 0$$

for $i \in \{1,2,\dots,j+1\}$. This means that $\varepsilon_j^{(k)}(H) = 0$ for $k \in \{1,2,\dots,j+1\}$, especially $\varepsilon_j(H) = \varepsilon_j^{(j+1)}(H) = 0$.

Assume that there is $i \in \{1,2,\dots,j+1\}$ such that $R_{i,j+1} < 0$. Let i_0 be the minimum element such that $R_{i_0,j+1} = \min\{R_{i,j+1} \mid i \in \{1,2,\dots,j+1\}\}$. For $i \in \{1,2,\dots,j+1\}$ and $l \in \{1,2,\dots,j+1-i\}$, we have

$$R_{i+l,j+1} - R_{i,j+1} = \sum_{k=i}^{i+l-1} U_{k,j} - \sum_{k=i+1}^{i+l} U_{k,j+1}.$$

By the choice of i_0, for $l = 1,2,\dots,i_0-1$, $R_{i_0,j+1} - R_{i_0-l,j+1} < 0$ holds, and hence $\varepsilon_j^{(k)}(H) > 0$ for $k = i_0, i_0+1,\dots,j+1$. Also, by the choice of i_0 again, for $l = 1,2,\dots,j+1-i_0$, $R_{i_0+l,j+1} - R_{i_0,j+1} \leq 0$, and hence $\varepsilon_j^{(i_0-1)}(H) = 0$. Thus, $i_0 = \min\{k \in [n] \mid \forall l \geq k, \varepsilon_j^{(l)}(H) > 0\}$. □

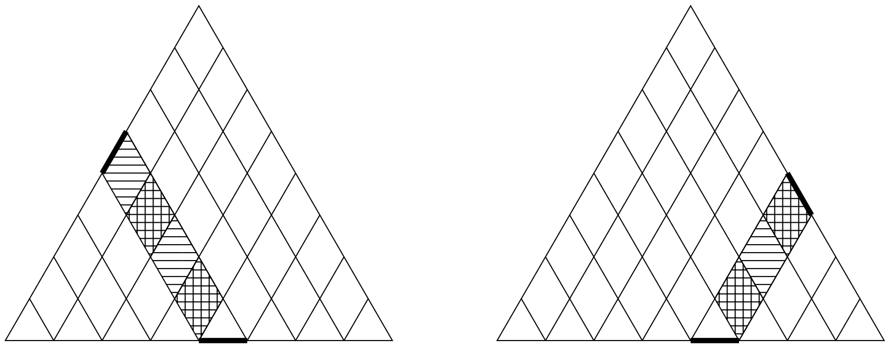
4. Tensor product decomposition map

In this section, we show the tensor product decomposition map in terms of K-hives. The decomposition map is computed by a graphical method, through the notion of path operators on K-hives. In 4.1, the notion of path operators on K-hives is defined, and some examples of them are given. In 4.2, the tensor product decomposition map is given using path operators. The main reference is [20].

4.1. **Path Operators.** In this subsection, we define the notion of a path operator on K-hives and give some examples. Then we investigate the relationship between such operators and the crystal structure of $\mathbb{H}(\lambda)$.

Definition 4.1. Let $\lambda \in P^+$. Let $H = (\lambda, \mu, \nu, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. For $1 \le j \le n$, let L_j denote the set of the thickened boundary edges and shaded upright rhombi shown in FIGURE 8 (A), and let R_j denote the set of the thickened boundary edges and shaded upright rhombus shown in FIGURE 8 (B). Let $p = (p_k)_{k=0,1\dots,m-1}$ be a sequence of upright rhombus and boundary edges of H. The sequence p is called a **path** on H if p satisfies the following: $p_k \in R_i \cap L_j$ implies $p_{k+1} \in R_i \cup L_j$ for $k = 0, 1, \dots, m-2$.

For $\lambda, \nu \in P^+$, let f be a map from $\mathbb{H}(\lambda)$ to $\bigsqcup_{\nu \in P^+} \mathbb{H}(\nu)$. Then f is called a **path operator** if for $H \in \mathbb{H}(\lambda)$ there is a path p such that $f(H)$ is obtained by reducing or increasing boundary edge labels or rhombus gradients specified by the path p.



(A) (B)

FIGURE 8. Example of L_i and R_i

Example 4.2. Let $n = 4$, $\lambda = (6, 4, 1, 0)$. Let $H = (\lambda, \mu, \nu, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$ as shown in FIGURE 9. Let p be a sequence of upright rhombus and boundary edges of H highlighted in red on the left side of FIGURE 9, and let q be a sequence of upright rhombus and boundary edges of H highlighted in red on the right side of FIGURE 9. Then p and q are a path on H, respectively.

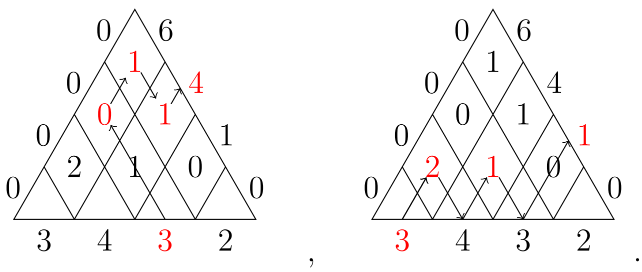


FIGURE 9. An example of a path on a K-hive

We define an operator on $\mathbb{H}(\lambda)$, which is a path operator as we will see later (Proposition 4.8). Note that for $\lambda \in P^+$ a partition $(\lambda_1, \dots, \lambda_n)$ representing λ is uniquely determined by $\lambda_n = 0$.

Definition 4.3. Let $\lambda = \sum_{k\in I} \lambda_k \epsilon_k \in P^+$ with $\lambda \neq 0$. Set $j_{\iota(H)} = \min\{i \in [n] \mid U_{\ell(\lambda),i} \neq 0\}$. The operator ι on $\mathbb{H}(\lambda)$ is defined as $\iota(H) = (\nu, \xi, 0, (V_{ij})_{i<j})$, where

$$\nu_k = \begin{cases} \lambda_k - 1 & \text{if } k = \ell(\lambda), \\ \lambda_k & \text{otherwise,} \end{cases}$$

$$\xi_k = \begin{cases} \mu_k - 1 & \text{if } k = j_{\iota(H)}, \\ \mu_k & \text{otherwise,} \end{cases}$$

$$V_{ij} = \begin{cases} U_{ij} - 1 & \text{if } (i,j) = (\ell(\lambda), j_{\iota(H)}), \\ U_{ij} & \text{otherwise.} \end{cases}$$

Remark 4.4. The operator ι is considered as a path operator as follows. Let $H \in \mathbb{H}(\lambda)$. Then the action of ι is obtained by decreasing boundary edge labels and rhombus gradients specified by a path by 1, hence ι is a path operator. Note that we have that $\iota(H)$ is a K-hive by Proposition 4.8.

Example 4.5. Let $n = 4$. Let $\lambda = (6, 4, 2, 0) \in P^+$. Let $H = (\lambda, \mu, \nu, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$ be as shown on the left of FIGURE 10, and then the path for ι on H is as illustrated in blue. Then the action of ι for H is as shown in FIGURE 10.

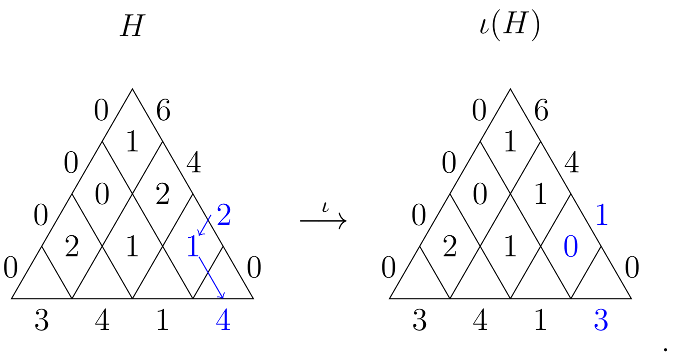


FIGURE 10. Action of ι

Example 4.6. Let $n = 4$. Let $\lambda = (6, 4, 2, 0) \in P^+$. Let $H = (\lambda, \mu, \nu, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$ be as shown to the left of FIGURE 11, and then the path for ι on H is as illustrated in blue. Then the action of ι for H is as shown in FIGURE 11.

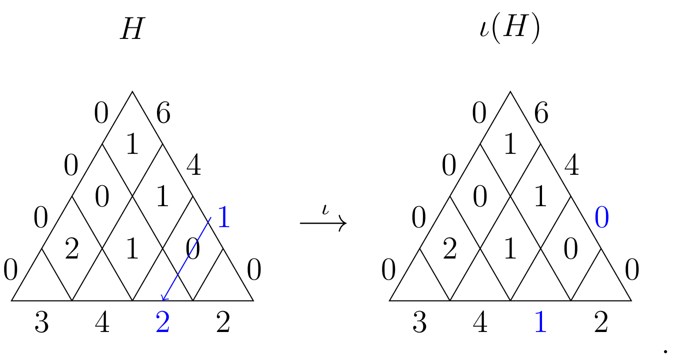


FIGURE 11. Action of ι

As we will see in Proposition 4.8, the result of the action of ι is a K-hive. Before this, we prepare the following lemma.

Lemma 4.7. Let $\lambda \in P^+$ with $\lambda \neq 0$. Let $H \in \mathbb{H}(\lambda)$. Then, $\iota(H)$ is an integer hive.

Proof. Let $\lambda = \sum_{i\in I} \lambda_i \epsilon_i \in P^+$ with $\lambda \neq 0$. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let $\nu = \sum_{i\in I} \nu_i \epsilon_i$, where $\nu_{\ell(\lambda)} = \lambda_{\ell(\lambda)} - 1$, $\nu_k = \lambda_k$ if $k \neq \ell(\lambda)$. By Definition 4.3, we can assume that $\iota(H) = (\nu, \xi, 0, (V_{ij})_{i<j})$.

To show that $\iota(H)$ is an integer hive, we need to show that

$$\xi_m - \sum_{k=1}^{m-1} V_{km} = \nu_m - \sum_{k=m+1}^{n} V_{mk} \quad (m = 1, 2, \ldots, n). \tag{4.6}$$

If $m \neq \ell(\lambda)$ and $m \neq j_{\iota(H)}$, then (4.6) is trivial by Definition 4.3. Suppose $m = \ell(\lambda)$. If $\ell(\lambda) = j_{\iota(H)}$, Definition 4.3 and the fact that $H \in \mathbb{H}(\lambda)$ show (4.6) as follows.

$$\begin{aligned}
\xi_m - \sum_{k=1}^{m-1} V_{km} &= \mu_m - 1 - \sum_{k=1}^{m-1} U_{km} \\
&= \lambda_m - 1 - \sum_{k=m+1}^{n} U_{mk} \\
&= \nu_m - \sum_{k=m+1}^{n} V_{mk}.
\end{aligned}$$

If $\ell(\lambda) \neq j_{\iota(H)}$, we know that $\ell(\lambda) < j_{\iota(H)}$ by the choice of $j_{\iota(H)}$. Then it follows from Definition 4.3 that

$$\begin{aligned}
\xi_{\ell(\lambda)} - \sum_{k=1}^{\ell(\lambda)-1} V_{k,\ell(\lambda)} &= \mu_{\ell(\lambda)} - \sum_{k=1}^{\ell(\lambda)-1} U_{k,\ell(\lambda)}, \\
\nu_{\ell(\lambda)} - \sum_{k=\ell(\lambda)+1}^{n} V_{\ell(\lambda),k} &= \lambda_{\ell(\lambda)} - 1 - \left(\sum_{k\neq j_{\iota(H)}}^{n} U_{\ell(\lambda),k} + U_{\ell(\lambda),j_{\iota(H)}} - 1 \right) \\
&= \lambda_{\ell(\lambda)} - \sum_{k=\ell(\lambda)+1}^{n} U_{\ell(\lambda),k}.
\end{aligned}$$

Since $H \in \mathbb{H}(\lambda)$, (4.6) holds. Suppose that $m = j_{\iota(H)} \neq \ell(\lambda)$. Since $\ell(\lambda) < j_{\iota(H)}$ and $H \in \mathbb{H}(\lambda)$, we have the following from Definition 4.3.

$$\begin{aligned}
\xi_{j_{\iota(H)}} - \sum_{k=1}^{j_{\iota(H)}-1} V_{k,j_{\iota(H)}} &= \mu_{j_{\iota(H)}} - 1 - \left(\sum_{k\neq \ell(\lambda)} U_{k,j_{\iota(H)}} + U_{\ell(\lambda),j_{\iota(H)}} - 1 \right) \\
&= \mu_{j_{\iota(H)}} - \sum_{k=1}^{j_{\iota(H)}-1} U_{k,j_{\iota(H)}} \\
&= \lambda_{j_{\iota(H)}} - \sum_{k=j_{\iota(H)}+1}^{n} U_{j_{\iota(H)},k} \\
&= \nu_{j_{\iota(H)}} - \sum_{k=j_{\iota(H)}+1}^{n} V_{j_{\iota(H)},k}.
\end{aligned}$$

Then (4.6) holds. Thus, $\iota(H)$ is an integer hive. $\square$

Proposition 4.8. Let $\lambda = \sum_{i\in I} \lambda_i \epsilon_i \in P^+$ with $\lambda \neq 0$. Let $\nu = \sum_{i\in I} \nu_i \epsilon_i \in P^+$, where $\nu_{\ell(\lambda)} = \lambda_{\ell(\lambda)} - 1$, $\nu_k = \lambda_k$ if $k \neq \ell(\lambda)$. Then, ι is a map from $\mathbb{H}(\lambda)$ to $\mathbb{H}(\nu)$.

Proof. Let $\lambda = \sum_{i\in I} \lambda_i \epsilon_i \in P^+$ with $\lambda \neq 0$. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let $\nu = \sum_{i\in I} \nu_i \epsilon_i \in P^+$, where $\nu_{\ell(\lambda)} = \lambda_{\ell(\lambda)} - 1$, $\nu_k = \lambda_k$ if $k \neq \ell(\lambda)$. By Definition 4.3, we can assume that $\iota(H) = (\nu, \xi, 0, (V_{ij})_{i<j})$. By Lemma 4.7, we know that $\iota(H)$ is an integer hive, and then it suffices to show that $\iota(H)$ is a K-hive.

By Definition 4.3, $\nu \in P^+$, $\xi \in P$, and $V_{kl} \geq 0$ hold. For $1 \leq i < j \leq n$, set $L_{ij} = \sum_{k=1}^{j-1} U_{ik} - \sum_{k=1}^{j} U_{i+1,k}$, $L'_{ij} = \sum_{k=1}^{j-1} V_{ik} - \sum_{k=1}^{j} V_{i+1,k}$. Then we show that $L'_{ij} \geq 0$. If $i \neq \ell(\lambda) - 1, \ell(\lambda)$, $j < j_{\iota(H)}$, or $(i,j) = (\ell(\lambda), j_{\iota(H)})$, we have $L'_{ij} = L_{ij} \geq 0$ since $U_{kl} = V_{kl}$ unless $(k,l) = (\ell(\lambda), j_{\iota(H)})$. Suppose that $i = \ell(\lambda) - 1$. In this case, it suffices to consider the case where $j \geq j_{\iota(H)}$. By Definition 4.3, we have the following.

$$\begin{aligned}
L'_{\ell(\lambda)-1,j} &= \sum_{k=1}^{j-1} V_{\ell(\lambda)-1,k} - \sum_{k=1}^{j} V_{\ell(\lambda),k} \\
&= \sum_{k=1}^{j-1} U_{\ell(\lambda)-1,k} - \left(\sum_{k\neq j_{\iota(H)}} U_{\ell(\lambda),k} + U_{\ell(\lambda),j_{\iota(H)}} - 1 \right) \\
&= \sum_{k=1}^{j-1} U_{\ell(\lambda)-1,k} - \sum_{k=1}^{j} U_{\ell(\lambda),k} + 1 \\
&= L_{\ell(\lambda)-1,j} + 1 \geq 0.
\end{aligned}$$

Suppose that $i = \ell(\lambda)$. In this case, it suffices to consider the case where $j > j_{\iota(H)}$. By Definition 4.3, we have the following.

$$\begin{aligned}
L'_{\ell(\lambda),j} &= \sum_{k=1}^{j-1} V_{\ell(\lambda),k} - \sum_{k=1}^{j} V_{\ell(\lambda)+1,k} \\
&= \sum_{k\neq j_{\iota(H)}} U_{\ell(\lambda),k} + (U_{\ell(\lambda),j_{\iota(H)}} - 1) - \sum_{k=1}^{j} U_{\ell(\lambda)+1,k} \\
&= L_{\ell(\lambda),j} - 1.
\end{aligned}$$

By Remark 2.11 and the definition of $\ell(\lambda)$, we have $\sum_{k=1}^{j} U_{\ell(\lambda)+1,k} = 0$. It follows that

$$\begin{aligned}
L_{\ell(\lambda),j} &= \sum_{k=1}^{j-1} U_{\ell(\lambda),k} - \sum_{k=1}^{j} U_{\ell(\lambda)+1,k} \\
&= \sum_{k\neq j_{\iota(H)}} U_{\ell(\lambda),k} + U_{\ell(\lambda),j_{\iota(H)}} > 0.
\end{aligned}$$

This implies $L'_{\ell(\lambda),j} \geq 0$. Thus, $L'_{ij} \geq 0$ holds for $1 \leq i < j \leq n$.

Finally, we show that $\xi_m - \sum_{k=1}^{m-1} V_{km} \geq 0$ for $m \in [n]$. If $m \neq j_{\iota(H)}$,

$$\xi_m = \mu_m \geq \sum_{k=1}^{m-1} U_{km} = \sum_{k=1}^{m-1} V_{km}$$

holds by Definition 4.3. Suppose $m = j_{\iota(H)}$. By the proof of Lemma 4.7, we have

$$\xi_{j_{\iota(H)}} - \sum_{k=1}^{j_{\iota(H)}-1} V_{k,j_{\iota(H)}} = \begin{cases} \mu_{j_{\iota(H)}} - 1 - \left(\sum_{k\neq\ell(\lambda)} U_{k,j_{\iota(H)}} + U_{\ell(\lambda),j_{\iota(H)}} - 1\right) & \text{if } \ell(\lambda) < j_{\iota(H)}, \\ \mu_{j_{\iota(H)}} - 1 - \sum_{k=1}^{j_{\iota(H)}-1} U_{k,j_{\iota(H)}} & \text{if } \ell(\lambda) = j_{\iota(H)}, \end{cases}$$

$$= \begin{cases} \mu_{j_{\iota(H)}} - \sum_{k=1}^{j_{\iota(H)}-1} U_{k,j_{\iota(H)}} & \text{if } \ell(\lambda) < j_{\iota(H)}, \\ \mu_{j_{\iota(H)}} - \sum_{k=1}^{j_{\iota(H)}-1} U_{k,j_{\iota(H)}} - 1 & \text{if } \ell(\lambda) = j_{\iota(H)}. \end{cases}$$

If $\ell(\lambda) = j_{\iota(H)}$, by the definition of U_{ii} $(i \in [n])$ and the choice of $j_{\iota(H)}$, we have

$$U_{j_{\iota(H)},j_{\iota(H)}} = \mu_{j_{\iota(H)}} - \sum_{k=1}^{j_{\iota(H)}-1} U_{k,j_{\iota(H)}} > 0.$$

Thus, we have $\xi_m - \sum_{k=1}^{m-1} V_{km} \geq 0$ for $m \in [n]$. Therefore, $\iota(H) \in \mathbb{H}(\nu)$. □

We also define another operator on $\mathbb{H}(\lambda)$, which is a path operator as we will see later. The operator is defined with a sequence of indices of $H \in \mathbb{H}(\lambda)$, and we define it first. For $n, m \in \mathbb{Z}$, let $[n, m]_{\mathbb{Z}} = \{l \in \mathbb{Z} \mid n \leq l \leq m\}$.

Definition 4.9. Let $\lambda \in P^+$ and let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let $a \in [n]$. Set $(i_0, j_0) = (0, a)$. For $k \geq 1$, set

$$i_k = \begin{cases} i_{k-1} & \text{if } k \in 2\mathbb{Z}, \\ i_{k-1} + 1 & \text{if } k \in 2\mathbb{Z} + 1, \end{cases}$$

$$j_k = \begin{cases} \min\{j \in [j_{k-1} + 1, n]_{\mathbb{Z}} \mid U_{i_{k-1},j} > 0\} & \text{if } k \in 2\mathbb{Z}, \\ j_{k-1} & \text{if } k \in 2\mathbb{Z} + 1. \end{cases}$$

Let N be the minimum $k \in \mathbb{Z}$ such that $\{j \in [j_{k-1} + 1, n]_{\mathbb{Z}} \mid U_{i_{k-1},j} > 0\} = \varnothing$. Then set $j_N = n + 1$. Set $p_{a,k} = (i_k, j_k)$. Then we define

$$p_a(H) = (p_{a,k})_{k=0,\ldots,N}.$$

Remark 4.10. Let $\lambda \in P^+$ and let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let $a \in [n]$. Let $p_a(H) = (p_{a,m})_{m=0,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$. Then, $p_{a,m} = (i_m, j_m)$ for $m \neq 0, N$ represents the upright rhombus in $R_{i_m} \cap L_{j_m}$. Also, $p_{a,0} = (0, a)$ represents the a-th bottom boundary edge, and $p_{a,N} = (i_N, n + 1)$ represents the i_N-th right boundary edge. Definition 4.11 defines an operator on $\mathbb{H}(\lambda)$ which is obtained by increasing or decreasing the rhombus gradients and the boundary edge labels determined by $p_a(H)$.

Definition 4.11. Let $\lambda \in P^+$ and let $H \in \mathbb{H}(\lambda)$. Let $a \in [n]$. Let $p_a(H) = (p_{a,m})_{m=0,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$. The operator ρ_a on $\mathbb{H}(\lambda)$ is defined by $\rho_a(H) = (\nu, \xi, 0, (V_{kl})_{k<l})$, where

$$\nu_k = \begin{cases} \lambda_k + 1 & \text{if } k = i_N, \\ \lambda_k & \text{otherwise}, \end{cases}$$

$$\xi_k = \begin{cases} \mu_k + 1 & \text{if } k = j_0, \\ \mu_k & \text{otherwise}, \end{cases}$$

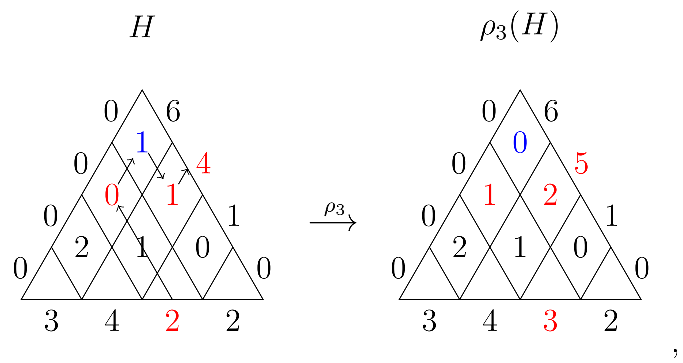
and for $1 \leq k < l \leq n$,

$$V_{kl} = \begin{cases} U_{kl} - 1 & \text{if } (k,l) = p_{a,m} \text{ for some } m \in 2\mathbb{Z}, \\ U_{kl} + 1 & \text{if } (k,l) = p_{a,m} \text{ for some } m \in 2\mathbb{Z}+1, \\ U_{kl} & \text{otherwise.} \end{cases}$$

If $i_N = n$, reduce $\nu_k, \xi_k (k \in [n])$ by 1.

Remark 4.12. The operator ρ_a is considered as a path operator as follows. Let $\lambda \in P^+$ and let $H \in \mathbb{H}(\lambda)$. Let $a \in [n]$. Then $\rho_a(H)$ is obtained by increasing or decreasing the boundary edge labels and the rhombus gradients specified by the path determined by $p_a(H)$. Hence, ρ_a is a path operator. Note that $\rho_a(H)$ is a K-hive according to Proposition 4.18.

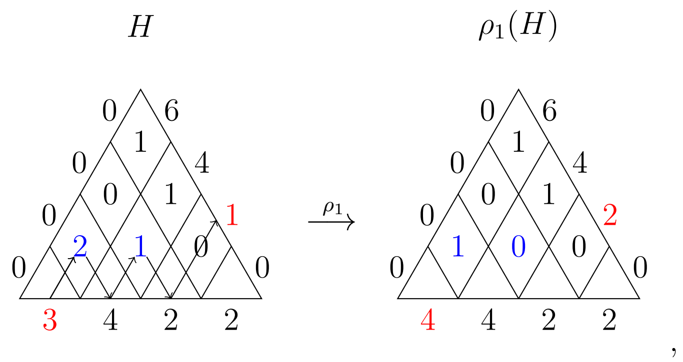
Example 4.13. Let $n = 4$ and $\lambda = (6, 4, 1, 0) \in P^+$. Let $H = (\lambda, \mu, \nu, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$ as shown on the left of FIGURE 12, and then the path on H specified by $p_3(H)$ is illustrated in blue and red. Then the action of ρ_3 for H is as shown in FIGURE 12.



$$p_3(H) = \{(0,3), (1,3), (1,4), (2,4), (2,5)\}.$$

FIGURE 12. Action of ρ_3

Example 4.14. Let $n = 4$ and $\lambda = (6, 4, 1, 0) \in P^+$. Let $H = (\lambda, \mu, \nu, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$ as shown on the left of FIGURE 12, and then the path on H specified by $p_1(H)$ is illustrated in red and blue. Then the action of ρ_1 for H is as shown in FIGURE 13.



$$p_1(H) = \{(0,1), (1,1), (1,2), (2,2), (2,3), (3,3), (3,5)\}.$$

FIGURE 13. Action of ρ_1

In Examples 4.13, 4.14, we can confirm that the action of ρ_a generates a K-hive. As we will see in the following, this observation holds in general.

Lemma 4.15. Let $\lambda \in P^+$ and $H \in \mathbb{H}(\lambda)$. Let $a \in [n]$ and $p_a(H) = (p_{a,m})_{m=0,1,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$. Suppose $i_N = n$. Then $\rho_a(H)$ is an integer hive.

Proof. Let $\lambda \in P^+$ and let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let $a \in [n]$. Let $p_a(H) = (p_{a,m})_{m=0,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$. Suppose $i_N = n$. Let $\rho_a(H) = (\nu, \xi, 0, (V_{ij})_{i<j})$. We show that

$$\xi_m - \sum_{k=1}^{m-1} V_{km} = \nu_m - \sum_{k=m+1}^{n} V_{mk} \quad (m = 1, 2, \ldots, n). \tag{4.7}$$

By Definition 4.11, we have $\nu_k = \lambda_k$ if $k = n$, otherwise $\nu_k = \lambda_k - 1$, and $\xi_k = \mu_k$ if $k = a$, otherwise $\xi_k = \mu_k - 1$. Since $i_N = n$, $i_k = j_k$ for $k \in [0, N]_{\mathbb{Z}} \cap (2\mathbb{Z} + 1)$, and $i_k = j_k - 1$ for $k \in [0, N]_{\mathbb{Z}} \cap 2\mathbb{Z}$ by Definition 4.9. In particular, $a = 1$ holds. This implies that $\sum_{k=1}^{m-1} V_{km} = \sum_{k=1}^{m-1} U_{km} - 1$ and $\sum_{k=m+1}^{n} V_{mk} = \sum_{k=m+1}^{n} U_{mk} - 1$. Then we have

$$\xi_m - \sum_{k=1}^{m-1} V_{km} = \mu_m - \sum_{k=1}^{m-1} U_{km},$$
$$\nu_m - \sum_{k=m+1}^{n} V_{mk} = \lambda_m - \sum_{k=m+1}^{n} U_{mk}.$$

Since $H \in \mathbb{H}(\lambda)$, we have (4.7). □

Lemma 4.16. Let $\lambda \in P^+$ and $H \in \mathbb{H}(\lambda)$. Let $a \in [n]$. Let $p_a(H) = (p_{a,m})_{m=0,1,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$. Suppose $i_N \neq n$. Then $\rho_a(H)$ is an integer hive.

Proof. Let $\lambda \in P^+$ and let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let $a \in [n]$. Let $p_a(H) = (p_{a,m})_{m=0,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$. Suppose $i_N \neq n$. Let $\rho_a(H) = (\nu, \xi, 0, (V_{ij})_{i<j})$. We show that

$$\xi_m - \sum_{k=1}^{m-1} V_{km} = \nu_m - \sum_{k=m+1}^{n} V_{mk} \quad (m = 1, 2, \ldots, n). \tag{4.8}$$

By Definition 4.11, we have $\nu_k = \lambda_k + 1$ if $k = i_N$, otherwise $\nu_k = \lambda_k$, and $\xi_k = \mu_k + 1$ if $k = a$, otherwise $\xi_k = \mu_k$. We first consider the left side of (4.8). If $m = a$, then

$$\sum_{k=1}^{a-1} V_{ka} = \begin{cases} \sum_{k=1}^{a-1} U_{ka} + 1 & \text{if } a \neq 1, \\ \sum_{k=1}^{a-1} U_{ka} & \text{otherwise} \end{cases}$$

since $V_{1a} = U_{1a} + 1$ and $V_{ka} = U_{ka}$ if $k \neq 1$. Suppose $m \neq a$. If there exists $l \in [0, N]_{\mathbb{Z}}$ such that $j_l = m$, let l_0 be the smallest such l. It follows from Definition 4.9 that $l_0 \in 2\mathbb{Z}$, $(i_{l_0+1}, j_{l_0+1}) = (i_{l_0} + 1, j_{l_0})$, and $j_l \neq m$ if $k \neq l_0, l_0 + 1$. This implies that

$$\sum_{k=1}^{m-1} V_{km} = \begin{cases} \sum_{k=1}^{m-1} U_{km} - 1 & \text{if } i_{l_0} = m - 1, \\ \sum_{k=1}^{m-1} U_{km} & \text{otherwise.} \end{cases}$$

Note that the case where $m = a$ and $a = 1$ and the case where $m \neq a$ and $i_{l_0} = m - 1$ means that there is $l \in [0, N]_{\mathbb{Z}}$ such that $i_l = j_l = m$. Then we have

$$\xi_m - \sum_{k=1}^{m-1} V_{km} = \begin{cases} \mu_m - \sum_{k=1}^{m-1} U_{km} + 1 & \text{if there is } l \text{ such that } i_l = j_l, \\ \mu_m - \sum_{k=1}^{m-1} U_{km} & \text{otherwise.} \end{cases}$$

Then we consider the right side of (4.8). If $m = i_N$, then

$$\sum_{k=i_N+1}^{n} V_{i_N,k} = \begin{cases} \sum_{k=i_N+1}^{n} U_{i_N,k} & \text{if } j_{N-1} = i_N, \\ \sum_{k=i_N+1}^{n} U_{i_N,k} + 1 & \text{otherwise.} \end{cases}$$

Note that $i_{N-1} = i_N$ holds by Definition 4.9. Suppose $m \neq i_N$. If there exists $k \in [0, N]_{\mathbb{Z}}$ such that $i_k = m$, let k_0 be the smallest such k. It follows from Definition 4.9 that $k_0 \in 2\mathbb{Z}+1$, $(i_{k_0+1}, j_{k_0+1}) = (i_{k_0}, j_{k_0+1})$, and $i_k \neq m$ if $k \neq k_0, k_0+1$. This implies that

$$\sum_{k=m+1}^{n} V_{m,k} = \begin{cases} \sum_{k=m+1}^{n} U_{m,k} - 1 & \text{if } j_{k_0} = m, \\ \sum_{k=m+1}^{n} U_{m,k} & \text{otherwise.} \end{cases}$$

Note that the case where $m = i_N$ and $j_{N-1} = i_N$ and the case where $m \neq i_N$ and $j_{k_0} = m$ means that there is $k \in [0, N]_{\mathbb{Z}}$ such that $i_k = j_k = m$. Then we have

$$\nu_m - \sum_{k=m+1}^{n} V_{m,k} = \begin{cases} \lambda_m - \sum_{k=m+1}^{n} U_{m,k} + 1 & \text{if there is } k \text{ such that } i_k = j_k = m, \\ \lambda_m - \sum_{k=m+1}^{n} U_{m,k} & \text{otherwise.} \end{cases}$$

Thus, we have (4.8). □

By Lemmas 4.15, 4.16, we immediately obtain the following.

Lemma 4.17. Let $\lambda \in P^+$ and $H \in \mathbb{H}(\lambda)$. Let $a \in [n]$. Then $\rho_a(H)$ is an integer hive.

Proposition 4.18. Let $\lambda \in P^+$. Let $a \in [n]$. Then we have that ρ_a is a map from $\mathbb{H}(\lambda)$ to $\bigsqcup_{\nu \in P^+} \mathbb{H}(\nu)$.

Proof. Let $\lambda = \sum_{i \in I} \lambda_i \epsilon_i \in P^+$. Let $a \in [n]$. We show that for all $H \in \mathbb{H}(\lambda)$ there exist $\nu \in P^+$ such that $\rho_a(H) \in \mathbb{H}(\nu)$. Fix $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. Let $p_a(H) = (p_{a,m})_{m=0,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$. Let $\rho_a(H) = (\nu, \xi, 0, (V_{ij})_{i<j})$. By Lemma 4.26, we know $\rho_a(H)$ is an integer hive. Then we show that $\rho_a(H)$ is a K-hive.

First, we show $\nu \in P^+$. If $i_N = n$, then $\nu \in P^+$ is trivial. Suppose $i_N \neq n$. It suffices to show that $\nu_{i_N-1} \geq \nu_{i_N}$, which implies that $\lambda_{i_N-1} > \lambda_{i_N}$. Suppose $\lambda_{i_N-1} = \lambda_{i_N}$. By the choice of p_{a,i_N}, $U_{i_N,j} = 0$ if $j > j_{N-1}$. Then we have that

$$\lambda_{i_N} = \sum_{k=1}^{n} U_{i_N,j} = \sum_{k=1}^{j_{N-1}} U_{i_N,j}.$$

Note that $i_{N-1} = i_N - 1$ holds by Definition 4.9. For $1 \leq i < j \leq n$, set $L_{ij} = \sum_{k=1}^{j-1} U_{ik} - \sum_{k=1}^{j} U_{i+1,k}$. It follows that

$$\begin{aligned} L_{i_{N-1},j_{N-1}} &= \sum_{k=1}^{j_{N-1}-1} U_{i_{N-1},k} - \sum_{k=1}^{j_{N-1}} U_{i_{N-1}+1,k} \\ &= \sum_{k=1}^{j_{N-1}-1} U_{i_N-1,k} - \sum_{k=1}^{j_{N-1}} U_{i_N,k} \\ &= \sum_{k=1}^{j_{N-1}-1} U_{i_N-1,k} - \lambda_{i_N} \\ &< \lambda_{i_N-1} - \lambda_{i_N} = 0. \end{aligned}$$

This is a contradiction for $H \in \mathbb{H}(\lambda)$. Then we obtain $\lambda_{i_N-1} > \lambda_{i_N}$, hence we have $\nu_{i_N-1} \geq \nu_{i_N}$. By Definition 4.9 and Definition 4.11, $\xi \in P$ and $V_{ij} \geq 0$ is clear.

For $1 \le i < j \le n$, set $L'_{ij} = \sum_{k=1}^{j-1} V_{ik} - \sum_{k=1}^{j} V_{i+1,k}$. Next, we show that $L'_{ij} \ge 0$. If $i > i_N$, then $V_{ij} = U_{ij}$ $(i < j)$ by Definition 4.11, hence $L'_{ij} = L_{ij}$ $(i < j)$. If $i = i_N$, then $\sum_{k=1}^{j-1} V_{i_N,k} = \sum_{k=1}^{j-1} U_{i_N,k} + 1$ if $j - 1 \ge j_{N-1}$, otherwise $\sum_{k=1}^{j-1} V_{i_N,k} = \sum_{k=1}^{j-1} U_{i_N,k} + 1$. Then $L'_{i_N,j} \ge L_{i_N,j} \ge 0$ holds. Suppose $i < i_N$. In this case, from Definition 4.9 it follows that there exists $k \in [0, N]_{\mathbb{Z}}$ such that $i_k = i$. Let k_0 be the smallest such k. For $l = 1, 2, 3$, set $k_l = k_0 + l$. By Definition 4.11, $k_0 \in 2\mathbb{Z} + 1$, $i_{k_0} = i_{k_1} = i$, $i_{k_2} = i_{k_3} = i + 1$, and $j_{k_0} < j_{k_1} = j_{k_2} < j_{k_3}$ holds. Then we have the following.

$$\sum_{k=1}^{j-1} V_{ik} = \begin{cases} \sum_{k=1}^{j-1} U_{ik} + 1 & \text{if } j_{k_0} < j \le j_{k_1}, \\ \sum_{k=1}^{j-1} U_{ik} & \text{otherwise,} \end{cases}$$

$$\sum_{k=1}^{j} V_{i+1,k} = \begin{cases} \sum_{k=1}^{j-1} U_{ik} + 1 & \text{if } j_{k_1} \le j < j_{k_3}, \\ \sum_{k=1}^{j-1} U_{ik} & \text{otherwise.} \end{cases}$$

Hence, $L'_{ij} \ge 0$ is clear unless $j_{k_1} < j < j_{k_3}$. Suppose that $j_{k_1} < j < j_{k_3}$. In this case, $L'_{ij} = L_{ij} - 1$, and then we would like to show $L_{ij} > 0$. By the construction of $p_a(H)$, we know that $U_{i+1,j} = 0$ if $j_{k_1} < j < j_{k_3}$ and $U_{i,j_{k_1}} > 0$. It follows that

$$\begin{aligned} L_{ij} &= \sum_{k=1}^{j-1} U_{ik} - \sum_{k=1}^{j} U_{i+1,k} \\ &= \sum_{k=1}^{j-1} U_{ik} - \sum_{k=1}^{j_{m_1}} U_{i+1,k} \\ &> L_{i,j_{k_1}} \ge 0. \end{aligned}$$

Thus, $L'_{ij} \ge 0$ holds for $1 \le i < j \le n$.

Finally, we show that $\xi_m - \sum_{k=1}^{m-1} V_{km} \ge 0$ for $m \in [n]$. If $i_N \ne n$, then $\xi_m - \sum_{k=1}^{m-1} V_{km} \ge \mu_m - \sum_{k=1}^{m-1} U_{km}$ holds by the proof of Lemma 4.17. If $i_N = n$, then $i_k = j_k$ holds for $k \in [0, N]_{\mathbb{Z}} \cap (2\mathbb{Z} + 1)$ by Definition 4.9. In particular, $a = 1$ holds. It follows that $U_{mm} = \mu_m - \sum_{k=1}^{m-1} U_{km} > 0$, hence $V_{mm} = \xi_m - \sum_{k=1}^{m-1} V_{km} \ge 0$ holds. Therefore, $\rho_a(H)$ is a K-hive. □

Remark 4.19. By Remark 2.13, the operator ρ_a can be viewed as an analogue to the operation for a semi-standard tableau which inserts a box with a into the 1st row, then move the leftmost box in the 1st row containing a number greater than a to the 2nd row, and so on. This means that ρ_a is an analogue of the insertion algorithm in K-hives.

There exists a path operator which can be viewed as a reverse operator of ρ_a defined as follows.

Definition 4.20. Let $\lambda \in P^+$. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. If $\lambda \ne 0$, then let $b \in \{i \in I \mid \lambda_i \ne 0\} \cup \{n\}$, otherwise let $b = n$. Set $(\bar{i}_0, \bar{j}_0) = (b, n + 1)$. For $k \ge 1$, set

$$\bar{i}_k = \begin{cases} \bar{i}_{k-1} - 1 & \text{if } k \in 2\mathbb{Z}, \\ \bar{i}_{k-1} & \text{if } k \in 2\mathbb{Z} + 1, \end{cases}$$

$$\bar{j}_k = \begin{cases} \bar{j}_{k-1} & \text{if } k \in 2\mathbb{Z}, \\ \max\{j \in [1, \bar{j}_{k-1} - 1]_{\mathbb{Z}} \mid U_{\bar{i}_{k-1},j} > 0\} & \text{if } k \in 2\mathbb{Z} + 1. \end{cases}$$

Let M be the minimum $k \in [0, M]_{\mathbb{Z}}$ such that $i_k = 0$. Set $\bar{p}_{b,k} = (\bar{i}_k, \bar{j}_k)$. Then we define

$$\bar{p}_b(H) = (\bar{p}_{b,k})_{k=0,\ldots,M}.$$

Remark 4.21. Let $\lambda \in P^+$. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. If $\lambda \neq 0$, then let $b \in \{i \in I \mid \lambda_i \neq 0\} \cup \{n\}$, otherwise let $b = n$. Let $\bar{p}_b(H) = (\bar{p}_{b,m})_{m=0,1,\ldots,M}$, where $\bar{p}_{b,m} = (\bar{i}_m, \bar{j}_m)$. Then, $\bar{p}_{b,m} = (\bar{i}_m, \bar{j}_m)$ for $m \neq 0, M$ represents the upright rhombus in $R_{\bar{i}_m} \cap L_{\bar{j}_m}$. Also, $\bar{p}_{b,0} = (b, n+1)$ represents the b-th right boundary edge, and $\bar{p}_{b,M} = (0, \bar{j}_M)$ represents the $\bar{j}_M$-th bottom boundary edge. Definition 4.22 defines an operator on $\mathbb{H}(\lambda)$ which is obtained by increasing or decreasing the rhombus gradients and the boundary edge labels determined by $\bar{p}_b(H)$.

Definition 4.22. Let $\lambda \in P^+$. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. If $\lambda \neq 0$, then let $b \in \{i \in I \mid \lambda_i \neq 0\} \cup \{n\}$, otherwise let $b = n$. If $b = n$, then increase λ_k, μ_k $(k \in [n])$ by 1. Let $\bar{p}_b(H) = (\bar{p}_{b,m})_{m=0,1,\ldots,M}$, where $\bar{p}_{b,m} = (\bar{i}_m, \bar{j}_m)$. An operator $\bar{\rho}_b$ on $H \in \mathbb{H}(\lambda)$ is defined by $\bar{\rho}_b(H) = (\nu, \xi, 0, (V_{kl})_{k<l})$ as follows. Then

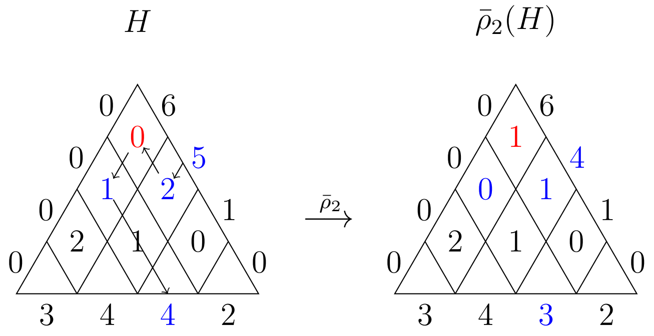
$$\nu_k = \begin{cases} \lambda_k - 1 & \text{if } k = \bar{i}_0, \\ \lambda_k & \text{otherwise}, \end{cases}$$
$$\xi_k = \begin{cases} \mu_k - 1 & \text{if } k = \bar{j}_M, \\ \mu_k & \text{otherwise}, \end{cases}$$

and for $1 \leq k < l \leq n$,

$$V_{kl} = \begin{cases} U_{kl} + 1 & \text{if } (k, l) = \bar{p}_{b,m} \text{ for some } m \in 2\mathbb{Z}, \\ U_{kl} - 1 & \text{if } (k, l) = \bar{p}_{b,m} \text{ for some } m \in 2\mathbb{Z} + 1, \\ U_{kl} & \text{otherwise}. \end{cases}$$

Remark 4.23. The operator $\bar{\rho}_b$ is considered as a path operator as follows. Let $\lambda \in P^+$ and let $H \in \mathbb{H}(\lambda)$. If $\lambda \neq 0$, then let $b \in \{i \in I \mid \lambda_i \neq 0\} \cup \{n\}$, otherwise let $b = n$. Then $\bar{\rho}_b(H)$ is obtained by increasing or decreasing the boundary edge labels and the rhombus gradients specified by the path determined by $\bar{p}_b(H)$. Therefore, $\bar{\rho}_b$ is a path operator if $\bar{\rho}_b(H)$ is a K-hive. Note that $\bar{\rho}_b(H)$ is a K-hive under some conditions (Proposition 4.27).

Example 4.24. Let $n = 4$ and $\lambda = (6, 5, 1, 0) \in P^+$. Let $H = (\lambda, \mu, \nu, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$ as shown on the left of FIGURE 14, and then the path on H specified by $\bar{p}_2(H)$ is illustrated in blue and red in the figure. Then the action of $\bar{\rho}_2$ for H is as shown in FIGURE 14.

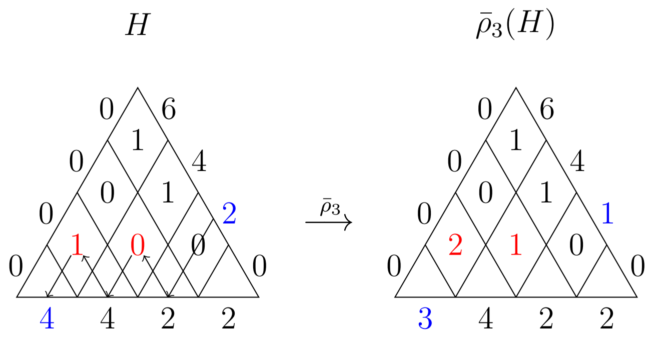


,

$$\bar{p}_2(H) = \{(2,5),(2,4),(1,4),(1,3),(0,3)\}.$$

FIGURE 14. Action of $\bar{\rho}_2$

Example 4.25. Let $n = 4$ and $\lambda = (6,4,3,0) \in P^+$. Let $H = (\lambda, \mu, \nu, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$ as shown on the left of FIGURE 15, and then the path on H specified by $\bar{p}_3(H)$ is illustrated in blue in the figure. Then the action of $\bar{\rho}_3$ for H is as shown in FIGURE 15.



,

$$\bar{p}_3(H) = \{(3,5),(3,3),(2,3),(2,2),(1,2),(1,1),(0,1)\}.$$

FIGURE 15. Action of $\bar{\rho}_3$

In Example 4.24, 4.25, the action of $\bar{\rho}_2$ generates a K-hive. Moreover, this action can be viewed as an inverse operator of ρ_a, see Example 4.13, 4.14. As we see in the following, this observation generally holds under some conditions.

Lemma 4.26. Let $\lambda \in P^+$. If $\lambda \neq 0$, then let $b \in \{i \in I \mid \lambda_i \neq 0\} \cup \{n\}$, otherwise let $b = n$. Then, $\bar{\rho}_b(H)$ is an integer hive.

Proof. Let $\lambda \in P^+$ and $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. If $\lambda \neq 0$, then let $b \in \{i \in I \mid \lambda_i \neq 0\} \cup \{n\}$, otherwise let $b = n$. Let $\bar{\rho}_b(H) = (\nu, \xi, 0, (V_{ij})_{i<j})$ and let $\bar{p}_b(H) = (\bar{p}_{b,m})_{m=0,\dots,M}$, where $\bar{p}_{b,m} = (\bar{i}_m, \bar{j}_m)$. Note that if $b = n$, then we consider $\lambda_k, \mu_k\,(k \in [n])$ by increasing by 1. By Definition 4.20, $\nu_k = \lambda_k - 1$ if $k = b$ otherwise $\nu_k = \lambda_k$, and $\xi_k = \mu_k - 1$ if $k = \bar{j}_M$ otherwise $\xi_k = \mu_k$. We need to show that

$$\xi_m - \sum_{k=1}^{m-1} V_{km} = \nu_m - \sum_{k=m+1}^{n} V_{mk} \quad (m = 1, \dots, n). \tag{4.9}$$

We first consider the left side of (4.9). If $m = \bar{j}_M$, then

$$\sum_{k=1}^{\bar{j}_M-1} V_{k,\bar{j}_M} = \begin{cases} \sum_{k=1}^{\bar{j}_M-1} U_{k,\bar{j}_M} - 1 & \text{if } \bar{j}_M \neq 1, \\ \sum_{k=1}^{\bar{j}_M-1} U_{k,\bar{j}_M} & \text{otherwise} \end{cases}$$

since $V_{1,\bar{j}_M} = U_{1,\bar{j}_M} - 1$ and $V_{k,\bar{j}_M} = U_{k,\bar{j}_M}$ if $k \neq 1$. Suppose $m \neq j_M$. If there exists $l \in [0, M]_{\mathbb{Z}}$ such that $\bar{j}_l = m$, let l_0 be the smallest such l. It follows from Definition 4.20 that $l_0 \in 2\mathbb{Z}+1$, $(\bar{i}_{l_0+1}, \bar{j}_{l_0+1}) = (\bar{i}_{l_0} - 1, \bar{j}_{l_0})$, and $\bar{j}_l \neq m$ if $l \neq l_0, l_0+1$. This implies that

$$\sum_{k=1}^{m-1} V_{km} = \begin{cases} \sum_{k=1}^{m-1} U_{km} + 1 & \text{if } \bar{i}_{l_0} = m, \\ \sum_{k=1}^{m-1} U_{km} & \text{otherwise.} \end{cases}$$

Note that the case where $m = \bar{j}_M$ and $\bar{j}_M = 1$ and the case where $m \neq \bar{j}_M$ and $\bar{i}_{l_0} = m$ implies that there is $l \in [0, M]_{\mathbb{Z}}$ such that $\bar{i}_l = \bar{j}_l = m$. Then we have

$$\xi_m - \sum_{k=1}^{m-1} V_{km} = \begin{cases} \mu_m - \sum_{k=1}^{m-1} U_{km} - 1 & \text{if there is } l \text{ such that } \bar{i}_l = \bar{j}_l = m, \\ \mu_m - \sum_{k=1}^{m-1} U_{km} & \text{otherwise.} \end{cases}$$

Then we consider the right side of (4.9). If $m = b$, then

$$\sum_{k=m+1}^{n} V_{bk} = \begin{cases} \sum_{k=b+1}^{n} U_{bk} + 1 & \text{if } \bar{j}_1 = b, \\ \sum_{k=b+1}^{n} U_{bk} & \text{otherwise} \end{cases}$$

since $V_{b,\bar{j}_1} = U_{b,\bar{j}_1} - 1$ and $V_{bl} = U_{bl}$ if $l \neq \bar{j}_1$. Suppose $m \neq b$. If there exists $k \in [0, M]_{\mathbb{Z}}$ such that $\bar{i}_k = m$, let k_0 be the smallest such k. It follows from Definition 4.20 that $k_0 \in 2\mathbb{Z}$, $(\bar{i}_{k_0+1}, \bar{j}_{k_0+1}) = (\bar{i}_{k_0}, \bar{j}_{k_0+1})$, and $\bar{j}_k \neq m$ if $k = k_0, k_0+1$. This implies that

$$\sum_{k=m+1}^{n} V_{mk} = \begin{cases} \sum_{k=m+1}^{n} V_{mk} + 1 & \text{if } \bar{j}_{k_0+1} = m, \\ \sum_{k=m+1}^{n} V_{mk} & \text{otherwise.} \end{cases}$$

Note that the case where $m = b$ and $j_1 = b$ and the case where $m \neq b$, $\bar{j}_{k_0+1} = m$ implies that there is $k \in [0, M]_{\mathbb{Z}}$ such that $i_k = j_k = m$. Then we have

$$\nu_m - \sum_{k=m+1}^{n} V_{mk} = \begin{cases} \lambda_m - \sum_{k=m+1}^{n} V_{mk} - 1 & \text{if there is } k \text{ such that } i_k = j_k = m, \\ \lambda_m - \sum_{k=m+1}^{n} V_{mk} & \text{otherwise.} \end{cases}$$

Thus, we have Lemma 4.26. □

Proposition 4.27. Let $\lambda = \sum_{i \in I} \lambda_i \epsilon_i \in P^+$. If $\lambda \neq 0$, then let $b \in \{i \in I \mid \lambda_i \neq 0\} \cup \{n\}$, otherwise let $b = n$. If $b = n$, consider λ_k ($k \in [n]$) by increasing by 1. Let $\nu = \sum_{i \in I} \nu_i \epsilon_i \in P^+$, where $\nu_b = \lambda_b - 1$, $\nu_k = \lambda_k$ if $k \neq b$. Suppose that $\lambda_b > \lambda_{b+1}$. Then $\bar{\rho}_b$ is a map from $\mathbb{H}(\lambda)$ to $\mathbb{H}(\nu)$.

Proof. Let $\sum_{i \in I} \lambda_i \epsilon_i \in P^+$ and let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. If $\lambda \neq 0$, then let $b \in \{i \in I \mid \lambda_i \neq 0\} \cup \{n\}$, otherwise let $b = n$. Let $\bar{\rho}_b(H) = (\nu, \xi, 0, (V_{ij})_{i<j})$ and let $\bar{p}_b(H) = (\bar{p}_{b,m})_{m=0,\dots,M}$, where $\bar{p}_{b,m} = (\bar{i}_m, \bar{j}_m)$. Note that if $b = n$, then we consider λ_k, μ_k ($k \in [n]$) by increasing by 1. Let $\nu = \sum_{i \in I} \nu_i \epsilon_i$, then it follows from Definition 4.22 that $\nu_b = \lambda_b - 1$ and $\nu_k = \lambda_k$ if $k \neq b$.

By Lemma 4.26, $\bar{\rho}_b(H)$ is an integer hive. Suppose that $\lambda_b > \lambda_{b+1}$. Then we show that $\bar{\rho}_b(H)$ is a K-hive.

Since $\lambda \in P^+$ and $\lambda_b > \lambda_{b+1}$, we have $\nu \in P^+$. It follows from Definition 4.22 that $\xi \in P$ and $V_{ij} \geq 0$.

For $1 \le k < l \le n$, set $L_{kl} = \sum_{m=1}^{l-1} U_{ml} - \sum_{m=1}^{l} U_{k+1,m}$, and $L'_{kl} = \sum_{m=1}^{l-1} V_{ml} - \sum_{m=1}^{l} V_{k+1,m}$. We would like to show $L'_{kl} \ge 0$. If $k > \bar{i}_0$, then $V_{kl} = U_{kl}$ holds for $k < l$ by Definition 4.22. Then we see $L'_{kl} = L_{kl} \ge 0$. Suppose $k = \bar{i}_0$. By Definition 4.22, we have that $V_{\bar{i}_0,\bar{j}_1} = U_{\bar{i}_0,\bar{j}_1} - 1$ and $V_{\bar{i}_0,l} = U_{\bar{i}_0,l}$ if $l \ne \bar{j}_1$. Note that $\bar{i}_0 = \bar{i}_1$ by Definition 4.20. Then we obtain the following.

$$
\begin{aligned}
L'_{\bar{i}_0,l} &= \sum_{m=1}^{l-1} V_{\bar{i}_0,m} - \sum_{m=1}^{l} V_{\bar{i}_0+1,m} \\
&= \begin{cases} \sum_{m=1}^{l-1} U_{\bar{i}_0,m} - \sum_{m=1}^{l} U_{\bar{i}_0+1,m} & \text{if } l-1 < \bar{j}_1, \\ \sum_{m\ne \bar{j}_1}^{l-1} U_{\bar{i}_0,m} + (U_{\bar{i}_0,\bar{j}_1} - 1) - \sum_{m=1}^{l} U_{\bar{i}_0+1,m} & \text{if } l-1 \ge \bar{j}_1, \end{cases} \\
&= \begin{cases} L_{\bar{i}_0,l} & \text{if } l-1 < \bar{j}_1, \\ L_{\bar{i}_0,l} - 1 & \text{if } l-1 \ge \bar{j}_1. \end{cases}
\end{aligned}
$$

Since $U_{\bar{i}_0,m} = 0$ holds for $m > \bar{j}_1$ by the choice of $\bar{j}_1$,

$$
\lambda_{\bar{i}_0} = \sum_{m=1}^{n} U_{\bar{i}_0,m} = \sum_{m=1}^{\bar{j}_1} U_{\bar{i}_0,m}.
$$

Then if $l - 1 \ge \bar{j}_1$,

$$
\begin{aligned}
L_{\bar{i}_0,l} &= \sum_{m=1}^{l-1} U_{\bar{i}_0,m} - \sum_{m=1}^{l} U_{\bar{i}_0+1,m} \\
&= \sum_{m=1}^{\bar{j}_1} U_{\bar{i}_0,m} - \sum_{m=1}^{l} U_{\bar{i}_0+1,m} \\
&= \lambda_{\bar{i}_0} - \sum_{m=1}^{l} U_{\bar{i}_0+1,m} \\
&> \lambda_{\bar{i}_0} - \lambda_{\bar{i}_0+1} > 0.
\end{aligned}
$$

This implies $L'_{\bar{i}_0,l} \ge 0$. Suppose $k < \bar{i}_0$. By construction of $\bar{p}_b(H)$, there exists $m \in \mathbb{Z}$ such that $\bar{i}_m = k$, let m_0 be the smallest such m. For $l = -2, -1, 1$, set $m_l = m_0 + l$. By Definition 4.20, $m_0 \in 2\mathbb{Z}$, $\bar{i}_{m_0} = \bar{i}_{m_1} = k$, $\bar{i}_{m_{-1}} = \bar{i}_{m_{-2}} = k+1$, $\bar{j}_{m_1} < \bar{j}_{m_0} = \bar{j}_{m_{-1}} < \bar{j}_{m_{-2}}$. By Definition 4.22, we have the following.

$$
\sum_{m=1}^{l-1} V_{km} = \begin{cases} \sum_{k=1}^{l-1} U_{ik} - 1 & \text{if } \bar{j}_{m_1} < l \le \bar{j}_{m_0}, \\ \sum_{k=1}^{l-1} U_{ik} & \text{otherwise,} \end{cases}
$$

$$
\sum_{m=1}^{l} V_{k+1,m} = \begin{cases} \sum_{m=1}^{l} U_{k+1,m} - 1 & \text{if } \bar{j}_{m_0} \le l < \bar{j}_{m_{-2}}, \\ \sum_{m=1}^{l} U_{k+1,m} & \text{otherwise.} \end{cases}
$$

Therefore, $L'_{kl} \ge 0$ is clear unless $\bar{j}_{m_1} < l < \bar{j}_{m_0}$. Suppose $\bar{j}_{m_1} < l < \bar{j}_{m_0}$. In this case, $L'_{kl} = L_{kl} - 1$ holds, then it suffices to show that $L_{kl} > 0$. By the choice of $\bar{j}_{m_0}$, we know that $U_{kl} = 0$ for $\bar{j}_{m_1} < l < \bar{j}_{m_0}$. Then we obtain

$$
\begin{aligned}
L_{kl} &= \sum_{m=1}^{l-1} U_{km} - \sum_{m=1}^{l} U_{k+1,m} \\
&= \sum_{m=1}^{\bar{j}_{m_1}} U_{km} - \sum_{m=1}^{l} U_{k+1,m}.
\end{aligned}
$$

This implies that $L_{kl} \geq L_{k,l'}$ if $\bar{j}_{m_1} < l \leq l' \leq \bar{j}_{m_0}$. In particular, since $U_{k+1,\bar{j}_{m_0}} > 0$ by the choice of m_0, $L_{k,\bar{j}_{m_0}-1} > L_{k,\bar{j}_{m_0}}$. Thus, we have $L_{kl} \geq 0$ for $1 \leq k < l \leq n$.

By the proof of Lemma 4.26, it suffices to check the case where there exists $l \in [0,N]_{\mathbb{Z}}$ such that $\bar{i}_l = \bar{j}_l = m$. In this case, $m \in 2\mathbb{Z}+1$ holds by Definition 4.22, and hence $U_{mm} > 0$. Then $\mu_m - \sum_{k=1}^{m-1} U_{km} > 0$, which implies $\xi_m - \sum_{k=1}^{m-1} V_{km} \geq 0$. Therefore, $\bar{\rho}_b(H)$ is a K-hive. □

The relation between ρ_a and $\bar{\rho}_b$ can be viewed as an inverse operator under some conditions, as we see in the following. See Examples 4.13, 4.14, 4.24, 4.25.

Proposition 4.28. (1) Let $\lambda \in P^+$ and $H \in \mathbb{H}(\lambda)$. Let $a \in [n]$ and $p_a(H) = (p_{a,m})_{m=0,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$. Set $b = i_N$ and $K = \rho_a(H)$. Then, we obtain $\bar{\rho}_b(K) = H$.

(2) Let $\nu \in P^+$ and $K \in \mathbb{H}(\nu)$. If $\lambda \neq 0$, then let $b \in \{i \in I \mid \lambda_i \neq 0\} \cup \{n\}$, otherwise let $b = n$. Set $H = \bar{\rho}_b(K)$, and let $\bar{p}_b(K) = (\bar{p}_{b,m})_{m=0,\ldots,M}$, where $\bar{p}_{b,m} = (\bar{i}_m, \bar{j}_m)$. Set $a = \bar{j}_M$. Suppose that $H \in \mathbb{H}(\lambda)$ for some $\lambda \in P^+$. Then, we obtain $\rho_a(H) = K$.

Proof. (1) Let $\lambda \in P^+$ and $H = (\lambda, \mu, 0, (U_{kl})_{k<l}) \in \mathbb{H}(\lambda)$. Let $a \in [n]$ and $p_a(H) = (p_{a,m})_{m=0,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$. Set $K = \rho_a(H)$ and $b = i_N$. Let $K = (\nu, \xi, 0, (V_{kl})_{k<l}) \in \mathbb{H}(\nu)$. Let $\bar{p}_b(K) = (\bar{p}_{b,m})_{m=1,\ldots,M}$, where $\bar{p}_{b,m} = (\bar{i}_m, \bar{j}_m)$.

By Definition 4.11, we have $\nu_b = \lambda_b + 1$, $\nu_k = \lambda_k$ $(k \neq b)$. We would like to show $\rho_b(K) = H$, then it suffices to show that $p_a(H) = \bar{p}_b(K)$ by Definition 4.11 and Definition 4.22. Note that if $b = n$, then we consider ν_k, ξ_k $(k \in [n])$ by increasing by 1. By the construction of $p_a(H)$ and $\bar{p}_b(K)$, we have $N = 2i_N$ and $M = 2\bar{i}_0$. By the choice of b, $i_N = \bar{i}_0 = b$ holds, and hence $N = M$. We claim $p_{a,N-m} = \bar{p}_{b,m}$ for $m = 0, 1, \ldots, N$, and show this by induction on m. By the definition of $p_a(H)$ and $\bar{p}_b(K)$, $j_N = \bar{j}_0 = 0$, and hence $p_{a,N} = \bar{p}_{b,0}$.

Suppose $m \in 2\mathbb{Z}_{>0} \setminus \{N\}$. Note that $N - m \in 2\mathbb{Z}_{>0}$. By the induction hypothesis and Definition 4.20,

$$i_{N-m} = i_{N-m+1} - 1 = \bar{i}_{m-1} - 1 = \bar{i}_m,$$
$$j_{N-m} = j_{N-m+1} = \bar{j}_{m-1} = \bar{j}_m.$$

Then $p_{a,N-m} = \bar{p}_{b,m}$ holds for $m \in 2\mathbb{Z}_{>0} \setminus \{N\}$. If $m = N$, then

$$i_0 = \bar{i}_N = 0,$$
$$j_0 = j_1 = \bar{j}_{N-1} = \bar{j}_N$$

holds by the definition of $p_a(H)$ and $\bar{p}_b(H)$. Then $p_{a,0} = \bar{p}_{b,N}$ holds.

Assume that $m \in 2\mathbb{Z}_{\geq 0} + 1$. In this case, $N - m \in 2\mathbb{Z}_{\geq 0} + 1$. Also, note that $i_m = i_{m+1}$ and $U_{i_{m+1},j} = 0$ if $j_m < j < j_{m+1}$ hold, and hence

$$j_m = \max\{j \in [1, j_{m+1} - 1]_{\mathbb{Z}} \mid U_{i_{m+1},j} \neq 0\}.$$

Then, by the induction hypothesis and Definition 4.20,

$$\begin{aligned} i_{N-m} &= i_{N-m+1} = \bar{i}_{m-1} = \bar{i}_m, \\ j_{N-m} &= \max\{j \in [1, j_{N-m+1} - 1]_{\mathbb{Z}} \mid U_{i_{N-m+1},j} \neq 0\} \\ &= \max\{j \in [1, \bar{j}_{m-1} - 1]_{\mathbb{Z}} \mid U_{\bar{i}_{m-1},j} \neq 0\} \\ &= \bar{j}_m. \end{aligned}$$

Then we have $p_{a,N-m} = \bar{p}_{b,m}$ for $m \in 2\mathbb{Z}_{\geq 0} + 1$, thus the claim holds for $m = 0, 1, \ldots, N$. Thus, $p_a(H) = \bar{p}_b(H)$ holds, and hence $\bar{\rho}_b(K) = H$.

(2) Let $\nu \in P^+$ and $K = (\nu, \xi, 0, (V_{kl})_{k<l}) \in \mathbb{H}(\nu)$. If $\nu \neq 0$, then let $b \in \{i \in I \mid \nu_i \neq 0\} \cup \{n\}$, otherwise let $b = n$. Set $H = \bar{\rho}_b(K)$, and let $\bar{p}_b(K) = (\bar{p}_{b,m})_{m=0,\ldots,M}$, where $\bar{p}_{b,m} = (\bar{i}_m, \bar{j}_m)$. Note that if $b = n$, then we consider ν_k, ξ_k $(k \in [n])$ by increasing by 1. Suppose $H \in \mathbb{H}(\lambda)$ for some $\lambda \in P^+$. By Definition 4.22, we know that $\lambda_b = \nu_b - 1$ and $\lambda_k = \nu_k$ if $k \neq b$. Set $a = \bar{j}_M$. Let $H = (\lambda, \mu, 0, (U_{kl})_{k<l})$. Let $p_a(H) = (p_{a,m})_{m=1,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$.

To show $\rho_a(H) = K$, it suffices to show that $\bar{p}_b(K) = p_a(H)$ by Definition 4.11 and Definition 4.22. Set $L = \min\{M, N\}$. We claim that $\bar{p}_{b,M-m} = p_{a,m}$ for $m = 0, 1, \ldots, L$. By the construction of $\bar{p}_b(K)$, $p_a(H)$ and the choice of a,

$$\bar{p}_{b,M} = (0, a) = p_{a,0}.$$

Also, we have

$$\begin{aligned} \bar{i}_{M-1} &= \bar{i}_M + 1 = i_0 + 1 = i_1, \\ \bar{j}_{M-1} &= \bar{j}_M = j_0 = j_1, \end{aligned}$$

hence we have $\bar{p}_{b,M-1} = p_{a,1}$.

Suppose $m \in 2\mathbb{Z}_{>0} \setminus \{L\}$. Note that $L - m \in 2\mathbb{Z}_{>0}$. By the induction hypothesis, we have $i_m = i_{m-1} = \bar{i}_{N-m+1} = \bar{i}_{N-m}$ and

$$\begin{aligned} j_m &= \max\{j \in [j_{m-1} + 1, n]_{\mathbb{Z}} \mid U_{i_{m-1},j} \neq 0\} \\ &= \max\{j \in [\bar{j}_{N-m+1} + 1, n]_{\mathbb{Z}} \mid U_{\bar{i}_{N-m+1},j} \neq 0\} \\ &= \bar{j}_{N-m}. \end{aligned}$$

Then $\bar{p}_{b,N-m} = p_{a,m}$.

Suppose $m \in 2\mathbb{Z}_{>0} + 1$. Note that $N - m \in 2\mathbb{Z}_{>0}$. By the induction hypothesis,

$$\begin{aligned} i_m &= i_{m-1} + 1 = \bar{i}_{N-m+1} + 1 = \bar{i}_{N-m}, \\ j_m &= j_{m-1} = \bar{j}_{N-m+1} = \bar{j}_{N-m}. \end{aligned}$$

Then $\bar{p}_{b,M-m} = p_{a,m}$ holds for $m = 0, 1, \ldots, L - 1$. By the definition of L, $\bar{j}_{M-L} = n + 1$ or $j_L = n + 1$ holds. Since $\bar{p}_{b,M-m} = p_{a,m}$ for $m = 0, 1, \ldots, L - 1$, we have $\bar{j}_{M-L} = j_L = n + 1$. This implies $M = N$. Thus, we have $\bar{p}_b(K) = p_a(H)$. Therefore, we obtain $\rho_a(H) = K$. □

Now, we investigate the relation between path operators and the crystal structure of $\mathbb{H}(\lambda)$.

Lemma 4.29. Let $\lambda \in P^+$ with $\lambda \neq 0$. Let $H \in \mathbb{H}(\lambda)$. For $i \in I$ and $j \in [i]$, set $\widehat{\varphi}_i^{(j)}(H) = U_{j,i} - U_{j+1,i+1} + \varphi_i^{(j-1)}(H)$. Then we have the following.

$$\varphi_i(\iota(H)) = \begin{cases} \varphi_i(H) - 1 & \text{if } j_{\iota(H)} = i, \\ \varphi_i(H) + 1 & \text{if } j_{\iota(H)} = i+1,\ \widehat{\varphi}_i^{(\ell(\lambda)-1)}(H) \geq 0, \\ \varphi_i(H) & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda \in P^+$ with $\lambda \neq 0$ and let $H = (\lambda, \mu, 0, (U_{kl})_{k<l}) \in \mathbb{H}(\lambda)$. Let $\nu \in P^+$ and let $\iota(H) = (\nu, \xi, 0, (V_{kl})_{k<l}) \in \mathbb{H}(\nu)$. Fix $i \in I$. Note that it follows from Remark 2.11 that $\varphi_i(H) = \varphi_i^{(\ell(\lambda))}(H)$ and $\varphi_i(\iota(H)) = \varphi_i^{(\ell(\lambda))}(\iota(H))$ since $\ell(\nu) \leq \ell(\lambda)$. Since $V_{kl} = U_{kl}$ if $(k,l) \neq (\ell(\lambda), j_{\iota(H)})$, we obtain $\varphi_i^{(k)}(H) = \varphi_i^{(k)}(\iota(H))$ for $k < \ell(\lambda) - 1$.

Suppose $j_{\iota(H)} = i$. In this case, $U_{\ell(\lambda),i} > 0$ and $U_{\ell(\lambda),i+1} = 0$. Then the following holds.

$$\begin{aligned} \varphi_i(H) &= \max\{U_{\ell(\lambda),i} - U_{\ell(\lambda)+1,i+1} + \varphi_i^{(\ell(\lambda)-1)}(H), 0\} \\ &= U_{\ell(\lambda),i} + \varphi_i^{(\ell(\lambda)-1)}(H) > 0. \end{aligned}$$

Since $j_{\iota(H)} = i$, we have $\varphi_i^{(\ell(\lambda)-1)}(\iota(H)) = \varphi_i^{(\ell(\lambda)-1)}(H)$. Then we obtain

$$\begin{aligned} \varphi_i(\iota(H)) &= \max\{V_{\ell(\lambda),i} - V_{\ell(\lambda)+1,i+1} + \varphi_i^{(\ell(\lambda)-1)}(\iota(H)), 0\} \\ &= \max\{U_{\ell(\lambda),i} - 1 + \varphi_i^{(\ell(\lambda)-1)}(H), 0\} \\ &= U_{\ell(\lambda),i} - 1 + \varphi_i^{(\ell(\lambda)-1)}(H). \end{aligned}$$

Thus, $\varphi_i(\iota(H)) = \varphi_i(H) - 1$ holds.

Suppose $j_{\iota(H)} = i+1$. In this case, $U_{\ell(\lambda),i} - U_{\ell(\lambda)+1,i+1} = 0$ holds. Then we have

$$\varphi_i(H) = \max\{U_{\ell(\lambda)-1,i} - U_{\ell(\lambda),i+1} + \varphi_i^{(\ell(\lambda)-2)}(H), 0\}.$$

Also, the following holds.

$$\begin{aligned} \varphi_i(\iota(H)) &= \max\{V_{\ell(\lambda)-1,i} - V_{\ell(\lambda),i+1} + \varphi_i^{(\ell(\lambda)-2)}(\iota(H)), 0\} \\ &= \max\{U_{\ell(\lambda)-1,i} - (U_{\ell(\lambda),i+1} - 1) + \varphi_i^{(\ell(\lambda)-2)}(H), 0\}. \end{aligned}$$

For $i \in I$ and $j \in [i]$, set $\widehat{\varphi}_i^{(j)}(H) = U_{j,i} - U_{j+1,i+1} + \varphi_i^{(j-1)}(H)$. Then we have $\varphi_i(\iota(H)) = \varphi_i(H) + 1$ if $\widehat{\varphi}_i^{(\ell(\lambda)-1)} \geq 0$ otherwise $\varphi_i(\iota(H)) = \varphi_i(H)$.

Suppose $j_{\iota(H)} \neq i, i+1$. By Definition 4.3, $V_{ki} = U_{ki}, V_{k,i+1} = U_{k,i+1}$ holds for $k \in [n]$. Thus, we have $\varphi_i(H) = \varphi_i(\iota(H))$. □

Remark 4.30. Suppose $j_{\iota(H)} = i + 1$. In this case, $U_{\ell(\lambda),i} = 0$. Then $\varphi_i(H) = \max\{U_{\ell(\lambda)-1,i} - U_{\ell(\lambda),i+1} + \varphi_i^{(\ell(\lambda)-2)}(H), 0\}$. Thus, if $j_{\iota(H)} = i+1$ and $\varphi_i(\iota(H)) = \varphi_i(H)$, then $\varphi_i(H) = 0$ holds.

Lemma 4.31. Let $\lambda \in P^+$ with $\lambda \neq 0$. Let $H \in \mathbb{H}(\lambda)$. For $i \in I$ and $j \in [i]$, Set $\widehat{\varphi}_i^{(j)}(H) = U_{j,i} - U_{j+1,i+1} + \varphi_i^{(j-1)}(H)$. Then we have the following.

$$\varepsilon_i(\iota(H)) = \begin{cases} \varepsilon_i(H) + 1 & \text{if } j_{\iota(H)} = i+1,\ \widehat{\varphi}_i^{(\ell(\lambda)-1)}(H) < 0, \\ \varepsilon_i(H) & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda \in P^+$ with $\lambda \neq 0$ and let $H \in \mathbb{H}(\lambda)$. By Definition 4.3, we have

$$\langle h_i, \mathrm{wt}(\iota(H))\rangle = \begin{cases} \langle h_i, \mathrm{wt}H\rangle - 1 & \text{if } j_{\iota(H)} = i, \\ \langle h_i, \mathrm{wt}H\rangle + 1 & \text{if } j_{\iota(H)} = i+1, \\ \langle h_i, \mathrm{wt}H\rangle & \text{otherwise.} \end{cases}$$

Then the statement follows from Definition 2.1 and Lemma 4.29. □

Lemma 4.32. Let $\lambda \in P^+$ and let $H \in \mathbb{H}(\lambda)$. Let $a \in [n]$. Then, for $i \in I$, we have the following.

$$\varepsilon_i(\rho_a(H)) = \begin{cases} \varepsilon_i(H) + 1 & \text{if } a = i+1, \\ \varepsilon_i(H) - 1 & \text{if } a = i, \varepsilon_i(H) > 0, \\ \varepsilon_i(H) & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda \in P^+$ and let $H = (\lambda, \mu, 0, (U_{kl})_{k<l}) \in \mathbb{H}(\lambda)$. Let $a \in [n]$ and let $p_a(H) = (p_{a,m})_{m=0,1,\dots,N}$, where $p_{a,m} = (i_k, j_k)$. Let $\rho_a(H) = (\nu, \xi, 0, (V_{kl})_{k<l}) \in \mathbb{H}(\nu)$ for $\nu \in P^+$.

Suppose $a = i+1$. By Definition 4.11, $V_{ki} = U_{ki}$ for $k \in [n]$. Also, if $k = 1$ then $V_{k,i+1} = U_{k,i+1} + 1$ otherwise $V_{k,i+1} = U_{k,i+1}$ holds. It follows that $\varepsilon_i^{(k)}(\rho_{i+1}(H)) = \varepsilon_i^{(k)}(H)$ for $k \leq i$. Then we have

$$\begin{aligned} \varepsilon_i(\rho_{i+1}(H)) &= \varepsilon_i^{(i+1)}(\rho_{i+1}(H)) \\ &= \max\{V_{1,i+1} + \varepsilon_i^{(i)}(\rho_{i+1}(H)), 0\} \\ &= \max\{U_{1,i+1} + 1 + \varepsilon_i^{(i)}(H), 0\} \\ &= U_{1,i+1} + \varepsilon_i^{(i)}(H) + 1 \\ &= \varepsilon_i(H) + 1. \end{aligned}$$

Suppose that $a = i$ and $\varepsilon_i(H) > 0$. By Definition 4.11, if $k = 1$ then $V_{k,i} = U_{k,i} + 1$ otherwise $V_{k,i} = U_{k,i}$ holds. Suppose $U_{1,i+1} = 0$. In this case, $V_{k,i+1} = U_{k,i+1}$ for $k \in [n]$ by Definition 4.11. It follows that $\varepsilon_i^{(k)}(\rho_i(H)) = \varepsilon_i^{(k)}(H)$ if $k \leq i-1$. Since $\varepsilon_i(H) > 0$, $\varepsilon_i(H) = U_{2,i+1} - U_{1,i} + \varepsilon_i^{(i-1)}(H)$ holds. Then we have

$$\begin{aligned} \varepsilon_i(\rho_i(H)) &= \max\{V_{1,i+1} + \varepsilon_i^{(i)}(\rho_i(H)), 0\} \\ &= \max\{V_{2,i+1} - V_{1,i} + \varepsilon_i^{(i-1)}(\rho_i(H)), 0\} \\ &= \max\{U_{2,i+1} - (U_{1,i} + 1) + \varepsilon_i^{(i-1)}(H), 0\} \\ &= U_{2,i+1} - U_{1,i} + \varepsilon_i^{(i-1)}(H) - 1 \\ &= \varepsilon_i(H) - 1. \end{aligned}$$

Suppose $U_{1,i+1} > 0$. By Definition 4.11, we have

$$V_{k,i+1} = \begin{cases} U_{1,i+1} - 1 & \text{if } k = 1, \\ U_{2,i+1} + 1 & \text{if } k = 2, \\ U_{k,i+1} & \text{otherwise.} \end{cases}$$

Then $\varepsilon_i^{(k)}(\rho_{i+1}(H)) = \varepsilon_i^{(k)}(H)$ holds if $k \leq i-1$. Moreover, the following holds.

$$\begin{aligned}\varepsilon_i^{(i)}(\rho_i(H)) &= \max\{V_{2,i+1} - V_{1,i} + \varepsilon_i^{(i-1)}(\rho_i(H)), 0\} \\ &= \max\{U_{2,i+1} - U_{1,i} + \varepsilon_i^{(i-1)}(H), 0\} \\ &= \varepsilon_i^{(i)}(H).\end{aligned}$$

It follows that

$$\begin{aligned}\varepsilon_i(\rho_a(H)) &= \max\{V_{1,a+1} + \varepsilon_i^{(i)}(\rho_a(H)), 0\} \\ &= \max\{U_{1,a+1} - 1 + \varepsilon_i^{(i)}(H), 0\} \\ &= \varepsilon_i(H) - 1.\end{aligned}$$

Suppose that $a = i$ and $\varepsilon_i(H) = 0$. By the above discussion, we have $\varepsilon_i(\rho_a(H)) = \max\{U_{2,a+1} - U_{1,a} + \varepsilon_i^{(i-1)}(H) - 1, 0\}$. Since $\varepsilon_i(H) = 0$, we obtain $U_{2,a+1} - U_{1,a} + \varepsilon_i^{(i-1)}(H) \geq 0$. Thus, $\varepsilon_i(\rho_a(H)) = 0 = \varepsilon_i(H)$.

Suppose that $a \neq i, i+1$. If $a > i+1$, then $V_{ki} = U_{ki}$, $V_{k,i+1} = U_{k,i+1}$ holds for $k \in [n]$. Thus, we have $\varepsilon_i(\rho_a(H)) = \varepsilon_i(H)$. Suppose that $a < i$. In this case, there are four cases as follows.

(1) There is no $k \in [0, N]_{\mathbb{Z}}$ such that $j_k \in \{i, i+1\}$,
(2) there exists $k, l \in [0, N]_{\mathbb{Z}}$ such that $j_k = i$, $j_l = i+1$,
(3) there exists $k \in [0, N]_{\mathbb{Z}}$ such that $j_k = i$ and there is no $l \in [0, N]_{\mathbb{Z}}$ such that $j_l = i+1$,
(4) there is no $k \in [0, N]_{\mathbb{Z}}$ such that $j_k = i$ and there exists $l \in [0, N]_{\mathbb{Z}}$ such that $j_l = i+1$.

In case (1), $V_{ki} = U_{ki}$, $V_{k,i+1} = U_{k,i+1}$ for $k \in [n]$ by Definition 4.11. Thus, we have $\varepsilon_i(\rho_a(H)) = \varepsilon_i(H)$.

In case (2), let k_0 be the smallest $k \in [0, N]_{\mathbb{Z}}$ such that $j_k = i$. For $l = 1, 2, 3, 4$, set $k_l = k_0 + l$. By Definition 4.9, we have that $k_0 \in 2\mathbb{Z}$, $p_{a,k_0} = (i_{k_0}, i)$, $p_{a,k_1} = (i_{k_0}+1, i)$, $p_{a,k_2} = (i_{k_0}+1, i+1)$, $p_{a,k_3} = (i_{k_0}+2, i+1)$, and $j_{k_4} > i+1$. By Definition 4.11, $V_{kl} = U_{kl} - 1$ if $(k, l) = p_{a,k_0}, p_{a,k_2}$, and $V_{kl} = U_{kl} + 1$ if $(k, l) = p_{a,k_1}, p_{a,k_3}$. Then $V_{k+1,i+1} - V_{k,i} = U_{k+1,i+1} - U_{k,i}$ holds for $k = 0, 1 \ldots, i$. Thus, $\varepsilon_i(\rho_a(H)) = \varepsilon_i(H)$.

In case (3), let k_0 be the smallest $k \in [0, N]_{\mathbb{Z}}$ such that $j_k = i$. By Definition 4.9, $k_0 \in 2\mathbb{Z}$, $p_{a,k_0} = (i_{k_0}, i)$, $p_{a,k_0+1} = (i_{k_0}+1, i)$, and $j_{k_0+2} > i+1$. Note that $j_{k_0+2} > i+1$ means $U_{i_{k_0}+1,i+1} = 0$ by Definition 4.9. Also, $U_{i_{k_0},i} > 0$ holds by the choice of k_0. Then we have

$$\begin{aligned}V_{i_{k_0}+1,i+1} - V_{i_{k_0},i} &= U_{i_{k_0}+1,i+1} - (U_{i_{k_0},i} - 1) \\ &= -U_{i_{k_0},i} + 1 \leq 0.\end{aligned}$$

Also, we have

$$\begin{aligned}V_{i_{k_0}+2,i+1} - V_{i_{k_0}+1,i} &= U_{i_{k_0}+2,i+1} - (U_{i_{k_0}+1,i} + 1) \\ &= U_{i_{k_0}+2,i+1} - U_{i_{k_0}+1,i} - 1.\end{aligned}$$

Note that $\varepsilon_i^{(k)}(\rho_a(H)) = \varepsilon_i^{(k)}(H)$ if $k < i - i_{k_0}$ by Definition 4.11. Then we obtain $\varepsilon_i^{(i-i_{k_0})}(\rho_a(H)) \leq \varepsilon_i^{(i-i_{k_0})}(H)$, and hence $\varepsilon_i^{(i+1-i_{k_0})}(\rho_a(H)) = \varepsilon_i^{(i+1-i_{k_0})}(H)$. By Definition 4.11, we have $\varepsilon_i(\rho_a(H)) = \varepsilon_i(H)$.

In case (4), let l_0 be the smallest $l \in [0,N]_{\mathbb{Z}}$ such that $j_l = i+1$. By Definition 4.9, $l_0 \in 2\mathbb{Z}+1$, $j_{l_0-1} < i$, $p_{a,l_0} = (i_{l_0}, i+1)$, $p_{a,l_0+1} = (i_{l_0}+1, i+1)$, and $j_{l_0+2} > i+1$. By the choice of l_0, we have $U_{i_{l_0},i} = 0$. Then

$$\begin{aligned} V_{i_{l_0+1},i+1} - V_{i_{l_0},i} &= U_{i_{l_0+1},i+1} + 1 - U_{i_{l_0},i} \\ &= U_{i_{l_0},i+1} + 1 > 1. \end{aligned}$$

Note that $\varepsilon_i^{(l)}(\rho_a(H)) = \varepsilon_i^{(l)}(H)$ if $l < i+1-i_{l_0}$ by Definition 4.11. Then we obtain $\varepsilon_i^{(i+1-i_{l_0})}(\rho_a(H)) \geq \varepsilon_i^{(i+1-i_{l_0})}(H) > 0$. Also, we have

$$V_{i_{l_0},i+1} - V_{i_{l_0}-1,i} = U_{i_{l_0+1},i+1} - U_{i_{l_0},i} - 1.$$

Hence, $\varepsilon_i^{(i+2-i_{l_0})}(\rho_a(H)) = \varepsilon_i^{(i+2-i_{l_0})}(H)$. Thus, by Definition 4.11, we have $\varepsilon_i(\rho_a(H)) = \varepsilon_i(H)$. □

Lemma 4.33. Let $\lambda \in P^+$ and let $H \in \mathbb{H}(\lambda)$. Let $a \in [n]$. Then, for $i \in I$, we have the following.

$$\varphi_i(\rho_a(H)) = \begin{cases} \varphi_i(H) + 1 & \text{if } a = i,\ \varepsilon_i(H) = 0, \\ \varphi_i(H) & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda \in P^+$ and let $H \in \mathbb{H}(\lambda)$. Let $a \in [n]$. By Definition 4.11,

$$\langle h_i, \mathrm{wt}(\rho_a(H)) \rangle = \begin{cases} \langle h_i, \mathrm{wt}H \rangle + 1 & \text{if } a = i, \\ \langle h_i, \mathrm{wt}H \rangle - 1 & \text{if } a = i+1, \\ \langle h_i, \mathrm{wt}H \rangle & \text{otherwise.} \end{cases}$$

Thus, the statement holds from Definition 2.1 and Lemma 4.32. □

Remark 4.34. Since $\varepsilon_i(H) = 0$ implies that $\sum_{k=1}^{m}(U_{i+2-k,i+1} - U_{i+1-k,i}) \leq 0$ for $m \in [i+1]$, we have $\sum_{k=m}^{i}(U_{k,i} - U_{k+1,i+1}) \geq 0$ for $m \in [i+1]$. Also, we have $U_{1,i+1} = 0$. Then if $\varepsilon_i(H) = 0$ and $a = i$, then $\sum_{k=1}^{m}(U_{k,i} - U_{k+1,i+1}) > 0$ holds for $m \in [i]$ by Definition 4.11. This means $k_{f_i\rho_a(H)} = 1$.

4.2. **The tensor product decomposition.** In this subsection, we define an operator Θ on the tensor product of a set of K-hives and show that Θ is a crystal embedding. Then we give the decomposition theorem.

Definition 4.35. Let $\lambda, \mu \in P^+$ with $\lambda \neq 0$. Let $H \otimes K \in \mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$. Then Θ is defined as an operator on $\mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$ by $\Theta(H \otimes K) = \iota(H) \otimes \rho_{j_{\iota(H)}}(K)$.

Example 4.36. Let $n = 4$. Let $\lambda = (3,2,1,0), \mu = (6,4,1,0) \in P^+$. Let $H \otimes K \in \mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$ as shown in FIGURE 16. We have $j_{\iota(H)} = 3$. Then the action of Θ for $H \otimes K$ is as shown in FIGURE 16.

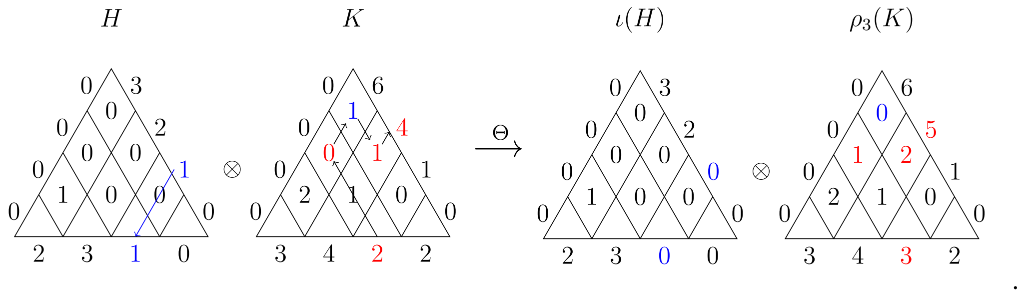


FIGURE 16. Action of Θ

Proposition 4.37. Let $\lambda, \mu \in P^+$ with $\lambda \neq 0$. Let $\nu \in P^+$, where $\nu_k = \lambda_k - 1$ if $k = \ell(\lambda)$ otherwise $\nu_k = \lambda_k$. Then we obtain

$$\Theta\colon \mathbb{H}(\lambda) \otimes \mathbb{H}(\mu) \to \bigsqcup_{\xi \in P^+} \mathbb{H}(\nu) \otimes \mathbb{H}(\xi).$$

Proof. Let $\lambda, \mu \in P^+$ with $\lambda \neq 0$. Let $\nu \in P^+$, where $\nu_k = \lambda_k - 1$ if $k = \ell(\lambda)$ otherwise $\nu_k = \lambda_k$. Let $H_1 \otimes H_2 \in \mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$. It follows from Proposition 4.8 that $\iota(H_1) \in \mathbb{H}(\nu)$. Set $a = j_{\iota(H_1)}$. By Proposition 4.18, $\rho_a(H_2) \in \mathbb{H}(\xi)$ for some $\xi \in P^+$. Then $\Theta(H_1 \otimes H_2) \in \mathbb{H}(\nu) \otimes \mathbb{H}(\xi)$ holds. Since $\xi \in P^+$ is determined by the choice of $H_1 \otimes H_2$, we have that Θ is a map from $\mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$ to $\bigsqcup_{\xi \in P^+} \mathbb{H}(\nu) \otimes \mathbb{H}(\xi)$. $\square$

In the following, we show that Θ is a crystal embedding (Proposition 4.42).

Proposition 4.38. Let $\lambda, \mu \in P^+$ with $\lambda \neq 0$. Let $H_1 \otimes H_2 \in \mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$. Then, the following holds.

$$\langle h_i, \mathrm{wt}(\Theta(H_1 \otimes H_2)) \rangle = \langle h_i, \mathrm{wt}(H_1 \otimes H_2) \rangle \quad (i \in I) \tag{4.10}$$

Proof. Let $\lambda^{(1)}, \lambda^{(2)} \in P^+$ with $\lambda^{(1)} \neq 0$ and let $H_1 \otimes H_2 \in \mathbb{H}(\lambda^{(1)}) \otimes \mathbb{H}(\lambda^{(2)})$. Set $K_1 \otimes K_2 = \Theta(H_1 \otimes H_2)$. For $\nu^{(1)}, \nu^{(2)} \in P^+$, let $K_1 \otimes K_2 \in \mathbb{H}(\nu^{(1)}) \otimes \mathbb{H}(\nu^{(2)})$. Let $H_m = (\lambda^{(m)}, \mu^{(m)}, 0, (U_{ij}^{(m)})_{i<j})$, $K_m = (\nu^{(m)}, \xi^{(m)}, 0, (V_{ij}^{(m)})_{i<j})$. By Definition 4.3,

$$\langle h_i, \mathrm{wt} K_1 \rangle = \begin{cases} \langle h_i, \mathrm{wt} H_1 \rangle + 1 & \text{if } i = j_{\iota(H_1)} - 1, \\ \langle h_i, \mathrm{wt} H_1 \rangle - 1 & \text{if } i = j_{\iota(H_1)}, \\ \langle h_i, \mathrm{wt} H_1 \rangle & \text{otherwise.} \end{cases}$$

On the other hand,

$$\langle h_i, \mathrm{wt} K_2 \rangle = \begin{cases} \langle h_i, \mathrm{wt} H_2 \rangle - 1 & \text{if } i = j_{\iota(H_1)} - 1, \\ \langle h_i, \mathrm{wt} H_2 \rangle + 1 & \text{if } i = j_{\iota(H_1)}, \\ \langle h_i, \mathrm{wt} H_2 \rangle & \text{otherwise.} \end{cases}$$

Thus we have the following.

$$\begin{aligned} \langle h_i, \mathrm{wt}(K_1 \otimes K_2) \rangle &= \langle h_i, \mathrm{wt} K_1 \rangle + \langle h_i, \mathrm{wt} K_2 \rangle \\ &= \langle h_i, \mathrm{wt} H_1 \rangle + \langle h_i, \mathrm{wt} H_2 \rangle \\ &= \langle h_i, \mathrm{wt}(H_1 \otimes H_2) \rangle. \end{aligned}$$

$\square$

Proposition 4.39. Let $\lambda, \mu \in P^+$ with $\lambda \neq 0$. Let $H_1 \otimes H_2 \in \mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$. Then the following holds.

$$\varphi_i(H_1 \otimes H_2) = \varphi_i(\Theta(H_1 \otimes H_2)) \quad (i \in I), \tag{4.11}$$

$$\varepsilon_i(H_1 \otimes H_2) = \varepsilon_i(\Theta(H_1 \otimes H_2)) \quad (i \in I). \tag{4.12}$$

Proof. Let $\lambda^{(m)}, \nu^{(m)} \in P^+$ $(m = 1, 2)$ with $\lambda^{(1)}, \nu^{(1)} \neq 0$. Let $H_1 \otimes H_2 \in \mathbb{H}(\lambda^{(1)}) \otimes \mathbb{H}(\lambda^{(2)})$, and let $\Theta(H_1 \otimes H_2) \in \mathbb{H}(\nu^{(1)}) \otimes \mathbb{H}(\nu^{(2)})$. Set $K_1 \otimes K_2 = \Theta(H_1 \otimes H_2)$. For $m = 1, 2$, let $H_m = (\lambda^{(m)}, \mu^{(m)}, 0, (U_{ij}^{(m)})_{i<j})$, $K_m = (\nu^{(m)}, \xi^{(m)}, 0, (V_{ij}^{(m)})_{i<j})$. If $\varphi_i(H_1 \otimes H_2) = \varphi_i(K_1 \otimes K_2)$ holds, then it follows from Proposition 4.38 that $\varepsilon_i(H_1 \otimes H_2) = \varepsilon_i(K_1 \otimes K_2)$ since

$$\begin{aligned} \varepsilon_i(K_1 \otimes K_2) &= \varphi_i(K_1 \otimes K_2) - \langle h_i, \mathrm{wt}(K_1 \otimes K_2) \rangle \\ &= \varphi_i(H_1 \otimes H_2) - \langle h_i, \mathrm{wt}(H_1 \otimes H_2) \rangle \\ &= \varepsilon_i(H_1 \otimes H_2). \end{aligned}$$

Then it suffices to show that $\varphi_i(H_1 \otimes H_2) = \varphi_i(K_1 \otimes K_2)$.

First, we consider the case where $i < j_{\iota(H_1)} - 1$. By Definitions 4.11, 4.3, $V_{kl}^{(m)} = U_{kl}^{(m)}$ $(m = 1, 2)$ if $l = i, i+1$. Then we have $\varphi_i(H_m) = \varphi_i(K_m)$ $(m = 1, 2)$. Thus, (4.11) holds by Proposition 4.38.

Next, we consider the case where $i = j_{\iota(H_1)} - 1$. In this case, we have $\varphi_i(K_2) = \varphi_i(H_2)$ by Lemma 4.33. Suppose that $\varphi_i(K_1) = \varphi_i(H_1) + 1$. By Definitions 4.3, 4.11,

$$\begin{aligned} \varphi_i(K_1) + \langle h_i, \mathrm{wt} K_2 \rangle &= \varphi_i(H_1) + 1 + \langle h_i, \mathrm{wt} H_2 \rangle - 1 \\ &= \varphi_i(H_1) + \langle h_i, \mathrm{wt} H_2 \rangle. \end{aligned}$$

Thus, (4.11) holds. Suppose that $\varphi_i(K_1) = \varphi_i(H_1)$. By Remark 4.30, we have $\varphi_i(H_1) = 0$. This means

$$\mu_i^{(1)} - \mu_{i+1}^{(1)} = \sum_{k=1}^{i} (U_{ki}^{(1)} - U_{k+1,i+1}^{(1)}) < 0.$$

Then $\xi_i^{(1)} - \xi_{i+1}^{(1)} \leq 0$ holds by Definition 4.3. Then we have $\langle h_i, \mathrm{wt} H_1 \rangle, \langle h_i, \mathrm{wt} K_1 \rangle \leq 0$. Hence, (4.11) holds since

$$\varphi_i(H_1 \otimes H_2) = \varphi_i(H_2) = \varphi_i(K_2) = \varphi_i(K_1 \otimes K_2).$$

Next, we consider the case where $i = j_{\iota(H_1)}$. By Lemma 4.29, $\varphi_i(K_1) = \varphi_i(H_1) - 1$. By Definition 4.3, $\langle h_i, \mathrm{wt}(K_2) \rangle = \langle h_i, \mathrm{wt}(H_2) \rangle + 1$. Then we have

$$\varphi_i(H_1) + \langle h_i, \mathrm{wt}(H_2) \rangle = \varphi_i(K_1) + \langle h_i, \mathrm{wt}(K_2) \rangle.$$

If $\varphi_i(K_2) = \varphi_i(H_2)$, then (4.11) is trivial. Suppose that $\varphi_i(K_2) = \varphi_i(H_2) + 1$. By Lemma 4.33, we have $\varepsilon_i(H_2) = 0$. Then $\varphi_i(H_2) = \mu_i^{(2)} - \mu_{i+1}^{(2)}$ holds by Definition 2.1. It follows that

$$\varphi_i(H_1) + \langle h_i, \mathrm{wt} H_2 \rangle = \varphi_i(H_1) + \mu_i^{(2)} - \mu_{i+1}^{(2)} > \varphi_i(H_2).$$

Then $\varphi_i(K_1) + \langle h_i, \mathrm{wt} K_2 \rangle \geq \varphi_i(K_2)$. Thus, we obtain

$$\varphi_i(H_1 \otimes H_2) = \varphi_i(H_1) + \langle h_i, \mathrm{wt} H_2 \rangle = \varphi_i(K_1) + \langle h_i, \mathrm{wt} K_2 \rangle = \varphi_i(K_1 \otimes K_2).$$

Finally, we consider the case where $i > j_{\iota(H_1)}$. In this case, $\varphi_i(H_1) = \varphi_i(K_1)$ holds by Lemma 4.29. By Definition 4.11, we have $\langle h_i, \mathrm{wt}(H_2)\rangle = \langle h_i, \mathrm{wt}(K_2)\rangle$. Also, by Lemma 4.33, $\varphi_i(H_2) = \varphi_i(K_2)$ holds. Then we have $\varphi_i(H_1 \otimes H_2) = \varphi_i(K_1 \otimes K_2)$. □

From Lemma 4.29, Lemma 4.32, and Definition 2.3, we can investigate the relation between Θ and f_i, e_i.

Lemma 4.40. Let $\lambda, \mu \in P^+$ with $\lambda \neq 0$. Let $H_1 \otimes H_2 \in \mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$. Then we have the following.

(1) If $f_i(H_1 \otimes H_2) = f_i H_1 \otimes H_2$, then

$$f_i\left(\iota(H_1) \otimes \rho_{j_{\iota(H_1)}}(H_2)\right) = \begin{cases} \iota(H_1) \otimes f_i \rho_{j_{\iota(H_1)}}(H_2) & \text{if } j_{\iota(H_i)} = i, \varphi_i(H_1) = 1, \\ f_i \iota(H_1) \otimes \rho_{j_{\iota(H_1)}}(H_2) & \text{otherwise.} \end{cases}$$

(2) If $f_i(H_1 \otimes H_2) = H_1 \otimes f_i H_2$, then

$$f_i\left(\iota(H_1) \otimes \rho_{j_{\iota(H_1)}}(H_2)\right) = \iota(H_1) \otimes f_i \rho_{j_{\iota(H_1)}}(H_2).$$

(3) If $e_i(H_1 \otimes H_2) = e_i H_1 \otimes H_2$, then

$$e_i\left(\iota(H_1) \otimes \rho_{j_{\iota(H_1)}}(H_2)\right) = \begin{cases} \iota(H_1) \otimes e_i \rho_{j_{\iota(H_1)}}(H_2) & \text{if } j_{\iota(H_1)} = i+1, \varphi_i(\iota(H_1)) = 0, \\ e_i \iota(H_1) \otimes \rho_{j_{\iota(H_1)}}(H_2) & \text{otherwise.} \end{cases}$$

(4) If $e_i(H_1 \otimes H_2) = H_1 \otimes e_i H_2$, then

$$e_i\left(\iota(H_1) \otimes \rho_{j_{\iota(H_1)}}(H_2)\right) = \iota(H_1) \otimes e_i \rho_{j_{\iota(H_1)}}(H_2).$$

Proof. Let $\lambda^{(1)} \in P^+$ with $\lambda^{(1)} \neq 0$ and let $H_1 = (\lambda^{(1)}, \mu^{(1)}, 0, (U_{ij}^{(1)})_{i<j}) \in \mathbb{H}(\lambda^{(1)})$. Let $\lambda^{(2)} \in P^+$ and let $H_2 = (\lambda^{(2)}, \mu^{(2)}, 0, (U_{ij}^{(1)})_{i<j}) \in \mathbb{H}(\lambda^{(2)})$.

(1) Suppose $f_i(H_1 \otimes H_2) = f_i H_1 \otimes H_2$. By Definition 2.3, $\varphi_i(H_1) > \varepsilon_i(H_2)$ holds. If $j_{\iota(H_1)} = i$ and $\varphi_i(H_1) = 1$, then $\varphi_i(\iota(H_1)) = 0 \leq \varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2))$ holds.

Suppose $j_{\iota(H_1)} \neq i$. If $j_{\iota(H_1)} \neq i+1$, $\varphi_i(\iota(H_1)) = \varphi_i(H_1) > \varepsilon_i(H_2) = \varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2))$. If $j_{\iota(H_1)} = i+1$, then $\varphi_i(H_1) = U_{\ell(\lambda)-1,i}^{(1)} - U_{\ell(\lambda),i+1}^{(1)} + \varphi_i^{(\ell(\lambda)-2)}(H_1) > 0$. Then $\varphi_i(\iota(H_1)) = \varphi_i(H_1) + 1 > \varepsilon_i(H_2) + 1 = \varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2))$.

Suppose $\varphi_i(H_1) \neq 1$. Then we have $\varphi_i(H_1) > 1$. It suffices to consider the case where $j_{\iota(H_1)} = i$. In this case, $\varphi_i(\iota(H_1)) = \varphi_i(H_1) - 1$. If $\varepsilon_i(H_2) > 0$, then $\varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2)) = \varepsilon_i(H_2) - 1$, otherwise $\varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2)) = \varepsilon_i(H_2)$. Then $\varphi_i(\iota(H_1)) > \varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2))$ holds. Thus, the statement holds by Definition 2.3.

(2) Suppose $f_i(H_1 \otimes H_2) = H_1 \otimes f_i H_2$. By Definition 2.3, $\varphi_i(H_1) \leq \varepsilon_i(H_2)$ holds. By Lemmas 4.29, 4.32, we immediately obtain $\varphi_i(\iota(H_1)) \leq \varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2))$. Thus, the statement holds by Definition 2.3.

(3) Suppose $e_i(H_1 \otimes H_2) = e_i H_1 \otimes H_2$. By Definition 2.3, $\varphi_i(H_1) \geq \varepsilon_i(H_2)$ holds. Suppose that $j_{\iota(H_1)} = i+1$ and $\varphi_i(\iota(H_1)) = 0$. Since $j_{\iota(H_1)} = i+1$, then we have $\varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2)) = \varepsilon_i(H_2) + 1$ and $\varphi_i(\iota(H_1)) \geq \varphi_i(H_1)$ from Lemma 4.32 and Lemma 4.29. Since $\varphi_i(\iota(H_1)) = 0$, we have $\varphi_i(H) = 0$. Hence, we obtain $\varphi_i(\iota(H_1)) = \varphi_i(H_1) = 0 < \varepsilon_i(H_2) + 1 = \varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2))$.

If $j_{\iota(H_1)} \neq i+1$, then $\varphi_i(\iota(H_1)) \geq \varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2))$ is clear by Lemmas 4.29, 4.32. Suppose $\varphi_i(\iota(H_1)) > 0$. It suffices to consider the case where $j_{\iota(H_1)} = i+1$. In this case, by Lemma 4.29, $\varphi_i(\iota(H_1)) = \varphi_i(H_1) + 1$. Then we have $\varphi_i(\iota(H_1)) \geq \varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2))$ by Lemmas 4.29, 4.32. Thus, the statement holds by Definition 2.3.

(4) Suppose that $e_i(H_1 \otimes H_2) = H_1 \otimes e_i H_2$. By Definition 2.3, $\varphi_i(H_1) < \varepsilon_i(H_2)$ holds. By Lemmas 4.29, 4.32, we immediately obtain $\varphi_i(\iota(H_1)) < \varepsilon_i(\rho_{j_{\iota(H_1)}}(H_2))$. Thus, the statement holds by Definition 2.3. □

Proposition 4.41. Let $\lambda^{(1)}, \lambda^{(2)} \in P^+$ with $\lambda^{(1)} \neq 0$ and let $H_1 \otimes H_2 \in \mathbb{H}(\lambda^{(1)}) \otimes \mathbb{H}(\lambda^{(2)})$. Then the following holds.

$$(\Theta \circ f_i)(H_1 \otimes H_2)) = (f_i \circ \Theta)(H_1 \otimes H_2)) \quad (i \in I), \tag{4.13}$$

$$(\Theta \circ e_i)(H_1 \otimes H_2)) = (e_i \circ \Theta)(H_1 \otimes H_2)) \quad (i \in I). \tag{4.14}$$

Proof. Let $\lambda^{(m)}, \nu^{(m)} \in P^+$ $(m = 1, 2)$ with $\lambda^{(1)}, \nu^{(1)} \neq 0$. Let $H_1 \otimes H_2 \in \mathbb{H}(\lambda^{(1)}) \otimes \mathbb{H}(\lambda^{(2)})$, and let $\Theta(H_1 \otimes H_2) = \mathbb{H}(\nu^{(1)}) \otimes \mathbb{H}(\nu^{(2)})$.

If we show $(\Theta \circ f_i)(H_1 \otimes H_2) = (f_i \circ \Theta)(H_1 \otimes H_2)$, then we can obtain $(\Theta \circ e_i)(H_1 \otimes H_2) = (e_i \circ \Theta)(H_1 \otimes H_2)$ as follows. Since $(f_i \circ \Theta)(e_i(H_1 \otimes H_2)) = \Theta(f_i(e_i(H_1 \otimes H_2)) = \Theta(H_1 \otimes H_2)$ and $e_i(f_i(\Theta(e_i(H_1 \otimes H_2)))) = \Theta(e_i(H_1 \otimes H_2))$, we have $\Theta(e_i(H_1 \otimes H_2)) = e_i(\Theta(H_1 \otimes H_2))$. Then it suffices to show that $(\Theta \circ f_i)(H_1 \otimes H_2) = (f_i \circ \Theta)(H_1 \otimes H_2)$.

Set $K_1 \otimes K_2 = \Theta(H_1 \otimes H_2)$. For $m = 1, 2$, let $H_m = (\lambda^{(m)}, \mu^{(m)}, 0, (U_{ij}^{(m)}))$, $K_m = (\nu^{(m)}, \xi^{(m)}, 0, (V_{ij}^{(m)}))$.

Suppose that $(\Theta \circ f_i)(H_1 \otimes H_2) = \iota(f_i H_1) \otimes \rho_{j_{\iota(f_i H_1)}}(H_2)$. Note that $\varphi_i(H_1) > \varepsilon_i(H_2)$ by Definition 2.3. By Lemma 4.40, we have

$$(f_i \circ \Theta)(H_1 \otimes H_2) = \begin{cases} K_1 \otimes f_i K_2 & \text{if } j_{K_1} = i,\ \varphi_i(H_1) = 1, \\ f_i K_1 \otimes K_2 & \text{otherwise.} \end{cases}$$

Then we consider the case where $(f_i \circ \Theta)(H_1 \otimes H_2) = K_1 \otimes f_i K_2$. In this case, we have that $j_{K_1} = i$ and $\varphi_i(H_1) = 1$, and hence $\varphi_i(K_1) = 0$ holds by Lemma 4.29. Since $j_{K_1} = i$ and $\varphi_i(H_1) = 1$, we have that $\varphi_i(H_1) = U^{(1)}_{\ell(\lambda^{(1)}),i} = 1$. This means $k_{f_i H_1} = \ell(\lambda^{(1)})$, and hence $j_{\iota(f_i H_1)} = i + 1$. Thus, we have $K_1 = \iota(f_i H_1)$ by Definition 4.3. Let $p_i(H_2) = (p_{i,k})_{k=1,\dots,N}$, where $p_{i,k} = (i_k, j_k)$. Let $p_{i+1}(H_2) = (p_{i+1,k})_{k=1,\dots,M}$, where $p_{i+1,k} = (s_k, t_k)$. Then we have

$$p_{i,0} = (0, i), p_{i,1} = (1, i), p_{i,2} = (1, j_2),$$
$$p_{i+1,0} = (0, i+1), p_{i+1,1} = (1, i+1), p_{i+1,2} = (1, t_2).$$

Since $\varphi_i(H_1) = 1$, we have $\varepsilon_i(H_2) = 0$ and, in particular, $U^{(2)}_{1,i+1} = 0$. Then we obtain

$$\begin{aligned} j_2 &= \min\{j \in [i+1, n]_{\mathbb{Z}} \mid U^{(2)}_{1,j} \neq 0\} \\ &= \min\{j \in [i+2, n]_{\mathbb{Z}} \mid U^{(2)}_{1,j} \neq 0\} \\ &= t_2. \end{aligned}$$

Then we have $p_{i,k} = p_{i+1,k}$ for $k \geq 2$. It means $M = N$. Let $\rho_{i+1}(H_2) = (\pi^{(2)}, o^{(2)}, 0, (W^{(2)}_{kl})_{k<l})$. Since $p_{i,k} = p_{i+1,k}$ for $k \geq 2$, we have that $\pi^{(2)} = \nu^{(2)}$ and $W^{(2)}_{kl} = V^{(2)}_{kl}$ unless $(k, l) = (1, i), (1, i+1)$. Thus, to show that $f_i K_2 = \rho_{i+1}(H_2)$, it suffices to show that $k_{f_i K_2} = 1$. Since $\varepsilon_i(H_2) = 0$, we have

$$\sum_{k=m}^{i+1} (U^{(2)}_{i+2-k,i+1} - U^{(2)}_{i+1-k,i}) \leq 0 \quad (m = 1, 2, \dots, i+1).$$

This implies that

$$\sum_{k=0}^{m}(U^{(2)}_{k+1,i+1} - U^{(2)}_{k,i}) \leq 0 \quad (m = 0, 1, \dots, i).$$

Since $U^{(2)}_{1,i+1} = 0$, we have

$$\sum_{k=1}^{m}(U^{(2)}_{k,i} - U^{(2)}_{k+1,i+1}) \geq 0 \quad (m = 1, 2, \dots, i).$$

Since $\varepsilon_i(H_2) = 0$ and $j_{\iota(H_1)} = i$, we have $\varepsilon_i(K_2) = 0$ by Lemma 4.32. Since $U^{(2)}_{1,i+1} = 0$ and $j_2 > i+1$, we obtain $V^{(2)}_{1,i+1} = 0$. Then we also have

$$\sum_{k=1}^{m}(V^{(2)}_{k,i} - V^{(2)}_{k+1,i+1}) \geq 0 \quad (m = 1, 2, \dots, i).$$

By Definition 4.11, we have $V^{(2)}_{1,i} - V^{(2)}_{2+1,i+1} = U^{(2)}_{1,i} - U^{(2)}_{2+1,i+1} + 1 > 0$ and $V^{(2)}_{k,i} - V^{(2)}_{k+1,i+1} = U^{(2)}_{k,i} - U^{(2)}_{k+1,i+1}$ for $k > 1$. It follows that

$$\sum_{k=1}^{m}(V^{(2)}_{k,i} - V^{(2)}_{k+1,i+1}) > 0 \quad (m = 1, 2, \dots, i).$$

This implies $\varphi^{(k)}_i(K_2) > 0$ for $k \geq 1$, then we have $k_{f_iK_2} = 1$. Hence, $f_i\rho_{j_{\iota(H_1)}}(H_2) = \rho_{j_{\iota(f_iH)}}(H_2)$ holds.

Next, we consider the case where $(f_i \circ \Theta)(H_1 \otimes H_2) = f_iK_1 \otimes K_2$. In this case, $\varphi_i(K_1) > 0$ holds. If $j_{\iota(H_1)} \neq j_{\iota(f_iH_1)}$, then $j_{\iota(H_1)} = i$, $k_{f_iH_1} = \ell(\lambda^{(1)})$, and $U^{(1)}_{\ell(\lambda^{(1)}),i} = 1$ hold by Theorem 3.36 and Definition 4.3. However, this leads to

$$\varphi_i(K_1) = \varphi_i(H_1) - 1 = U^{(1)}_{\ell(\lambda^{(1)}),i} - 1 = 0.$$

Then we have $j_{\iota(H_1)} = j_{\iota(f_iH_1)}$. Hence, we obtain $K_2 = \rho_{j_{\iota(f_iH)}}(H_2)$. To show that $f_iK_1 = \iota(f_iH_1)$, it suffices to show that $k_{f_iH_1} = k_{f_i\iota(H_1)}$. Since $j_{\iota(H_1)} = j_{\iota(f_iH_1)}$, we have $j_{\iota(H_1)} \neq i$, $k_{f_iH} \neq \ell(\lambda^{(1)})$, or $U^{(1)}_{\ell(\lambda^{(1)}),i} \neq 1$. Note that

$$\varphi^{(k)}_i(H_1) = \varphi^{(k)}_i(\iota(H_1)) \quad (k \in [\ell(\lambda^{(1)}) - 2])$$

holds by Definition 4.3. If $j_{\iota(H_1)} \neq i$, we have

$$V^{(1)}_{\ell(\lambda^{(1)})-1,i} - V^{(1)}_{\ell(\lambda^{(1)}),i+1} \geq U^{(1)}_{\ell(\lambda^{(1)})-1,i} - U^{(1)}_{\ell(\lambda^{(1)}),i+1},$$
$$V^{(1)}_{\ell(\lambda^{(1)}),i} - V^{(1)}_{\ell(\lambda^{(1)})+1,i+1} = U^{(1)}_{\ell(\lambda^{(1)}),i} - U^{(1)}_{\ell(\lambda^{(1)})+1,i+1}.$$

Then $\varphi^{(k)}_i(\iota(H_1)) \geq \varphi^{(k)}_i(H_1)$ holds for $k \in [i]$. If $j_{\iota(H_1)} = i$ and $U^{(1)}_{\ell(\lambda^{(1)}),i} \neq 1$, then we have $U^{(1)}_{\ell(\lambda^{(1)}),i} > 1$ by the choice of $j_{\iota(H_1)}$. Then we obtain that

$$\varphi^{(\ell(\lambda^{(1)})-1)}_i(\iota(H_1)) = \varphi^{(\ell(\lambda^{(1)})-1)}_i(H_1), \quad \varphi^{(\ell(\lambda^{(1)}))}_i(\iota(H_1)) = U^{(1)}_{\ell(\lambda^{(1)}),i} - 1 > 0.$$

If $j_{\iota(H_1)} = i$, $U^{(1)}_{\ell(\lambda^{(1)}),i} = 1$, and $k_{f_iH_1} \neq \ell(\lambda^{(1)})$, then $\varphi^{(\ell(\lambda^{(1)})-1)}_i(\iota(H_1)) = \varphi^{(\ell(\lambda^{(1)})-1)}_i(H_1) > 0$. Also, we have

$$\begin{aligned}\varphi^{(\ell(\lambda^{(1)}))}_i(\iota(H_1)) &= U^{(1)}_{\ell(\lambda^{(1)}),i} - 1 + \varphi^{(\ell(\lambda^{(1)})-1)}_i(H_1) \\ &= \varphi^{(\ell(\lambda^{(1)})-1)}_i(H_1) > 0.\end{aligned}$$

Hence, we have $k_{f_iH_1} = k_{f_i\iota(H_1)}$. Thus, $f_iK_1 = \iota(f_iH_1)$ holds by Definition 4.3.

Suppose $(\Theta \circ f_i)(H_1 \otimes H_2) = \iota(H_1) \otimes \rho_{j_{\iota(H_1)}}(f_i H_2)$. By Lemma 4.40, we have $(f_i \circ \Theta)(H_1 \otimes H_2) = K_1 \otimes f_i K_2$. Note that $\varphi_i(H_1) \leq \varepsilon_i(H_2)$, $\varphi_i(K_1) \leq \varepsilon_i(K_2)$ holds by Definition 2.3.

Suppose $f_i H_2 = 0$. If $j_{\iota(H_1)} \neq i$, then $\varphi_i(K_2) = \varphi_i(H_2) = 0$ holds by Lemma 4.33. If $j_{\iota(H_1)} = i$, then $\varphi_i(H_1) > 0$ holds. This implies that $\varepsilon_i(H_2) > 0$ by Definition 2.3, and hence $\varphi_i(K_2) = \varphi_i(H_2) = 0$ holds from Lemma 4.33. Then we obtain $f_i K_2 = 0$.

Suppose $f_i H_2 \neq 0$. It suffices to show that $\rho_{j_{\iota(H_1)}}(f_i H_2) = f_i K_2$. Let $p_{j_{\iota(H_1)}}(H_2) = (p_{j_{\iota(H_1)},m})_{m=0,1,\dots,N}$, where $p_{j_{\iota(H_1)},m} = (i_m, j_m)$. Let $p_{j_{\iota(H_1)}}(f_i H_2) = (p'_{j_{\iota(H_1)},m})_{m=0,1,\dots,M}$, where $p'_{j_{\iota(H_1)},m} = (i'_m, j'_m)$. Let $f_i H_2 = (\lambda^{(2)}, \mu^{(i,2)}, 0, (U^{(i,2)}_{kl})_{k<l})$. Note that it follows from Theorem 3.36 that

$$U^{(i,2)}_{kl} = U^{(2)}_{kl} \quad \text{if } (k,l) \neq (k_{f_i H_2}, i), (k_{f_i H_2}, i+1). \tag{4.15}$$

Suppose $j_{\iota(H_1)} < i$. Let m_0 be the largest $m \in [0,N]_{\mathbb{Z}}$ such that $j_m < i$. We have $p_{j_{\iota(H_1)},m} = p'_{j_{\iota(H_1)},m}$ for $m \in [0, m_0]_{\mathbb{Z}}$ from (4.15). By Lemma 4.33, we know that

$$\varphi_i(K_2) = \varphi_i(H_2). \tag{4.16}$$

Suppose $i_{m_0} \neq k_{f_i H_2}, k_{f_i H_2} - 1$. By Definition 4.11, $p_{j_{\iota(H_1)}}(H_2) = p_{j_{\iota(H_1)}}(f_i H_2)$ holds. Also, by Definition 4.11 and the choice of i_{m_0}, we have that

$$\sum_{k=k_{f_i H_2}}^{n} (V^{(2)}_{ki} - V^{(2)}_{k+1,i+1}) = \sum_{k=k_{f_i H_2}}^{n} (U^{(2)}_{ki} - U^{(2)}_{k+1,i+1}). \tag{4.17}$$

In particular, we obtain

$$V^{(2)}_{k_{f_i H_2},i} - V^{(2)}_{k_{f_i H_2}+1,i+1} = U^{(2)}_{k_{f_i H_2},i} - U^{(2)}_{k_{f_i H_2}+1,i+1} > 0.$$

Then it follows from (4.16) that $k_{f_i H_2} = k_{f_i K_2}$. Then $f_i K_2 = \rho_{j_{\iota(H_1)}}(f_i H_2)$ holds.

Suppose $i_{m_0} = k_{f_i H_2} - 1$. By the definition of $k_{f_i H_2}$, $U^{(2)}_{k_{f_i H_2}-1,i} - U^{(2)}_{k_{f_i H_2},i+1} \leq 0$. If $U^{(2)}_{k_{f_i H_2}-1,i} = 0$, then $p_{j_{\iota(H_1)}}(H_2) = p_{j_{\iota(H_1)}}(f_i H_2)$ is clear by Definition 4.11 and Theorem 3.36. If $U^{(2)}_{k_{f_i H_2}-1,i} > 0$, then $U^{(2)}_{k_{f_i H_2},i+1} > 0$ holds. Also, we have $U^{(i,2)}_{k_{f_i H_2}-1,i}, U^{(i,2)}_{k_{f_i H_2},i+1} > 0$ by Theorem 3.36. Then we obtain that

$$\begin{gathered}
p_{j_{\iota(H_1)},m_0+1} = p'_{j_{\iota(H_1)},m_0+1} = (k_{f_i H_2} - 1, i), \\
p_{j_{\iota(H_1)},m_0+2} = p'_{j_{\iota(H_1)},m_0+2} = (k_{f_i H_2}, i), \\
p_{j_{\iota(H_1)},m_0+3} = p'_{j_{\iota(H_1)},m_0+3} = (k_{f_i H_2}, i+1), \\
p_{j_{\iota(H_1)},m_0+4} = p'_{j_{\iota(H_1)},m_0+4} = (k_{f_i H_2} + 1, i+1).
\end{gathered}$$

It follows that $p_{j_{\iota(H_1)}}(H_2) = p_{j_{\iota(H_1)}}(f_i H_2)$. Also, by Definition 4.11, we have that $V^{(2)}_{k_{f_i H_2},i} - V^{(2)}_{k_{f_i H_2}+1,i+1} = U^{(2)}_{k_{f_i H_2},i} - U^{(2)}_{k_{f_i H_2}+1,i+1}$ and (4.17). Hence, $k_{f_i H_2} = k_{f_i K_2}$ holds from (4.16). Then $\rho_{j_{\iota(H_1)}}(f_i H_2) = f_i K_2$ holds.

Suppose $i_{m_0} = k_{f_i H_2}$. If $U^{(i,2)}_{k_{f_i H_2},i} > 0$, $p_{j_{\iota(H_1)}}(H_2) = p_{j_{\iota(H_1)}}(f_i H_2)$ and (4.17) hold by Definition 4.11. Also, $U^{(i,2)}_{k_{f_i H_2},i} > 0$ implies $U^{(2)}_{k_{f_i H_2},i} > 1$. By Definition 4.11, we have

$$V^{(2)}_{k_{f_i H_2},i} - V^{(2)}_{k_{f_i H_2}+1,i+1} = \begin{cases} U^{(2)}_{k_{f_i H_2},i} - U^{(2)}_{k_{f_i H_2}+1,i+1} & \text{if } U^{(2)}_{k_{f_i H_2}+1,i+1} > 0, \\ U^{(2)}_{k_{f_i H_2},i} - U^{(2)}_{k_{f_i H_2}+1,i+1} - 1 & \text{otherwise.} \end{cases}$$

Then we have $V^{(2)}_{k_{f_iH_2},i} - V^{(2)}_{k_{f_iH_2}+1,i+1} > 0$. It follows from (4.16) and (4.17) that $k_{f_iH_2} = k_{f_iK_2}$. Thus, $\rho_{j_{\iota(H_1)}}(f_iH_2) = f_iK_2$. Suppose $U^{(i,2)}_{k_{f_iH_2},i} = 0$. This means $U^{(2)}_{k_{f_iH_2},i} = 1$, and hence $U^{(2)}_{k_{f_iH_2}+1,i+1} = 0$ holds by the definition of $k_{f_iH_2}$. Then we have

$$\begin{aligned} p_{j_{\iota(H_1)},m_0+1} &= (k_{f_iH_2}, i),\\ p_{j_{\iota(H_1)},m_0+2} &= (k_{f_iH_2}+1, i),\\ p_{j_{\iota(H_1)},m_0+3} &= (k_{f_iH_2}+1, j_{m_0+3}), \end{aligned}$$

and

$$\begin{aligned} p'_{j_{\iota(H_1)},m_0+1} &= (k_{f_iH_2}, i+1),\\ p'_{j_{\iota(H_1)},m_0+2} &= (k_{f_iH_2}+1, i+1),\\ p'_{j_{\iota(H_1)},m_0+3} &= (k_{f_iH_2}+1, j'_{m_0+3}), \end{aligned}$$

where $j_{m_0+3}, j'_{m_0+3} > i+1$. From (4.15), $j_{m_0+3} = j'_{m_0+3}$ holds. In particular, we have $p_{j_{\iota(H_1)},m} = p'_{j_{\iota(H_1)},m}$ for $m \geq m_0+3$. Since

$$\varphi_i^{(k_{f_iH_2})}(H_2) = U^{(2)}_{k_{f_iH_2},i} - U^{(2)}_{k_{f_iH_2}+1,i+1} = 1$$

and $\varphi_i^{(k_{f_iH_2})+1}(H_2) > 0$, we have $U^{(2)}_{k_{f_iH_2}+1,i} - U^{(2)}_{k_{f_iH_2}+2,i+1} \geq 0$. It follows that

$$\begin{aligned} V^{(2)}_{k_{f_iH_2},i} - V^{(2)}_{k_{f_iH_2}+1,i+1} &= U^{(2)}_{k_{f_iH_2},i} - 1 - U^{(2)}_{k_{f_iH_2}+1,i+1} = 0,\\ V^{(2)}_{k_{f_iH_2}+1,i} - V^{(2)}_{k_{f_iH_2}+2,i+1} &= U^{(2)}_{k_{f_iH_2}+1,i} + 1 - U^{(2)}_{k_{f_iH_2}+2,i+1} > 0. \end{aligned}$$

This implies $k_{f_iK_2} = k_{f_iH_2} + 1$. Then $\rho_{j_{\iota(H_1)}}(f_iH_2) = f_iK_2$ holds by Theorem 3.36 and Definition 4.11.

Suppose $j_{\iota(H_1)} = i$. In this case, we have $\varphi_i(H_1) > 0$, and hence $\varepsilon_i(H_2) > 0$ holds. If $U^{(2)}_{1,i+1} > 0$, then $p_{j_{\iota(H_1)}}(H_2) = p_{j_{\iota(H_1)}}(f_iH_2)$ holds from $U^{(i,2)}_{1,i+1} > 0$. By Definition 4.11, $\varphi_i^{(k)}(K_2) = \varphi_i^{(k)}(H_2)$ holds for $k \in [i]$, and hence we have that (4.16) and $k_{f_iK_2} = k_{f_iH_2}$. Then, we have $\rho_{j_{\iota(H_1)}}(f_iH_2) = f_iK_2$.

Suppose $U^{(2)}_{1,i+1} = 0$, then $k_{e_iH_2} < i+1$. By the definition of $k_{e_iH_2}$, we have

$$\sum_{k=m}^{i+1-k_{e_iH_2}} (U^{(2)}_{ki} - U^{(2)}_{k+1,i+1}) < 0 \quad (m \in [0, i+1-k_{e_iH_2}]_{\mathbb{Z}}).$$

By Definition 4.11, we have that

$$V^{(2)}_{k,i} - V^{(2)}_{k+1,i+1} = \begin{cases} U^{(2)}_{1,i} - U^{(2)}_{2,i+1} + 1 & \text{if } k = 1,\\ U^{(2)}_{k,i} - U^{(2)}_{k+1,i+1} & \text{otherwise,} \end{cases}$$

then we obtain that

$$\sum_{k=m}^{i+1-k_{e_iH_2}} (V^{(2)}_{ki} - V^{(2)}_{k+1,i+1}) \leq 0 \quad (m \in [0, i+1-k_{e_iH_2}]_{\mathbb{Z}}).$$

This implies $\varphi^{(i+1-k_{e_iH_2})}(K_2) = \varphi_i^{(i+1-k_{e_iH_2})}(H_2) = 0$, and hence $\varphi_i^{(k)}(K_2) = \varphi_i^{(k)}(H_2)$ holds for $k > i+1-k_{e_iH_2}$. By the definition of $k_{f_iH_2}$, we have $k_{f_iH_2} \geq i+1-k_{e_iH_2}$. Thus,

$k_{f_iK_2} = k_{f_iH_2} > 1$. Then $p_{j_{\iota(H_1)}}(H_2) = p_{j_{\iota(H_1)}}(f_iH_2)$. Therefore, $\rho_{j_{\iota(H_1)}}(f_iH_2) = f_iK_2$ holds.

Suppose $j_{\iota(H_1)} \geq i+1$. By Definition 4.11 and (4.15), $p_{j_{\iota(H_1)}}(H_2) = p_{j_{\iota(H_1)}}(f_iH_2)$ holds. Also, we have $k_{f_iK_2} = k_{f_iH_2}$. Thus, $\rho_{j_{\iota(H_1)}}(f_iH_2) = f_iK_2$ holds. □

From Propositions 4.38, 4.39, and 4.41, we know that Θ is a crystal morphism. Furthermore, we have the following proposition.

Proposition 4.42. Θ is a crystal embedding.

Proof. From Propositions 4.38, 4.39, and 4.41, Θ is a crystal morphism. It then suffices to show that $\Theta\colon \mathbb{H}(\lambda)\otimes\mathbb{H}(\mu)\cup\{0\} \to \bigsqcup_{\xi\in P^+}\mathbb{H}(\nu)\otimes\mathbb{H}(\xi)\cup\{0\}$ is an embedding. For $i = 1, 2$, let $H_i = (\lambda^{(i)}, \mu^{(i)}, 0, (U^{(i)}_{kl})_{k<l}) \in \mathbb{H}(\lambda^{(i)})$, $K_i = (\lambda^{(i)}, \xi^{(i)}, 0, (V^{(i)}_{kl})_{k<l}) \in \mathbb{H}(\lambda^{(i)})$. Set $a = j_{\iota(H_1)}$ and $b = j_{\iota(K_1)}$. Suppose that $\Theta(H_1\otimes H_2) = \Theta(K_1\otimes K_2)$. Let $\Theta(H_1\otimes H_2) \in \mathbb{H}(\nu^{(1)})\otimes \mathbb{H}(\nu^{(2)})$. Then we have $\iota(H_1) = \iota(K_1)$, $\rho_a(H_2) = \rho_b(K_2)$. Let $p_a(H_2) = (p_{a,m})_{m=0,1,\ldots,N}$, where $p_{a,m} = (i_m, j_m)$. Let $p_b(K_2) = (p_{b,m})_{m=0,1,\ldots,M}$, where $p_{b,m} = (s_m, t_m)$.

Suppose $i_N \neq n$. By Definition 4.11, there exists $c \in I$ such that $\nu^{(2)}_c = \lambda^{(2)}_c + 1$, $\nu^{(2)}_k = \lambda^{(2)}_k$ $(k \neq c)$. It follows from Proposition 4.28 that $\bar{\rho}_c\rho_a(H_2) = H_2$ and $\bar{\rho}_c\rho_b(K_2) = K_2$. Since $\rho_a(H_2) = \rho_b(K_2)$, $H_2 = K_2$ holds. Suppose $i_N = n$. By Definition 4.11, $\nu^{(2)}_k = \lambda^{(2)}_k - 1$ for $k \in I$. Let $L = (\lambda, \mu, 0, (U_{kl})_{k<l})$, where $\lambda_k = \lambda^{(2)}_k + 1$, $\mu_k = \mu^{(2)}_k + 1$ for $k \in [n]$, and $U_{kl} = U^{(2)}_{kl}$ for $1 \leq k < l \leq n$. Let $L' = (\nu, \xi, 0, (V_{kl})_{k<l})$, where $\nu_k = \lambda^{(2)}_k + 1$, $\xi_k = \xi^{(2)}_k + 1$ for $k \in [n]$, and $V_{kl} = V^{(2)}_{kl}$ for $1 \leq k < l \leq n$. By the proof of Proposition 4.28, $\bar{\rho}_n(L) = H_2$ and $\bar{\rho}_n(L') = K_2$. Since $\rho_a(H_2) = \rho_b(K_2)$, $L = L'$ holds. Hence, we have $H_2 = K_2$. In particular, $a = b$ holds.

Since $\iota(H_1) = \iota(K_1)$, we have $U^{(1)}_{kl} = V^{(1)}_{kl}$ if $(k,l) \neq (\ell(\lambda^{(1)}), j_{\iota(H_1)}), (\ell(\lambda^{(1)}), j_{\iota(K_1)})$. Since $a = b$, we obtain $U^{(1)}_{kl} = V^{(1)}_{kl}$ for $1 \leq k < l \leq n$, and therefore $\mu^{(1)} = \xi^{(1)}$ holds. This implies $H_1 = K_1$. Thus, we have $H_1 \otimes H_2 = K_1 \otimes K_2$. □

For $H \otimes K \in \mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$, let $\overline{H \otimes K}$ be the highest weight vector of the connected component that contains $H \otimes K$. Let $\overline{}$ be a map from $\mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$ to $\mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$ that maps $H \otimes K$ to $\overline{H \otimes K}$.

Theorem 4.43. Let $\lambda, \mu \in P^+$ with $\lambda \neq 0$. Set $N = \sum_{i\in I}\lambda_i$. Let $M(\lambda,\mu)$ be the set of highest weight vectors in $\mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$. Then

$$(\overline{}, \Theta^N)\colon \mathbb{H}(\lambda) \otimes \mathbb{H}(\mu) \longrightarrow \bigsqcup_{L\in M(\lambda,\mu)} \mathbb{H}(\mathrm{wt}(L))$$
$$H \otimes K \longmapsto (\overline{H \otimes K}, \Theta^N(H \otimes K)).$$

is a crystal isomorphism. Note that the elements of the disjoint union are denoted by pairs (L, H'), where $L \in M(\lambda,\mu)$ and $H' \in \mathbb{H}(\mathrm{wt}(L))$.

Proof. Let $\lambda, \mu \in P^+$. Set $N = \sum_{i\in I}\lambda_i$. For $H \otimes K \in \mathbb{H}(\lambda) \otimes \mathbb{H}(\mu)$, let $\Theta(H \otimes K) \in \mathbb{H}(\nu) \otimes \mathbb{H}(\xi)$. Let $\nu \in P^+$. By Definition 4.35, we have $\sum_{i\in I}\nu_i = N - 1$. Then we can assume $\Theta^N(H\otimes K) \in \mathbb{H}(0)\otimes\mathbb{H}(\xi)$. Then $\Theta^N(H\otimes K)$ can be viewed as $\Theta^N(H\otimes K) \in \mathbb{H}(\xi)$. In particular, the highest weight vector in $\mathbb{H}(\xi)$ is obtained by $\Theta^N(\overline{H \otimes K})$. Thus, by Lemma 3.32 and Proposition 4.42, the statement holds. □

5. Algorithms and Implementations for the Crystal of K-hives

In this section, we give a set of algorithms to compute the crystal structure on $\mathbb{H}(\lambda)$ defined in Section 3 and show examples of the execution of the implementations of these algorithms. In 5.1, a set of algorithms for computing the crystal structure on $\mathbb{H}(\lambda)$ is given. In 5.2, examples of the execution of these algorithms by the Python implementation named *khive-crystal* are shown. The main reference is [15].

5.1. Algorithms for crystal of K-hives.

In this subsection, we give a set of algorithms to compute the components of the crystal structure on $\mathbb{H}(\lambda)\,(\lambda \in P^+)$ using two approaches. One approach is based on Definition 3.26, which implies that the crystal structure on $\mathbb{H}(\lambda)$ is regarded as a submodule of a tensor product of crystals of the form $\mathbb{H}(\Lambda_k)$. The other approach is based on Theorem 3.36, which is a more direct combinatorial description.

To consider algorithms, we regard $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$ as a hash table with keys λ, μ, γ, and $(U_{ij})_{i<j}$, where the value of λ is an array $[\lambda_1, \lambda_2, \ldots, \lambda_n]$, the value of μ is an array $[\mu_1, \mu_2, \ldots, \mu_n]$, the value of γ is an array $[0, 0, \ldots, 0]$, and the value of $(U_{ij})_{i<j}$ is a two-dimensional array $[[U_{12}, U_{13}, \ldots], [U_{23}, \ldots], \ldots, [U_{n-1,n}]]$.

To give algorithms for the crystal structure on $\mathbb{H}(\lambda)$ based on Definition 3.26, we first consider algorithms for the crystal structure on $\mathbb{H}(\Lambda_k)\,(k \in I)$. The maps f_i (resp. e_i) $(I \in I)$ for $\mathbb{H}(\Lambda_k)\,(k \in I)$ are computed by Algorithm 1 (resp. Algorithm 2). Note that the maps wt, $\varphi_i, \varepsilon_i\,(i \in I)$ are simply computed by Definition 3.2 as $\sum_{k\in I}(\mu_k - \mu_{k+1})\Lambda_k$, $\max(\mu_i - \mu_{i+1}, 0)$, $\max(\mu_{i+1} - \mu_i, 0)$, respectively for $H = (\Lambda_k, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_k)$.

Algorithm 1 Algorithm for f_i on $\mathbb{H}(\Lambda_k)$

Require: $H = (\Lambda_k, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_k)$, $i \in I$
Ensure: f_iH

```
if max(μ_i − μ_{i+1}, 0) = 0 then
    return 0
end if
Take k_0 from {k ∈ [i] | U_{k,i} > 0}
μ_i := μ_i − 1
μ_{i+1} := μ_{i+1} + 1
U_{k_0,i} := U_{k_0,i} − 1
U_{k_0,i+1} := U_{k_0,i+1} + 1
return (Λ_k, μ, 0, (U_{ij})_{i<j})
```

Let us give an example of the execution of Algorithm 1.

Example 5.1. The action of f_i on the $U_q(\mathfrak{sl}_4)$-crystal $\mathbb{H}(\Lambda_3)$ is computed as follows by Algorithm 1. Let $H = (\Lambda_3, \Lambda_3, (U_{kl})_{k<l}) \in \mathbb{H}(\Lambda_3)$, where $U_{kl} = 0$ for $1 \leq k < l \leq 4$. Set $\mu = \Lambda_3$. Note that Λ_3 corresponds to the partition $(1, 1, 1, 0)$. For $i = 1$, we have $f_1H = 0$ since $\max(\mu_1 - \mu_2, 0) = 0$. Also, for $i = 2$, we have $f_2H = 0$ since $\max(\mu_2 - \mu_3, 0) = 0$. Let $i = 3$. In this case, $\max(\mu_3 - \mu_4, 0) = 1$. Then we can proceed to the next step. Since $\{k \in [3] \mid U_{k,3} > 0\} = \{3\}$, k_0 is uniquely determined to 3. Then, set $\xi = \mu$, then set $\xi_3 = \mu_3 - 1 = 0$ and $\xi_4 = \mu_4 + 1 = 1$. Also, set $V_{ij} = U_{ij}$, and set $V_{3,3} = U_{3,3} - 1 = 0$ and $V_{3,4} = U_{3,4} + 1 = 1$. Then we have $f_iH = (\Lambda_3, \xi, 0, (V_{ij})_{i<j})$. See Fig. 17.

Algorithm 2 Algorithm for e_i on $\mathbb{H}(\Lambda_k)$

Require: $H = (\Lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\Lambda_k)$, $i \in I$
Ensure: $e_i H$
if $\max(\mu_{i+1} - \mu_i, 0) = 0$ **then**
  return 0
end if
Take k_0 from $\{k \in [i+1] \mid U_{k,i+1} > 0\}$
$\mu_i := \mu_i + 1$
$\mu_{i+1} := \mu_{i+1} - 1$
$U_{k_0,i} := U_{k_0,i} + 1$
$U_{k_0,i+1} := U_{k_0,i+1} - 1$
return $(\lambda, \mu, 0, (U_{ij})_{i<j})$

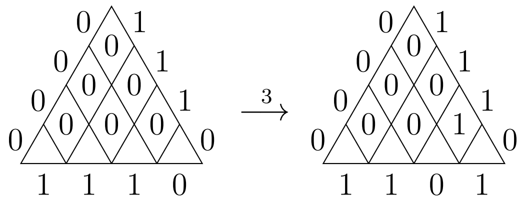


FIGURE 17. Action of f_3 on the $U_q(\mathfrak{sl}_4)$-crystal $\mathbb{H}(\Lambda_3)$

Algorithms 1 and 2 generate results that correspond to Definition 3.2 as follows.

Proposition 5.2. For $k \in I$, let $H \in \mathbb{H}(\Lambda_k)$. Let $i \in I$.

(1) Let K be the result of Algorithm 1 with inputs H and i. Then, $K = f_i H$,
(2) Let K be the result of Algorithm 2 with inputs H and i. Then, $K = e_i H$.

Proof. For $k \in I$, let $H \in \mathbb{H}(\Lambda_k)$. Let $i \in I$. (1) Let K be the result of Algorithm 1 with inputs H and i. By Lemma 3.1, k_0 in Algorithm 1 is uniquely determined. Then we have $K = f_i H$ from Definition 3.2. Similarly, (2) can be shown. □

For $\lambda \in P^+$, the map Ψ_λ is computed by Algorithm 3.

The following is an example of the execution of Algorithm 3.

Example 5.3. Let $n = 4$, $\lambda = (3, 2, 1, 0)$, and $\mu = (2, 3, 1, 0)$. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, where $U_{12} = 1$ and $U_{ij} = 0$ if $(i, j) \neq (1, 2)$ and $i < j$. Then $\Psi_\lambda(H)$ is computed by Algorithm 3 as follows. Set $\nu = \ell(\lambda) = 3$. Let $\lambda^{(2)} = (\lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_n^{(2)})$, where $\lambda_k^{(2)} = 1$ if $k \in [\nu]$ else $\lambda_k^{(2)} = 0$. Set $U_{ij}^{(2)} = U_{ij}$ for $1 \leq i < j \leq 4$. Since $\min\{l \in [4] \mid U_{1l} > 0\} = 1$, set $U_{11}^{(2)} = 1$ and $U_{12}^{(2)} = U_{13}^{(2)} = U_{14}^{(2)} = 0$. Since $\min\{l \in [4] \mid U_{2l} > 0\} = 2$, set $U_{22}^{(2)} = 1$ and $U_{23}^{(2)} = U_{24}^{(2)} = 0$. Since $\min\{l \in [4] \mid U_{3l} > 0\} = 3$, set $U_{33}^{(2)} = 1$ and $U_{34}^{(2)} = 0$. Set

$$\mu_1^{(2)} = U_{11}^{(2)} = 1, \quad \mu_2^{(2)} = U_{12}^{(2)} + U_{22}^{(2)} = 1,$$
$$\mu_2^{(2)} = U_{13}^{(2)} + U_{23}^{(2)} + U_{33}^{(2)} = 1, \quad \mu_4^{(2)} = U_{14}^{(2)} + U_{24}^{(2)} + U_{34}^{(2)} + U_{44}^{(2)} = 0.$$

Algorithm 3 Algorithm for Ψ_λ

Require: $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$
Ensure: $\Psi_\lambda(H)$

for $k = 1, 2, \ldots, n$ **do** ▷ Compute $\lambda^{(2)}$
  if $k \in [1, \ell(\lambda)]_{\mathbb{Z}}$ **then**
    $\lambda_k^{(2)} = 1$
  else
    $\lambda_k^{(2)} = 0$
  end if
end for
$\lambda^{(2)} := (\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_n^{(2)})$
$(U_{ij}^{(2)})_{i<j} := (U_{ij})_{i<j}$ ▷ Compute $(U_{ij}^{(2)})_{i<j}$
for $i = 1, 2, \ldots, n-1$ **do**
  for $j = i+1, i+2, \ldots, n$ **do**
    if $j = \min\{l \in [n] \mid U_{il} > 0\}$ **then**
      $U_{ij}^{(2)} := 1$
    else
      $U_{ij}^{(2)} := 0$
    end if
  end for
end for
for $k = 1, 2, \ldots, n$ **do** ▷ Compute $\mu^{(2)}$
  $\mu_k^{(2)} := \sum_{l=1}^{i} U_{li}^{(2)}$
end for
$\mu^{(2)} := (\mu_1^{(2)}, \mu_2^{(2)}, \ldots, \mu_n^{(2)})$
for $k = 1, 2, \ldots, n$ **do** ▷ Compute $\lambda^{(1)}$
  $\lambda_k^{(1)} := \lambda_k - \lambda_k^{(2)}$
end for
$\lambda^{(1)} := (\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_n^{(1)})$
$(U_{ij}^{(1)})_{i<j} := (U_{ij})_{i<j}$ ▷ Compute $(U_{ij}^{(1)})_{i<j}$
for $i = 1, 2, \ldots, n-1$ **do**
  for $j = i+1, i+2, \ldots, n$ **do**
    $U_{ij}^{(1)} := U_{ij} - U_{ij}^{(2)}$
  end for
end for
for $i = 1, 2, \ldots, n$ **do** ▷ Compute $\mu^{(1)}$
  $\mu_i^{(1)} = \sum_{l=1}^{i} U_{li}^{(1)}$
end for
return $(\lambda^{(1)}, \mu^{(1)}, 0, (U_{ij}^{(1)})_{i<j}) \otimes (\lambda^{(2)}, \mu^{(2)}, 0, (U_{ij}^{(2})_{i<j})$

Set

$$\lambda_1^{(1)} = \lambda_1 - \lambda_1^{(2)} = 2, \quad \lambda_2^{(1)} = \lambda_2 - \lambda_2^{(2)} = 1,$$
$$\lambda_3^{(1)} = \lambda_3 - \lambda_3^{(2)} = 0, \quad \lambda_4^{(1)} = \lambda_4 - \lambda_4^{(2)} = 0.$$

Set $U_{ij}^{(1)} = U_{ij} - U_{ij}^{(2)}$ for $1 \le i \le j \le 4$. Set

$$\mu_1^{(1)} = U_{11}^{(1)} = 1, \quad \mu_2^{(1)} = U_{12}^{(1)} + U_{22}^{(1)} = 2,$$
$$\mu_2^{(1)} = U_{13}^{(1)} + U_{23}^{(1)} + U_{33}^{(1)} = 0, \quad \mu_4^{(1)} = U_{14}^{(1)} + U_{24}^{(1)} + U_{34}^{(1)} + U_{44}^{(1)} = 0.$$

Then $\Psi_\lambda = (\lambda^{(1)}, \mu^{(1)}, 0, (U_{ij}^{(1)})) \otimes (\lambda^{(2)}, \mu^{(2)}, 0, (U_{ij}^{(2)}))$. See Fig. 18

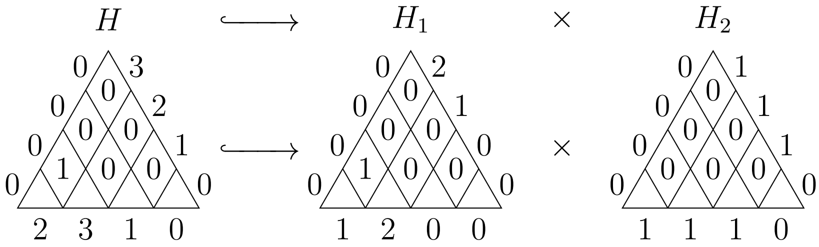


FIGURE 18. Action of Ψ_λ on $\mathbb{H}(\lambda)$

Algorithm 3 generates a result corresponding to an image of Ψ_λ.

Proposition 5.4. For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let K be the result of Algorithm 3 with input H. Then, $K = \Psi_\lambda(H)$.

Proof. The statement immediately follows from Definition 3.14. □

The map Ψ is defined to apply Ψ_λ ($\lambda \in P^+$) repeatedly, and note that the algorithm for Ψ_λ is given by Algorithm 3. Then, the map Ψ is computed using Algorithm 4.

Algorithm 4 Algorithm for Ψ

Require: $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$
Ensure: $\Psi(H)$
$H_1 \otimes H_2 := \Psi_\lambda(H)$
$N = 2$
while $H_1 \notin \mathbb{H}(\Lambda_k)$ for any $k \in I$ **do**
  $K_1 \otimes K_2 := \Psi(H_1)$
  $H := K_1 \otimes K_2 \otimes H_2 \otimes \cdots \otimes H_N$
  $N = N + 1$
  Rename H as $H = H_1 \otimes H_2 \otimes \cdots \otimes H_N$
end while
return $\bigotimes_{k \in N} H_k$

The following is an example of the execution of Algorithm 4.

Example 5.5. Let $n = 4$, $\lambda = (3, 2, 1, 0)$ and $\mu = (2, 3, 1, 0)$. Let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, where $U_{12} = 0$ and $U_{ij} = 0$ if $(i, j) \neq (1, 2)$ and $i < j$. By Algorithm 3,

$$\begin{aligned}\Psi_\lambda(H) &= ((2,1,0,0), (1,2,0,0), (0^4), (U_{ij}^{(1)})) \otimes ((1,1,1,0), (1,1,1,0), (0^4), (U_{ij}^{(2)})) \\ &:= H_1 \otimes H_2,\end{aligned}$$

where

$$U_{ij}^{(1)} = \begin{cases} 1 & \text{if } (i,j) = (1,2), \\ 0 & \text{otherwise}, \end{cases}$$

$$U_{ij}^{(2)} = 0 \quad (1 \leq i < j \leq 4).$$

Since $H_1 \in \mathbb{H}((2,1,0,0))$, we proceed with the algorithm.

$$\begin{aligned} \Psi_\lambda(H_1) &= ((1,0,0,0),(0,1,0,0),(0^4),(V_{ij}^1)) \otimes ((1,1,0,0),(1,1,0,0),(0^4),(V_{ij}^1)) \\ &:= K_1 \otimes K_2, \end{aligned}$$

where

$$V_{ij}^{(1)} = \begin{cases} 1 & \text{if } (i,j) = (1,2), \\ 0 & \text{otherwise}, \end{cases}$$

$$V_{ij}^{(2)} = 0 \quad (1 \leq i < j \leq 4).$$

Then rename $K_1 \otimes K_2 \otimes H_2$ as $H_1 \otimes H_2 \otimes H_3$. Then, we have

$$\Psi(H) = H_1 \otimes H_2 \otimes H_3.$$

See Fig. 19.

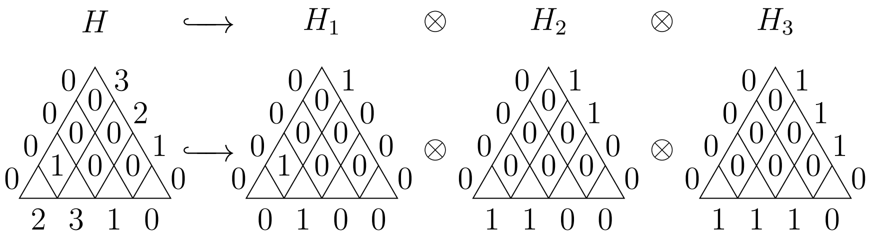


Figure 19. Action of Ψ on $\mathbb{H}(\lambda)$

The result of Algorithm 4 corresponds to the image of Ψ.

Proposition 5.6. For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let K be the result of Algorithm 4 for input H. Then, $K = \Psi(H)$.

Proof. By Proposition 3.19, it is clear that Algorithm 4 yields the image of Ψ if the while statement stops. For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Suppose that $H_1 \otimes H_2 \otimes \cdots \otimes H_{k+2}$ is obtained at the k-th step of the while statement in Algorithm 4, and $H_1 \notin \mathbb{H}(\Lambda_i)$ for all $i \in I$. Assume $H_1 \in \mathbb{H}(\lambda^{(1)})$ for $\lambda^{(1)} \in P^+$, where $\lambda^{(1)} \neq \Lambda_i$ for all $i \in I$. This means that there exists $m \in [n]$ such that $\lambda_m^{(1)} > 1$, especially $\lambda_1^{(1)} > 1$. Set $\lambda' = \lambda^{(1)}$ and $m_0 = \lambda_1^{(1)}$. Then at the $k + m_0 - 1$ step in the while statement, we have

$$H_1 \otimes H_2 \otimes \cdots \otimes H_{k+m_0+1}.$$

Assume $H_1 \in \mathbb{H}(\lambda^{(1)})$. Note that, since the indices are renamed, we retake H_1 and $\lambda^{(1)}$. By Algorithm 4, we have $\lambda_m^{(1)} = \max(\lambda_m^{(k)} - (m_0 - 1), 0)$ for $m \in [n]$. Since $\lambda' \in P^+$ and $m_0 = \lambda'_1$, $\lambda_m^{(1)} \in \{0,1\}$. Therefore, $H_1 \in \mathbb{H}(\Lambda_\nu)$ for $\nu \in I$. Thus, the while statement stops. $\square$

To compute $f_i, e_i\,(i \in I)$ on $\mathbb{H}(\lambda)$, we need the algorithm of Ψ^{-1} for the image of Ψ. Algorithm 5 computes Ψ^{-1} for $H \in \Psi(\mathbb{H}(\lambda)$.

Algorithm 5 Algorithm for Ψ^{-1}

Require: $H = H_1 \otimes H_2 \otimes \cdots \otimes H_N \in \bigotimes_k \mathbb{H}(\Lambda_k)$, $H_k = (\lambda^{(k)}, \mu^{(k)}, 0, (U_{ij}^{(k)})_{i<j}) \in \mathbb{H}(\lambda^{(k)})$.
Ensure: $\Psi^{-1}(H) \in \mathbb{H}(\lambda)$

for $i = 1, 2, \ldots, n$ **do**
  $\lambda_i := \sum_{k=1}^{N} \lambda_i^{(k)}$
end for
$\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_n)$
for $i = 1, 2, \ldots, n$ **do**
  $\mu_i := \sum_{k=1}^{N} \mu_i^{(k)}$
end for
$\mu := (\mu_1, \mu_2, \ldots, \mu_n)$
for $i = 1, 2, \ldots, n-1$ **do**
  for $j = i+1, i+2, \ldots, n$ **do**
    $U_{ij} := \sum_{k=1}^{N} U_{ij}^{(k)}$
  end for
end for
return $(\lambda, \mu, (0^n), (U_{ij})_{i<j})$

Proposition 5.7. For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let $\Psi(H) = H_1 \otimes H_2 \otimes \cdots \otimes H_N$. Let K be the result of Algorithm 5 with input $H_1 \otimes H_2 \otimes \cdots \otimes H_N$. Then, $K = H$.

Proof. For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let $\Psi(H) = H_1 \otimes H_2 \otimes \cdots \otimes H_N$. Let K be the result of Algorithm 5 with input $H_1 \otimes H_2 \otimes \cdots \otimes H_N$. Assume that $H = (\lambda, \mu, 0, (U_{ij})_{i<j})$ and $H_k = (\lambda^{(k)}, \mu^{(k)}, 0, (U_{ij}^{(k)})_{i<j})$ for $k = 1, 2, \ldots, N$. Let $\Psi_\lambda(H) = K_1 \otimes K_2$. Assume $K_m = (\nu^{(m)}, \xi^{(m)}, 0, (V_{ij}^{(k)}))$. By Definition 3.14, we have $\lambda_k = \nu_k^{(1)} + \nu_k^{(2)}$, $\mu_k = \xi_k^{(1)} + \xi_k^{(2)}$ for $k = 1, 2, \ldots, N$ and $U_{ij} = U_{ij}^{(1)} + U_{ij}^{(2)}$ for $1 \le i < j \le n$. By the construction of Ψ, we obtain

$$\begin{aligned} \lambda_k &= \lambda_k^{(1)} + \cdots + \lambda_k^{(N)} \quad (k = 1, 2, \ldots, N), \\ \mu_k &= \mu_k^{(1)} + \cdots + \mu_k^{(N)} \quad (k = 1, 2, \ldots, N), \\ U_{ij} &= U_{ij}^{(1)} + \cdots + U_{ij}^{(N)} \quad (1 \le i < j \le n). \end{aligned}$$

Thus, we have $K = H$. □

By Definition 3.26, the crystal structure on $\mathbb{H}(\lambda)$ is defined by considering $\mathbb{H}(\lambda)$ as a submodule of tensor products of $\mathbb{H}(\Lambda_k)$. In detail, embed $H \in \mathbb{H}(\lambda)$ into $\bigotimes_k \mathbb{H}(\Lambda_k)$ by Ψ, then compute the maps wt, $\varphi_i, \varepsilon_i, f_i, e_i\,(i \in I)$ by Definition 2.3, then pull it back into $\mathbb{H}(\lambda)$. Then, the maps wt, $\varphi_i, \varepsilon_i, f_i, e_i\,(i \in I)$ are computed by the following algorithms. For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let $\Psi(H) = H_1 \otimes H_2 \otimes \cdots \otimes H_N$, which is computed by Algorithm 4. Then $\mathrm{wt}(H)$ is computed by $\mathrm{wt}(H) = \sum_{k=1}^{N} \mathrm{wt}(H_k)$, where $\mathrm{wt}(H_k)$ is computed by an algorithm of wt for $\mathbb{H}(\Lambda_{k'})$ for some $k' \in I$. Then $\varphi_i(H)$ is computed by $\varphi_i(H) = \varphi_i(H_1 \otimes H_2 \otimes \cdots \otimes H_N)$, where $\varphi_i(H_1 \otimes H_2 \otimes \cdots \otimes H_N)$ is computed by Definition 2.3 and φ_i for $\mathbb{H}(\Lambda_k)\,(k \in I)$. Similarly, $\varepsilon_i(H)$ can be computed. Also, $f_i(H)$

is computed by $\Psi^{-1}(f_i(H_1 \otimes H_2 \otimes \cdots \otimes H_N))$, where $f_i(H_1 \otimes H_2 \otimes \cdots \otimes H_N))$ is computed by Definition 2.3 and Algorithm 1. Similarly, $e_i(H)$ can be computed.

Proposition 5.8. Let $\lambda \in P^+$. Let wt, $\varphi_i, \varepsilon_i, f_i, e_i \, (i \in I)$ be computed using the above algorithms for $\mathbb{H}(\lambda)$. Then, the crystal structure on $\mathbb{H}(\lambda)$ determined by these maps corresponds to the crystal structure defined by Definition 3.26.

Proof. By Definition 3.26, Proposition 5.6, and Proposition 5.2, the statement follows. □

The crystal structure on $\mathbb{H}(\lambda) \, (\lambda \in P^+)$ is also directly computed by Theorem 3.36. The following algorithms compute the maps $\varphi_i, \varepsilon_i, f_i, e_i \, (i \in I)$ based on Theorem 3.36. Note that the map wt is simply computed by $\sum_{k \in I}(\mu_k - \mu_{k+1})\Lambda_k$ for $H = (\lambda, \mu, 0, (U_{ij})_{i<j})$.

Algorithm 6 Algorithm for φ_i on $\mathbb{H}(\lambda)$

Require: $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, $i \in I$
Ensure: $\varphi_i(H)$
  $\varphi_i(H) := 0$
  for $k = 1, 2, \ldots, i$ **do**
    $\varphi_i(H) := \max(U_{ki} - U_{k+1,i+1} + \varphi_i(H), 0)$
  end for
  return $\varphi_i(H)$

Algorithm 7 Algorithm for ε_i on $\mathbb{H}(\lambda)$

Require: $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, $i \in I$
Ensure: $\varepsilon_i(H)$
  $\varepsilon_i(H) := 0$
  for $k = 1, 2, \ldots, i$ **do**
    $\varepsilon_i(H) := \max(U_{i+2-k,i+1} - U_{i+1-k,i} + \varepsilon_i(H), 0)$
  end for
  $\varepsilon_i(H) = \max(U_{1,i+1} + \varepsilon_i(H), 0)$ ▷ For $k = i + 1$
  return $\varepsilon_i(H)$

The following is an example of the execution of Algorithm 8.

Example 5.9. Let $n = 4$, $\lambda = \mu = \Lambda_1 + \Lambda_3$. Note that $\Lambda_1 + \Lambda_3$ corresponds to the partition $(2, 1, 1, 0)$. Let $H = (\lambda, \mu, 0, (U_{kl})_{k<l}) \in \mathbb{H}(\lambda)$, where $U_{kl} = 0$ for $1 \leq k < l \leq 4$. The action of f_1 on $\mathbb{H}(\lambda)$ is computed as follows by Algorithm 8. Let $i = 1$. Set $F = [0]$. Since $U_{11} - U_{22} + F[0] = 1$, set $F = [0, 1]$. Set $k_{f_iH} = 1$. Since $F[1] = 1 > 0$, we have $k_{f_iH} = 1$. Then set $\mu_1 = \mu_1 - 1 = 1$, $\mu_2 = \mu_2 + 1 = 2$, $U_{11} = U_{11} - 1 = 1$, and $U_{12} = U_{12} + 1 = 1$. Then we have $f_1H = (\lambda, \mu, 0, (U_{ij})_{i<j})$. See Fig. 20.

Algorithms 6, 7, 8, and 9 compute $\varphi_i, \varepsilon_i, f_i, e_i$, $(i \in I)$ according to Theorem 3.36.

Proposition 5.10. For $\lambda \in P^+$, let $H \in \mathbb{H}(\lambda)$. Let $i \in I$.

(1) Algorithm 6 with inputs H and i yields $\varphi_i(H)$.
(2) Algorithm 7 with inputs H and i yields $\varepsilon_i(H)$.
(3) Let K be the result of Algorithm 8 with inputs H and i. Then, $K = f_iH$.

Algorithm 8 Algorithm for f_i on $\mathbb{H}(\lambda)$

Require: $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, $i \in I$
Ensure: $f_i H$

if $\varphi_i(H) = 0$ **then**
  return 0
end if
$F := [0]$ ▷ Set an array
for $k = 1, 2, \ldots, i$ **do**
  $F := F.\mathrm{append}(\max(U_{ki} - U_{k+1,i+1} + F[k-1], 0))$
end for
$k_{f_i H} := 1$
for $k = i, i-1, \ldots, 1$ **do**
  if $F[k] < 0$ **then**
    $k_{f_i H} := k - 1$
    break
  end if
end for
$\mu_i := \mu_i - 1$
$\mu_{i+1} := \mu_{i+1} + 1$
$U_{k_{f_i},i} := U_{k_{f_i},i} - 1$
$U_{k_{f_i},i+1} := U_{k_{f_i},i+1} + 1$
return $(\lambda, \mu, 0, (U_{ij})_{i<j})$

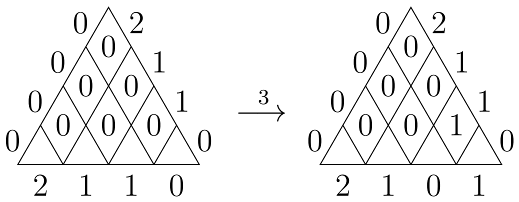


FIGURE 20. Action of f_3 on the $U_q(\mathfrak{sl}_4)$-crystal $\mathbb{H}(\Lambda_1 + \Lambda_3)$

(4) Let K be the result of Algorithm 9 with inputs H and i. Then, $K = e_i H$.

Proof. (1) and (2) immediately follow from Theorem 3.36. (3) is proved if $k_{f_i H}$ in Algorithm 8 corresponds to the one in Theorem 3.36.

For $\lambda \in P^+$, let $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$. We can assume $\varphi_i(H) > 0$. This means that $k_{f_i H}$ is defined and

$$\varphi_i(H) = \sum_{k=k_{f_i H}}^{n} (U_{ki} - U_{k+1,i+1}).$$

Algorithm 9 Algorithm for e_i on $\mathbb{H}(\lambda)$

Require: $H = (\lambda, \mu, 0, (U_{ij})_{i<j}) \in \mathbb{H}(\lambda)$, $i \in I$
Ensure: $e_i H$
  if $\varepsilon_i(H) = 0$ **then**
    return 0
  end if
  $E := [0]$
  for $k = 1, 2, \ldots, i+1$ **do**
    $E := E.\text{append}(\max(U_{i+2-k,i} - U_{i+1-k,i+1} + E[k-1], 0))$
  end for
  $k_{e_iH} := 1$
  for $k = i+1, i, \ldots, 1$ **do**
    if $E[k] < 0$ **then**
      $k_{e_iH} := k - 1$
      break
    end if
  end for
  $\mu_i := \mu_i + 1$
  $\mu_{i+1} := \mu_{i+1} - 1$
  $U_{k+2-k_{e_i},i} := U_{k+2-k_{e_i},i} + 1$
  $U_{k+2-k_{e_i},i+1} := U_{k+2-k_{e_i},i+1} - 1$
  return $(\lambda, \mu, 0, (U_{ij})_{i<j})$

In particular, $\varphi_i^{(k_{f_iH}-1)}(H) = 0$ and $\varphi_i^{(k_{f_iH})}(H) = U_{k_{f_iH},i} - U_{k_{f_iH}+1,i} > 0$ hold by the definition of k_{f_iH}. Then we have

$$\varphi_i^{(m)}(H) = \sum_{k=k_{f_iH}}^{m} (U_{ki} - U_{k+1,i+1}) > 0 \quad (m = k_{f_iH}, k_{f_iH}+1, \ldots, i).$$

By Theorem 3.36, F in Algorithm 8 is an array of $\varphi_i^{(l)}(H)$ such that $F[l] = \varphi_i^{(l)}(H)$ for $l \in [i]$. Then $max\{k \in [i] \mid F[k] < 0\} = k_{f_iH} - 1$ holds, and hence k_{f_iH} in Algorithm 8 corresponds to the one in Theorem 3.36. Similarly, (4) can be shown. □

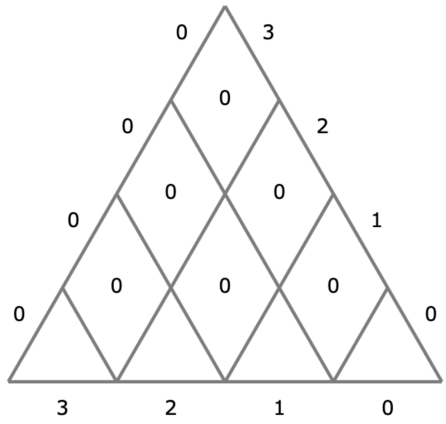
5.2. **Implementations and examples by *khive-crystal*.** In this subsection, we show some examples of executing the algorithms given in Section 5.1. These examples are computed using the Python package originally implemented named *khive-crystal* [22]. Then we also provide examples of the usage of *khive-crystal*.

In *khive-crystal*, K-hive can be declared by the function *khive*. Furthermore, we can show a K-hive as an image using the function *view*. The following code is an example of functions of *khive* and *view*.

```
>> from khive_crystal import khive, view
>> H = khive(
..   n=4, alpha=[3, 2, 1, 0], beta=[3, 2, 1, 0], gamma=[0, 0, 0, 0], Uij=[[0, 0, 0], [0, 0], [0]]
.. )
>> H
```

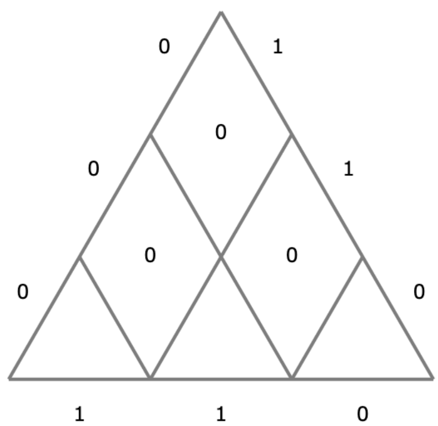
KHive(n=4, alpha=[3, 2, 1, 0], beta=[3, 2, 1, 0], gamma=[0, 0, 0, 0], Uij=[[0, 0, 0], [0, 0], [0]])

```
>> view(H)
```

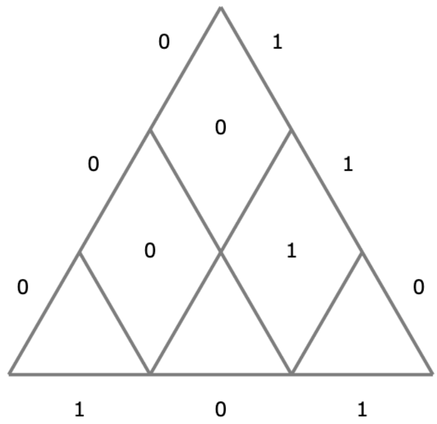


The following codes compute the crystal structure on $U_q(\mathfrak{sl}_3)$-crystal $\mathbb{H}(\Lambda_2)$ by Algorithms 1 and 2.

```
>> from khive_crystal import e, epsilon, f, khive, phi, view
>> H = khive(n=3, alpha=[1, 1, 0], beta=[1, 1, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
>> view(H)
```

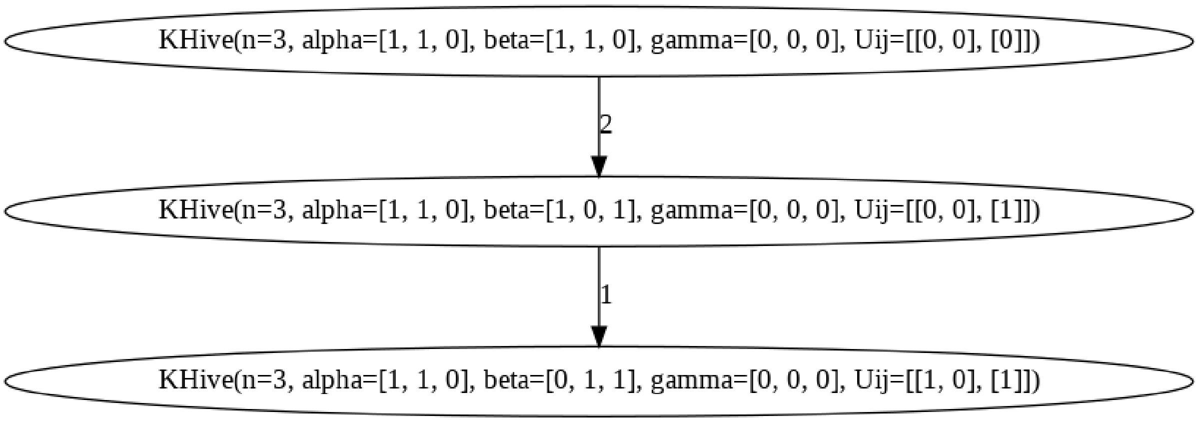



```
>> f(i=1)(H)
# None
>> view(f(i=2)(H))
```



The crystal graph of $\mathbb{H}(\Lambda_2)$ can be shown by the function called *crystal_graph*, where the function *khives* is the function to declare $\mathbb{H}(\Lambda_2)$.

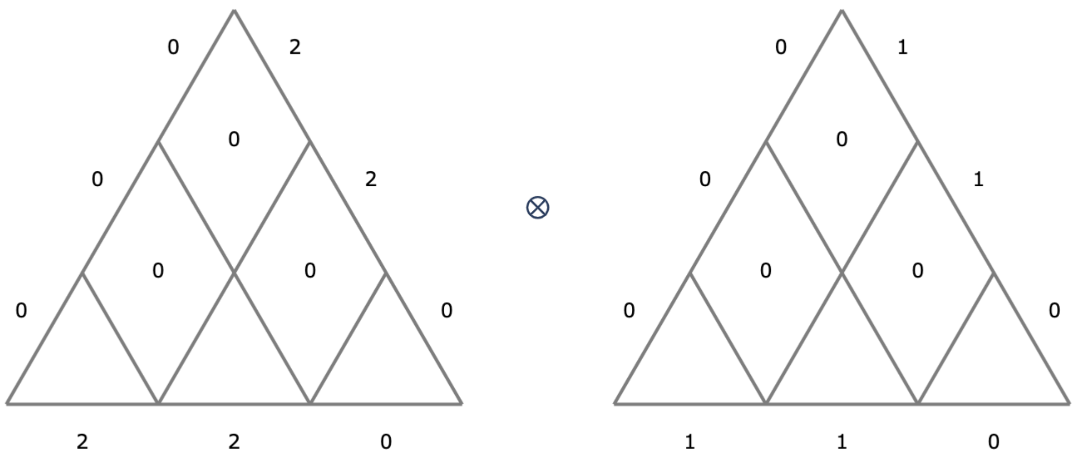
```
>> from khive_crystal import khives, crystal_graph
>> crystal_graph(khives(n=3, alpha=[1, 1, 0]))
```



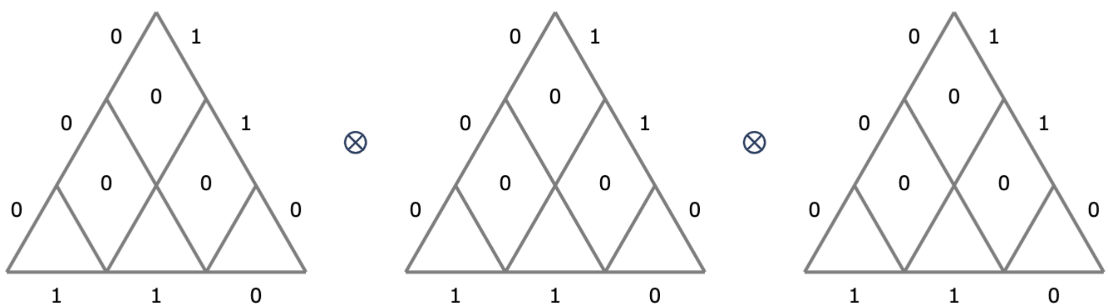
Note that the crystal graph is realized by the open source graph visualization software called *Graphviz*.

The crystal structure on $\mathbb{H}(\lambda)$ $(\lambda \in P^+)$ is defined by algorithms of the crystal structure of $\mathbb{H}(\Lambda_k)$ $(k \in I)$, Ψ_λ, Ψ, and Ψ^{-1}. Then we first show an example for Algorithms 3, 4, and 5, which are implemented as functions *psi_lambda*, *psi*, and *psi_inv*, respectively. The following code is an example for $\Psi_{(3,3,0)}$ and Ψ for $\mathbb{H}((3,3,0))$.

```
>> from khive_crystal import khive, psi, psi_lambda, view
>> H = khive(n=3, alpha=[3, 3, 0], beta=[3, 3, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
>> psi_lambda(H)
[
    KHive(n=3, alpha=[2, 2, 0], beta=[2, 2, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]]),
    KHive(n=3, alpha=[1, 1, 0], beta=[1, 1, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
]
>> view(psi_lambda(H))
```

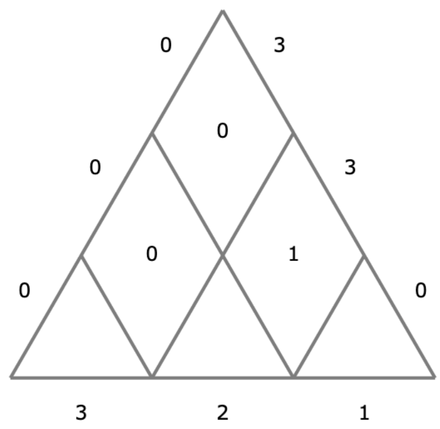



```
>> psi(H)
[
    KHive(n=3, alpha=[1, 1, 0], beta=[1, 0, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]]),
    KHive(n=3, alpha=[1, 1, 0], beta=[1, 0, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]]),
    KHive(n=3, alpha=[1, 1, 0], beta=[1, 1, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
]
>> view(psi(H))
```



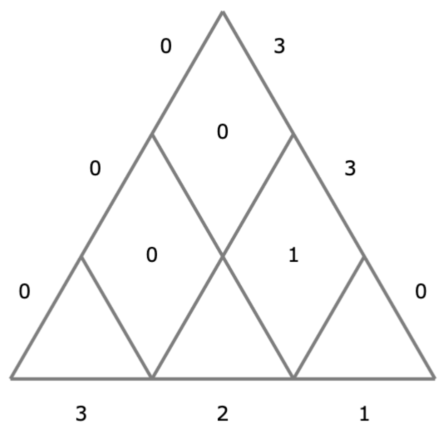
Then we show examples of algorithms of f_i for $\mathbb{H}(\lambda)$. The following code is an example of f_2 for $\mathbb{H}((3,3,0))$.

```
>> from khive_crystal import khive, psi, psi_inv, view
>> H = khive(n=3, alpha=[3, 3, 0], beta=[3, 3, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
>> psi_inv(f(i=2)(psi(H))) # = f_i(H)
```



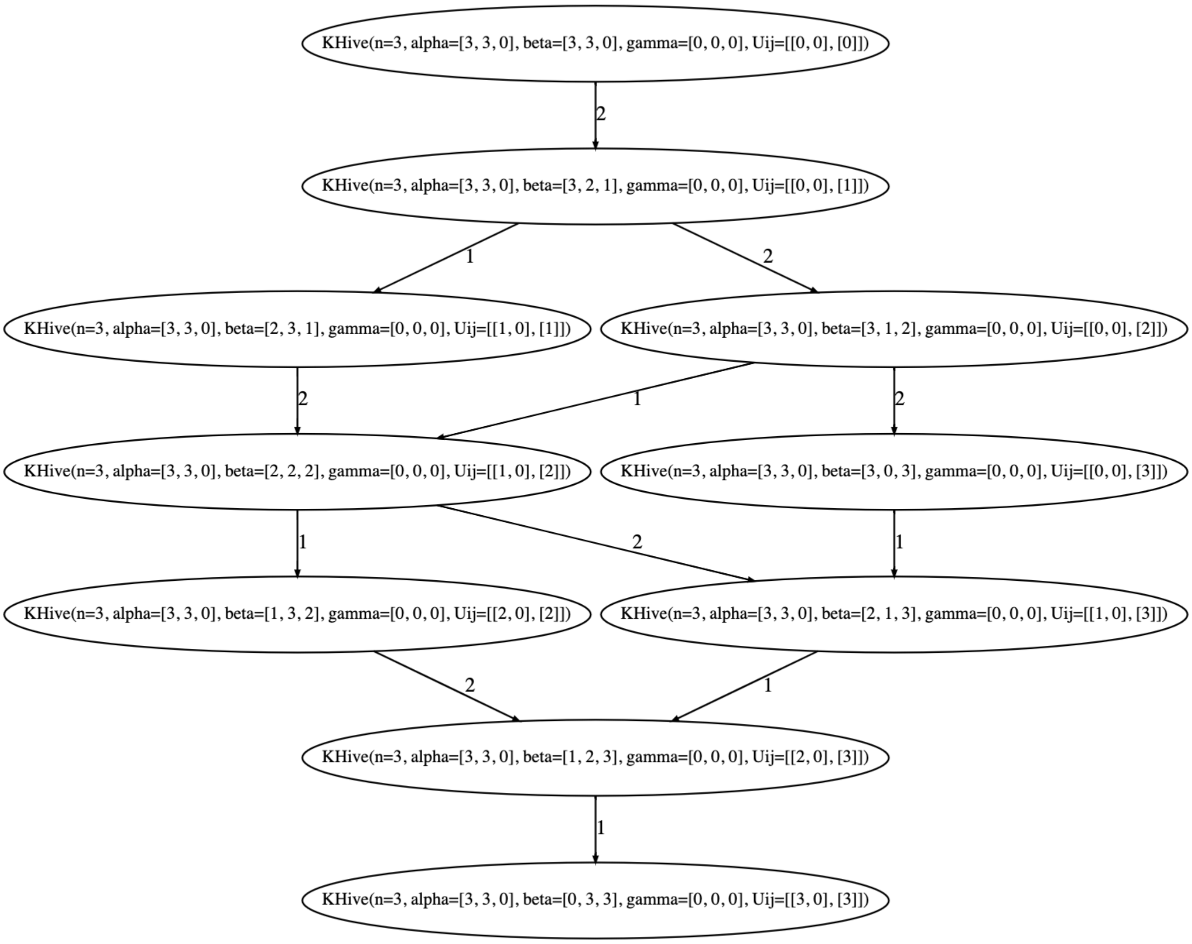
The crystal structure on $\mathbb{H}(\lambda)$ $(\lambda \in P^+)$ is also computed by Algorithms 8 and 9.

```
>> from khive_crystal import khive, e, epsilon, f, phi
>> H = khive(n=3, alpha=[3, 3, 0], beta=[3, 1, 0], gamma=[0, 0, 0], Uij=[[0, 0], [0]])
>> phi(i=2)(H)
3
>> view(f(i=2)(H))
```



The crystal graph of $\mathbb{H}((3, 3, 0))$ is the following.

```
>> from khive_crystal import khives, crystal_graph
>> crystal_graph(khives(n=3, alpha=[3, 3, 0]))
```



6. Concluding Remarks

In this thesis, we study the theory of A_{n-1}-crystal bases and K-hives. This thesis has three themes. The first theme is a combinatorial realization of crystal bases of highest weight modules over the quantized enveloping algebra of type A by K-hives. The second theme is the combinatorial tensor product decomposition rule of crystal bases by K-hives. The last theme is a set of algorithms for computing the crystal structure on K-hives and the implementation of these algorithms as a Python package.

We have obtained the results in the case of type A. The extension to other types is a remaining problem. In addition, affine crystal structures on K-hives should also be determined. It may also be possible to consider the Robinson-Schensted correspondence by K-hives using the tensor product decomposition map. In any case, the realization of the crystal structures is useful for considering these problems.

References

[1] D Bump and A Schilling. *Crystal bases*. Representations and combinatorics. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017, pp. xii+279. ISBN: 978-981-4733-44-1. DOI: `10.1142/9876`. URL: `https://doi.org/10.1142/9876`.

[2] V. G. Drinfeld. "Hopf algebras and the quantum Yang-Baxter equation". In: *Dokl. Akad. Nauk SSSR* 283.5 (1985), pp. 1060–1064. ISSN: 0002-3264.

[3] J. Hong and S.-J. Kang. *Introduction to Quantum Groups and Crystal Bases.* Vol. 42. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2002, pp. xviii+307. DOI: 10.1090/gsm/042. URL: https://doi.org/10.1090/gsm/042.

[4] M. Jimbo. "A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation". In: *Lett. Math. Phys.* 10.1 (1985), pp. 63–69. ISSN: 0377-9017. DOI: 10.1007/BF00704588. URL: https://doi.org/10.1007/BF00704588.

[5] M. Kashiwara. "Crystal bases of modified quantized enveloping algebra". In: *Duke Math. J.* 73.2 (1994), pp. 383–413. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-94-07317-1. URL: https://doi.org/10.1215/S0012-7094-94-07317-1.

[6] M. Kashiwara. "Crystalizing the q-analogue of universal enveloping algebras". In: *Comm. Math. Phys.* 133.2 (1990), pp. 249–260. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1104201397.

[7] M. Kashiwara. "On crystal bases of the Q-analogue of universal enveloping algebras". In: *Duke Math. J.* 63.2 (1991), pp. 465–516. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-91-06321-0. URL: https://doi.org/10.1215/S0012-7094-91-06321-0.

[8] M. Kashiwara and T. Nakashima. "Crystal graphs for representations of the q-analogue of classical Lie algebras". In: *J. Algebra* 165.2 (1994), pp. 295–345. ISSN: 0021-8693. DOI: 10.1006/jabr.1994.1114. URL: https://doi.org/10.1006/jabr.1994.1114.

[9] R. C. King, C. Tollu, and F. Toumazet. "Stretched Littlewood-Richardson and Kostka coefficients". In: *Symmetry in physics.* Vol. 34. CRM Proc. Lecture Notes. Amer. Math. Soc., Providence, RI, 2004, pp. 99–112. DOI: 10.1090/crmp/034/10. URL: https://doi.org/10.1090/crmp/034/10.

[10] R. C. King, C. Tollu, and F. Toumazet. "The hive model and the factorisation of Kostka coefficients". In: *Sém. Lothar. Combin.* 54A (2006), B54Ah, 22 pp.

[11] R. C. King, C. Tollu, and F. Toumazet. "The hive model and the polynomial nature of stretched Littlewood-Richardson coefficients". In: *Sém. Lothar. Combin.* 54A (2006), B54Ad, 19 pp.

[12] A. Knutson and T. Tao. "The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture". In: *J. Amer. Math. Soc.* 12.4 (1999), pp. 1055–1090. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-99-00299-4. URL: https://doi.org/10.1090/S0894-0347-99-00299-4.

[13] Allen Knutson and Terence Tao. "Apiary views of the Berenstein-Zelevinsky polytope, and Klyachko's saturation conjecture". In: *arXiv preprint arXiv:9807160v1* (1998).

[14] T Nakashima. "Crystal base and a generalization of the Littlewood-Richardson rule for the classical Lie algebras". In: *Comm. Math. Phys.* 154.2 (1993), pp. 215–243. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1104252969.

[15] S. Narisawa and K. Shirayanagi. "Algorithms for the crystal structure on K-hives of type A". In: *Communications of JSSAC* (in press).

[16] S. Narisawa and K. Shirayanagi. "Bender-Knuth transformation from a perspective of hives". In: *RIMS Kokyuroku, 2138, Computer Algebra – Theory and its Applications* (2019). (in Japanese), pp. 138–146.

[17] S. Narisawa and K. Shirayanagi. "Crystal bases of irreducible highest weight modules over a quantum group of type A and the hive model". In: *RIMS Kokyuroku, 2159, Computer Algebra – Theory and its Applications* (2020). (in Japanese), pp. 149–157.

[18] S. Narisawa and K. Shirayanagi. "Crystal structure of type A on hives". In: *RIMS Kokyuroku, 2185, Computer Algebra – Theory and its Applications* (2021). (in Japanese), pp. 57–70.

[19] S. Narisawa and K. Shirayanagi. "Crystal structure on K-hives of type A". In: *Communications in Algebra* 50.12 (2022), pp. 5266–5283. DOI: 10.1080/00927872.2022.2084101.

[20] S. Narisawa and K. Shirayanagi. "Tensor product decomposition of crystal bases of type A by K-hives". In: (submitted to Communications in Algebra).

[21] S. Narisawa and K. Shirayanagi. "The tensor product decomposition of crystal bases of type A and K-hives". In: *RIMS Kokyuroku, 2224, Computer Algebra – Foundations and Applications* (2022). (in Japanese), pp. 20–31.

[22] Shota Narisawa. *khive-crystal*. Version 0.1.0. URL: https://github.com/snrsw/khive-crystal.

[23] J. R. Stembridge. "A local characterization of simply-laced crystals". In: *Trans. Amer. Math. Soc.* 355.12 (2003), pp. 4807–4823. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-03-03042-3. URL: https://doi.org/10.1090/S0002-9947-03-03042-3.

[24] I. Terada, R. C. King, and O. Azenhas. "The symmetry of Littlewood-Richardson coefficients: a new hive model involutory bijection". In: *SIAM J. Discrete Math.* 32.4 (2018), pp. 2850–2899. ISSN: 0895-4801. DOI: 10.1137/17M1162834. URL: https://doi.org/10.1137/17M1162834.

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