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東邦大学審査学位論文（博士）

Asymptotic behavior of the resolvents on complete geodesic spaces

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Contents

Chapter 1

Introduction

The concept of resolvent is one of the important notions in convex analysis. To solve problems such as convex optimizations, many researchers have proposed resolvent mappings and studied its properties in the various settings of underlying spaces. Considering the asymptotic behavior is essential to understand resolvents. We describe the famous results about the asymptotic behavior of a resolvent for a convex function on Hilbert space below.

Theorem 1.1 ([23])**.** *Let X be a Hilbert space and $f\colon X \to\ ]-\infty, \infty]$ a proper convex lower semicontinuous function. For a positive number λ, define the resolvent $J_{\lambda f}$ of f by*

$$J_{\lambda f}(x) = \operatorname*{argmin}_{y \in X} \left\{ y \in X \mid \lambda f(y) + \|y - x\|^2 \right\}$$

for $x \in X$. If $\operatorname{argmin} f \neq \emptyset$, then for each $x \in X$,

$$\lim_{\lambda \to \infty} J_{\lambda f}(x) = P_{\operatorname{argmin} f}(x),$$

where $\operatorname{argmin} f$ is the set of all minimizers of f and P_C is the metric projection onto a nonempty closed convex subset C of X.

Theorem 1.2 ([23])**.** *Let X be a Hilbert space and $f\colon X \to\ ]-\infty, \infty]$ a proper convex lower semicontinuous function. For a positive number λ, define the resolvent $J_{\lambda f}$ of f by*

$$J_{\lambda f}(x) = \operatorname*{argmin}_{y \in X} \left\{ y \in X \mid \lambda f(y) + \|y - x\|^2 \right\}$$

for $x \in X$. Then for each $x \in X$,

$$\lim_{\lambda \searrow 0} J_{\lambda f}(x) = P_{\operatorname{cl} \operatorname{dom} f}(x);$$

$$\lim_{\lambda \to \lambda_0 \in]0, \infty[} J_{\lambda f}(x) = J_{\lambda_0 f}(x),$$

where $\operatorname{dom} f = \{y \in X \mid f(y) \in \mathbb{R}\}$, and P_K is the metric projection onto a nonempty closed convex subset K of X.

Geodesic spaces are the class of metric spaces which have some convex structures and have been studied actively in recent years. CAT(0) spaces and CAT(1) spaces are examples of geodesic spaces. They are defined by using the two dimensional plane and the two dimensional unit sphere, respectively, and they have similar properties to Hilbert spaces. Many issues addressed in Hilbert spaces are also considered in these spaces, and we often use resolvent mappings to solve them. For example, in the convex minimization problem, we define resolvent mappings for convex functions on geodesic spaces.

Equilibrium problems are important problems which contains optimization problems, complementarity problems, fixed point problems, variational inequalities, and Nash equilibria. For a bifunction $F\colon K \times K \to \mathbb{R}$ on a set K, these are the problems to find a point $z \in K$ which satisfies $F(z, y) \geq 0$ for all $y \in K$. To find the solution to the problem, we can use the resolvent. In geodesic spaces, resolvents are defined for an equilibrium problem.

Monotone operators theory is one of the central topics in nonlinear and convex analysis. It plays an important role in convex optimizations, variational inequalities, and evolution equations. It is especially important to consider zero points of a monotone operators, and resolvents are useful in such situations. Many researchers studies resolvents of a maximal monotone operator, for instance, see [7], [8], [9], [18], [20], [21], and [22]. For a complete CAT(0) space, the concept of dual space is introduced. Furthermore, monotone operators and their resolvents defined on a complete CAT(0) space.

In this thesis, we discuss the asymptotic behavior of these resolvents defined on complete geodesic spaces. In Section 2, we introduce basic concepts covered in subsequent chapters. In Section 3, we consider the asymptotic behavior of resolvent for convex functions. In Section 4, we consider the asymptotic behavior of resolvent for equilibrium problems. In Section 5, we consider the asymptotic behavior of resolvent for monotone operators. In Section 6, we summarize the results obtained in this thesis.

Chapter 2

Preliminaries

2.1 Geodesic spaces

Let (X, d) be a metric space and $x, y \in X$. A geodesic $c_{xy}\colon [0, d(x, y)] \to X$ with endpoint x, y is a mapping which satisfies

$$\begin{cases} c_{xy}(0) = x; \\ c_{xy}(d(x, y)) = y; \\ d(c_{xy}(s), c_{xy}(t)) = |s - t| \quad \text{for all } s, t \in [0, d(x, y)]. \end{cases}$$

X is called a uniquely geodesic space if there exists a unique geodesic c_{xy} for all two points $x, y \in X$. Let X be a uniquely geodesic space and $x, y \in X$. For $t \in [0, 1]$, a convex combination $tx \oplus (1 - t)y$ between x and y is defined as $tx \oplus (1 - t)y = c_{xy}((1 - t)d(x, y))$, and the geodesic segment $[x, y]$ joining x and y is the set of all convex combinations between x and y. We define the geodesic triangle $\triangle(a_1, a_2, a_3)$ for $a_1, a_2, a_3 \in X$ as

$$\triangle(a_1, a_2, a_3) = \bigcup_{i,j\in\{1,2,3\}} [a_i, a_j] .$$

For the geodesic triangle $\triangle(a_1, a_2, a_3)$ on X, a comparison triangle $\triangle_{\mathbb{R}^2}(A_1, A_2, A_3)$ on $\mathbb{R}^2$ is a plane triangle whose vertices A_1, A_2, A_3 satisfies $d(a_i, a_j) = d(A_i, A_j)_{\mathbb{R}^2}$ for $i, j \in \{1, 2, 3\}$. Further, for a point p on $[a_i, a_j]$, a comparison point P on A_iA_j is a point such that $d(a_i, p) = d(A_i, P)$. We call X a CAT(0) space if for any geodesic triangle $\triangle(a_1, a_2, a_3)$, any points $p, q \in \triangle(a_1, a_2, a_3)$ and their comparison points P, Q in $\triangle_{\mathbb{R}^2}(A_1, A_2, A_3)$, it hold that

$$d(p, q) \leq d(P, Q)_{\mathbb{R}^2}.$$

This inequality is called the the CAT(0) inequality. Let $b_1, b_2, b_3 \in X$ satisfying $d(b_1, b_2) + d(b_2, b_3) + d(b_3, b_1) < 2\pi$. For a geodesic triangle $\triangle(b_1, b_2, b_3)$ with such vertices, we take a spherical triangle $\triangle_{\mathbb{S}^2}(B_1, B_2, B_3)$ on $\mathbb{S}^2$ in the same way as comparison triangle on $\mathbb{R}^2$. We call X a CAT(1) space if for any geodesic triangle $\triangle(b_1, b_2, b_3)$ such that $d(b_1, b_2) + d(b_2, b_3) + d(b_3, b_1) < 2\pi$, any points $p, q \in \triangle(b_1, b_2, b_3)$ and their comparison points P, Q in $\triangle_{\mathbb{R}^2}(B_1, B_2, B_3)$, it hold that

$$d(p, q) \leq d(P, Q)_{\mathbb{S}^2}.$$

This inequality is called the CAT(1) inequality. If a CAT(1) space X satisfies $d(x, y) < \frac{\pi}{2}$ for all $x, y \in X$, we call X an admissible CAT(1) space. In regard to these spaces, we know some equivalence conditions. If X is a uniquely geodesic space, then the following statements are equivalent:

(i) X is a CAT(0) space;
(ii) $d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right)^2 \le \frac{1}{2}d(x,z)^2 + \frac{1}{2}d(y,z)^2 - \frac{1}{4}d(x,y)^2$ for all $x, y, z \in X$;
(iii) $d(tx \oplus (1-t)y, z)^2 \le td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2$ for all $t \in]0,1[$ and $x, y, z \in X$;
(iv) $d(x,w)^2 + d(y,z)^2 - d(x,z)^2 - d(y,x)^2 \le 2d(x,y)d(z,w)$ for all $x, y, z, w \in X$;
(v) $d(x,w)^2 + d(y,z)^2 - d(x,z)^2 - d(y,x)^2 \le d(x,y)^2 + d(z,w)^2$ for all $x, y, z, w \in X$;

see [2] for details. The inequality that appears in (ii) is called the parallelogram law in CAT(0) spaces, and the inequalities that appears in (iv) or (v) are called the four point inequalities in CAT(0) space. We also know that the equivalent conditions to be a CAT(1) space. Let X be a uniquely geodesic space such that $d(x,y) + d(y,z) + d(z,x) < 2\pi$ for all $x, y, z \in X$. Then X is a CAT(1) space if and only if

$$\cos d(tx \oplus (1-t)y, z) \sin d(x,y) \ge \cos d(x,z) \sin td(x,y) + \cos d(y,z) \sin(1-t)d(x,y)$$

holds for all $x, y, z \in X$, which is called a parallelogram law in CAT(1) spaces. In particular, for $t = \frac{1}{2}$, we have

$$\cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \cos \frac{d(x,y)}{2} \ge \frac{1}{2}\cos d(x,z) + \frac{1}{2}\cos d(y,z)$$

by the addition theorem of trigonometric functions. Further, from this inequality, we get

$$\begin{aligned} &-\log\left(\cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right)\right) \\ &\le \frac{1}{2}\left(-\log\left(\cos d(x,z)\right)\right) + \frac{1}{2}\left(-\log\left(\cos d(y,z)\right)\right) - \left(-\log\left(\cos \frac{d(x,y)}{2}\right)\right). \end{aligned}$$

which is the rewritten version of the parallelogram law adapting the function $-\log(\cos t)$, In the terms of the convexity, the function $-\log(\cos t)$ has many similarities to t^2, and this version is a similar shape to the inequality appearing in equivalent condition (ii) to be a CAT(0) space.

2.2 Sets and Convergence

Let X be a uniquely geodesic space. The subset C of X is said to be convex if $tx \oplus (1-t)y \in C$ for all $t \in [0,1]$ and $x, y \in C$. For a subset C of X, the convex hull $\operatorname{co} C$ is defined by

$$\operatorname{co} C = \bigcup_{i=1}^{\infty} C_i,$$

where $C_1 = C$ and $C_{i+1} = \{tx \oplus (1-t)y \mid x, y \in C_i, t \in [0,1]\}$. We say X has the convex hull finite property if for any finite subset C of X, any mapping $T\colon \operatorname{cl}\operatorname{co} A \to \operatorname{cl}\operatorname{co} A$ has a fixed point. Let $\{x_n\}$ be a bounded sequence of X. The asymptotic center $\operatorname{AC}(\{x_n\})$ of $\{x_n\}$ is the set such that

$$\operatorname{AC}(\{x_n\}) = \{z \in X \mid \limsup_{n\to\infty} d(z, x_n) = \inf_{y \in X} \limsup_{n\to\infty} d(y, x_n)\}.$$

We say that $\{x_n\}$ is Δ-convergent to x_0 if each subsequence $\{x_{n_i}\}$ of $\{x_n\}$, $\operatorname{AC}(\{x_{n_i}\}) = \{x_0\}$. We denote it by $x_n \overset{\Delta}{\rightharpoonup} x_0$. If X is a complete CAT(0) space for any bounded sequence $\{x_n\}$, AC($\{x_n\}$) consists of one point and $\{x_n\}$ has a Δ-convergent subsequence $\{x_{n_i}\}$. Any spherically bounded sequence $\{x_n\}$ on complete admissible CAT(1) space has the same property. The Mosco convergence of sets is firstly introduced by [3]. Let $\{C_n\}$ be a sequence of nonempty subsets of X. We define the set $\text{d-Li}\, C_n$ as follows; $x \in \text{d-Li}\, C_n$ if and only if there exists a sequence $\{x_n\}$ of X such that $x_n \in C_n$ for all $n \in \mathbb{N}$ and $x_n \to x$. We also define the set $\Delta\text{-Ls}\, C_n$; $x \in \Delta\text{-Ls}\, C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{x_i\}$ of X such that $x_i \in C_{n_i}$ for all $i \in \mathbb{N}$ and $x \in \text{AC}(\{x_i\})$. From the definitions above, $\Delta\text{-Ls}\, C_n \subset \text{d-Li}\, C_n$ always holds. If $\Delta\text{-Ls}\, C_n \supset C_0 \supset \text{d-Li}\, C_n$ holds for a nonempty subset C_0 of X, we say that $\{C_n\}$ is Δ-Mosco convergent to C_0 and denote it by $\text{M-lim}_{n\to\infty} C_n = C_0$. [17] define the set $\overline{\Delta}\text{-Ls}\, C_n$ as an alternative concept to $\Delta\text{-Ls}\, C_n$ for a sequence of $\{C_n\}$. $\overline{\Delta}\text{-Ls}\, C_n$ is defined as follows; $x \in \overline{\Delta}\text{-Ls}\, C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{x_i\}$ of X such that $x_i \in C_{n_i}$ for all $i \in \mathbb{N}$ and $x_i \overset{\Delta}{\rightharpoonup} x$. Then, $\Delta\text{-Ls}\, C_n \supset C_0 \supset \text{d-Li}\, C_n$ if and only if $\overline{\Delta}\text{-Ls}\, C_n \supset C_0 \supset \text{d-Li}\, C_n$, Therefore, we can use the set $\overline{\Delta}\text{-Ls}\, C_n$ to define the Δ-Mosco convergence of $\{C_n\}$.

2.3 Functions and Mappings

Let X be a uniquely geodesic space and $f\colon X \to \left]-\infty, \infty\right]$ a function. The effective domain $\text{dom}\, f$ of f is the set which is defined as $\text{dom}\, f = f^{-1}(\mathbb{R})$. A function $f\colon X \to \left]-\infty, \infty\right]$ is said to be proper if $\text{dom}\, f \neq \emptyset$. For a subset C of X, we express the set of all minimizers of f in C by the symbol $\text{argmin}_{y\in C} f(y)$. We sometimes denote $\text{argmin}_{y\in X} f(y)$ by $\text{argmin}\, f$ simply. We take a sequence $\{x_n\}$ arbitrarily to consider the continuity of a function. A function $f\colon X \to \left]-\infty, \infty\right]$ is said to be lower semicontinuous if

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n)$$

whenever $\{x_n\}$ is convergent to $x_0 \in X$. Similarly, it is said to be Δ-lower semicontinuous if it satisfies the inequality above whenever $\{x_n\}$ is Δ-convergent to $x_0 \in X$. Further, it is said to be upper hemicontinuous if

$$\limsup_{t\searrow 0} f(tx + (1-t)y) \leq f(y)$$

for $x, y \in X$. We also discuss the convexity of a function on a uniquely geodesic space. A function $f\colon X \to \left]-\infty, \infty\right]$ is said to be convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $t \in \left]0, 1\right[$ and $x, y \in X$. Moreover, f is said to be strictly convex if

$$f(tx \oplus (1-t)y) < tf(x) + (1-t)f(y)$$

for all $t \in \left]0, 1\right[$ and $x, y \in X$ with $x \neq y$. In a complete CAT(0) space, the function $d(\cdot, x)^2$ is continuous and strictly convex for any point x. In a complete admissible CAT(1) space, the function $-\log\cos(d(\cdot, x))$ is so. Since we know that all lower semicontinuous convex function is Δ-lower semicontinuous, these functions are also Δ-lower semicontinuous in each space. For a convex subset C of X, the indicator function i_C is defined by

$$i_C(x) = \begin{cases} 0 & (x \in C) \\ \infty & (x \notin C). \end{cases}$$

If C is nonempty closed convex, then i_C is a proper lower semicontinuous convex function.

Let $T\colon X \to X$ be a mapping. We denote the set of all fixed point of T by $\mathrm{Fix}(T)$. A mapping $T\colon X \to X$ is called a nonexpansive mapping if $d(x, y) \leq d(Tx, Ty)$ for all $x, y \in X$. Let X be a complete CAT(0) space or a complete admissible CAT(1) space and C a nonempty closed convex subset of X. Then, for $x \in X$, there exists a unique point $x_0 \in C$ such that

$$x_0 = \operatorname*{argmin}_{y \in C} d(y, x).$$

The metric projection P_C onto C is the mapping such that

$$P_C(x) = \operatorname*{argmin}_{y \in C} d(y, x).$$

2.4 Equilibrium problems

Let K be a set and $F\colon K \times K \to \mathbb{R}$ a bifunction. The equilibrium problem is formulated as follows:

$$\text{Find } z \in K \text{ such that } F(z, y) \geq 0 \text{ for all } y \in K.$$

We denote solutions to the equilibrium problem for f by $\mathrm{Equil}\, F$. That is,

$$\mathrm{Equil}\, F = \{z \in K \mid \inf_{y \in K} F(z, y) \geq 0\}.$$

In this thesis, we suppose that a bifunction $F\colon K \times K \to \mathbb{R}$ satisfies following four conditions in an equilibrium problems for F:

(E1) $F(y, y) = 0$ for all $y \in K$;
(E2) $F(y, z) + F(z, y) \leq 0$ for all $y, z \in K$;
(E3) $F(y, \cdot)\colon K \to \mathbb{R}$ is convex and lower semicontinuous for all $y \in K$;
(E4) $F(\cdot, y)\colon K \to \mathbb{R}$ is upper hemicontinuous for all $y \in K$.

The minimization problem is an example of equilibrium problems. Let $f\colon K \to \mathbb{R}$ and define $F\colon K \times K \to \mathbb{R}$ by $F(x, y) = f(y) - f(x)$ for all $x, y \in K$. Then, $z \in K$ is a solution to the equilibrium problem of F if and only if $f(z) \leq f(y)$ for all $y \in K$. Hence, it hold that $\mathrm{Equil}\, F = \operatorname{argmin} f$.

2.5 Dual spaces and Monotone operators

The concept of quazilinearization of CAT(0) spaces has introduced by [6]. Let X be a complete CAT(0) space. We denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and define the quazilinearization mapping $\langle \cdot, \cdot \rangle\colon (X \times X) \times (X \times X) \to \mathbb{R}$ by

$$\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \frac{1}{2}\{d(a, d)^2 + d(b, c)^2 - d(a, c)^2 - d(b, d)^2\}$$

for $a, b, c, d \in X$. From the definition above, we have the following equations;

(1) $\left\langle \overrightarrow{ab}, \overrightarrow{ab} \right\rangle = d(a, b)^2$ for $a, b \in X$;
(2) $\left\langle \overrightarrow{ba}, \overrightarrow{cd} \right\rangle = -\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle$ for $a, b, c, d \in X$;
(3) $\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \left\langle \overrightarrow{ae}, \overrightarrow{cd} \right\rangle + \left\langle \overrightarrow{eb}, \overrightarrow{cd} \right\rangle$ for $a, b, c, d, e \in X$.

Moreover, by the four point inequality, we get

$$\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \leq d(a,b)d(c,d)$$

for all $a, b, c, d \in X$, which is called the Cauchy-Schwarz inequality in CAT(0) spaces. The dual space of complete CAT(0) space is introduced by [6]. Let X be a Hadamard space and $C(X, \mathbb{R})$ the set of all continuous functions on X. We define the map $\Theta\colon \mathbb{R}\times X\times X \to C(X,\mathbb{R})$ by

$$\Theta(t,a,b)(x) = t\left\langle \overrightarrow{ab}, \overrightarrow{ax} \right\rangle$$

for all $t \in \mathbb{R}$ and $a, b, x \in X$. The Lipchitz seminorm is defined by

$$L(f) = \sup_{x\neq y} \frac{f(x) - f(y)}{d(x,y)}$$

for all continuous function $f\colon X \to \mathbb{R}$. From the Cauchy-Schwartz inequality, we get $L(\Theta(t,a,b)) = |t|d(a,b)$ for all $t \in \mathbb{R}$ and $a, b \in X$, and hence $\Theta(t,a,b)$ is a Lipchitz function. We define a function $d^*\colon (\mathbb{R}\times X\times X)\times(\mathbb{R}\times X\times X) \to \mathbb{R}$ by

$$d^*((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d))$$

for all $t, s \in \mathbb{R}$ and $a, b, c, d \in X$. Then, d^* is the pseudometric on $\mathbb{R}\times X\times X$ and the metric space $(\mathbb{R}\times X\times X, d^*)$ can be considered as subspace of pseudometric space of all Lipchitz functions. Since it holds that $d^*((t,a,b),(s,c,d)) = 0$ if and only if $t\left\langle \overrightarrow{ab}, \overrightarrow{xy} \right\rangle = s\left\langle \overrightarrow{cd}, \overrightarrow{xy} \right\rangle$ for $x, y \in X$, we consider an equivalence relation on $\mathbb{R}\times X\times X$ as follows;

$$\left[t\overrightarrow{ab}\right] = \left\{ s\overrightarrow{cd} \mid d^*((t,a,b),(s,c,d)) = 0 \right\}.$$

Under this equivalence relation, the set $X^* = \left\{ \left[t\overrightarrow{ab}\right] \mid (t,a,b) \in \mathbb{R}\times X\times X \right\}$ is a metric space with d^* and we call it the dual space of X. For simplicity, we use the following symbols.

(1) $\langle x^*, \overrightarrow{xy}\rangle = t\left\langle \overrightarrow{ab}, \overrightarrow{xy} \right\rangle$ for $x, y \in X$, $x^* = \left[t\overrightarrow{ab}\right] \in X^*$
(2) $\mathbf{0} = [t\overrightarrow{aa}]$ for $t \in \mathbb{R}$, $a \in X$
(3) $\langle tx^* + sy^*, \overrightarrow{xy}\rangle = t\langle x^*, \overrightarrow{xy}\rangle + s\langle y^*, \overrightarrow{xy}\rangle$ for $t, s \in \mathbb{R}$, $x, y \in X$, $x^*, y^* \in X^*$.

Let $A\colon X \rightrightarrows X^*$ be a multivalued operator. We define the domain $D(A)$ of A and the range $R(A)$ of A by

$$D(A) = \{x \in X \mid Ax \neq \emptyset\} \quad \text{and} \quad R(A) = \bigcup_{x\in D(A)} Ax.$$

In this thesis, we also use these symbols for a multivalued operator $A'\colon X \rightrightarrows X$. A multivalued operator A is called a monotone operator, if

$$\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0$$

for all $(x, x^*), (y, y^*) \in A$. Further, a monotone operator $A\colon X \to X$ is said to be maximal if no monotone operator B includes A properly. We know that if $(u, u^*) \in X \times X^*$ satisfies

$$\langle u^* - y^*, \overrightarrow{yu}\rangle \geq 0$$

for all $(y, y^*) \in A$, then the maximality of A implies $(u, u^*) \in A$.

2.6 Resolvents

2.6.1 Resolvents for convex functions

In this thesis, we use the term "increasing sequence" to mean "nondecreasing sequence". That is, the real sequence $\{\mu_n\}$ is said to be increasing if $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \dots$ holds. Similarly, "decreasing sequence" is also used for sequences that are not strictly decreasing. Let X be a complete CAT(0) space and $f\colon X \to]-\infty, \infty]$ a proper convex lower semicontinuous function. We consider a function $\varphi\colon [0, \infty[\to [0, \infty[$ which satisfies the following conditions;

(A1) $\varphi(0) = 0$;
(A2) φ is increasing;
(A3) φ is continuous;
(A4) $\varphi(d(\cdot, x))$ is strictly convex for all $x \in X$;
(A5) $\varphi(t) - kt \to \infty$ as $t \to \infty$, for all $k \in \mathbb{R}$.

If φ satisfies these conditions, the function $f(\cdot) + \varphi(d(\cdot, x))$ has a unique minimizer for $x \in X$. We define the resolvent $Q_f^\varphi\colon X \to X$ with perturbation φ by

$$Q_f^\varphi(x) = \operatorname*{argmin}_{y \in X} \{f(y) + \varphi(d(y, x))\}$$

for all $x \in X$. Then, it hold that $\operatorname{Fix}(Q_f^\varphi) = \operatorname{argmin} f$. The functions $\varphi_1(t) = t^2$ and $\varphi_2(t) = \tanh t \sinh t$ are examples of the functions satisfying (A1)-(A5). The resolvent with t^2 and $\tanh t \sinh t$ are defined by [19] and [5], respectively.

Similarly, we consider a resolvent for a proper convex lower semicontinuous function f on a complete admissible CAT(1) space X with a perturbation function $\psi\colon [0, \infty[\to \left[0, \frac{\pi}{2}\right[$ which satisfies following conditions:

(B1) $\psi(0) = 0$;
(B2) ψ is increasing;
(B3) ψ is continuous;
(B4) $\psi(d(\cdot, x))$ is strictly convex for all $x \in X$;
(B5) $\psi(t) \to \infty$ as $t \to \frac{\pi}{2}$.

We define the resolvent $Q_f^\psi\colon X \to X$ by

$$Q_f^\psi(x) = \operatorname*{argmin}_{y \in X} \{f(y) + \psi(d(y, x))\}$$

for all $x \in X$. Then $\operatorname{Fix}(Q_{\lambda f}^\psi) = \operatorname{argmin} f$. The functions $\psi_1(t) = -\log(\cos t)$ and $\psi_2(t) = \tan t \sin t$ satisfy (B1)-(B5). Resolvents with these functions are studied by [13] and [4].

2.6.2 Resolvents for equilibrium problems

For an equilibrium problem, we consider the resolvent. In a complete CAT(0) space, a resolvent is proposed by [12], and in a complete admissible CAT(1) space it is proposed by [11]. Let X be a complete CAT(0) space having the convex hull finite property, K a nonempty closed convex subset of X and $F\colon K \times K \to \mathbb{R}$ satisfying (E1)-(E4). We define a resolvent $R_F^{d^2}$ for equilibrium problems for F by

$$R_F^{d^2}(x) = \left\{ z \in K \mid \inf_{y \in K} \left(F(z,y) + d(y,x)^2 - d(z,x)^2 \right) \geq 0 \right\}$$

for all $x \in X$. We know that $\mathrm{Fix}(R_F^{d^2}) = \mathrm{Equil}\, F$.

Let X be a complete admissible CAT(1) space having the convex hull finite property and impose the same condition on K and F as in a CAT(0) space. We also define the resolvent $R_F^{-\log(\cos d)}$ by

$$R_F^{-\log(\cos d)}(x) = \left\{ z \in K \mid \inf_{y \in K} \left(f(z,y) - \log(\cos(d(y,x))) + \log(\cos(d(z,x))) \right) \geq 0 \right\}$$

for all $x \in X$. Then, $\mathrm{Fix}(R_F^{-\log(\cos d)}) = \mathrm{Equil}\, F$ holds.

2.6.3 Resolvents for monotone operators

Let X be a complete CAT(0) space, X^* its dual space, and $A\colon X \rightrightarrows X^*$ a multivalued mapping. We define the resolvent $S_A\colon X \rightrightarrows X$ for a multivalued operator A by

$$S_A(x) = \{ z \in X \mid [\overrightarrow{zx}] \in Az \}$$

for all $x \in X$. We know that properties of this resolvent as follows;

(1) $R(S_A) \subset D(A)$, and $\mathrm{Fix}(S_A) = A^{-1}(\mathbf{0})$.
(2) If A is monotone, S_A is a single-valued mapping.
(3) S_A is nonexpansive. That is, $d(S_A(x), S_A(y)) \leq d(x,y)$ for all $x, y \in D(A)$.

Therefore, if A is monotone and $D(S_A) = X$, then S_A is a single-valued mapping.

Chapter 3

Asymptotic behavior of resolvents for convex functions

In this chapter, we discuss the asymptotic behavior of resolvents for convex functions. In particular, we consider a resolvent for a sequence of convex functions with the reference in resolvents for a single convex function.

3.1 Resolvents on CAT(0) spaces

We consider the resolvents of a convex function on a complete CAT(0) space. Let X be a complete CAT(0) space, $f\colon X \to]-\infty,\infty]$ a proper convex lower semicontinuous function and $\varphi\colon [0,\infty[\to [0,\infty[$ satisfying (A1)-(A5). For a positive real number λ, $Q^{\varphi}_{\lambda f}\colon X \to X$ is the resolvent of the function λf, which is defined by

$$\begin{aligned} Q^{\varphi}_{\lambda f}(x) &= \operatorname*{argmin}_{y\in X} \{\lambda f(y) + \varphi(d(y,x))\} \\ &= \operatorname*{argmin}_{y\in X} \left\{ f(y) + \frac{1}{\lambda}\varphi(d(y,x)) \right\} \end{aligned}$$

for all $x \in X$. There are results for this resolvent.

Theorem 3.1 ([15])**.** *Let X be a complete CAT(0) space, $f\colon X \to]-\infty,\infty]$ a proper convex lower semicontinuous function and $x \in X$. If there exists a positive real sequence $\{\mu_n\}$ diverging to ∞ and $\{d(Q^{\varphi}_{\mu_n f}(x),x)\}$ is bounded, then* $\operatorname{argmin} f \neq \emptyset$ *and*

$$\lim_{\lambda\to\infty} Q^{\varphi}_{\lambda f}(x) = P_{\operatorname{argmin} f}(x).$$

Theorem 3.2 ([15])**.** *Let X be a complete CAT(0) space, $f\colon X \to]-\infty,\infty]$ a proper convex lower semicontinuous function and $x \in X$. Then*

$$\lim_{\lambda\searrow 0} Q^{\varphi}_{\lambda f}(x) = P_{\operatorname{cl}\operatorname{dom} f}(x)$$

and

$$\lim_{\lambda\to\lambda_0} Q^{\varphi}_{\lambda f}(x) = Q^{\varphi}_{\lambda_0 f}(x),$$

where λ_0 is a positive number.

Further, we discuss the resolvents for sequence of convex functions. For a sequence $\{f_n\}$ of proper convex lower semicontinuous functions and a sequence $\{\lambda_n\}$ of a positive real numbers, we define the sequence of resolvents $Q^{\varphi}_{\lambda_n f_n}: X \to X$ by

$$\begin{aligned} Q^{\varphi}_{\lambda_n f_n}(x) &= \operatorname*{argmin}_{y \in X} \{\lambda_n f_n(y) + \varphi(d(y,x))\} \\ &= \operatorname*{argmin}_{y \in X} \left\{ f_n(y) + \frac{1}{\lambda_n}\varphi(d(y,x)) \right\} \end{aligned}$$

for all $x \in X$ and consider the asymptotic behavior of this resolvent. To show the convergence of the resolvent, we impose suppositions about the set convergence of the minimizers on $\{f_n\}$.

Theorem 3.3. *Let X be a complete CAT(0) space and $f_0\colon X \to]-\infty, \infty]$ a proper convex lower semicontinuous function and $x \in X$. Suppose that $\{f_n\}$ satisfies the following conditions;*

(a) $\emptyset \neq \operatorname{argmin} f_0 \subset \text{d-Li} \operatorname{argmin} f_n$*;*
(b) $\overline{\Delta}\text{-Ls} \left\{ Q^{\varphi}_{\lambda_n f_n}(x) \right\} \subset \operatorname{argmin} f_0$.

Then

$$Q^{\varphi}_{\lambda_n f_n}(x) \to P_{\operatorname{argmin} f_0}(x).$$

Proof. Since $P_{\operatorname{argmin} f_0}(x)$ belongs to $\operatorname{argmin} f_0$, there exists a sequence $\{a_n\}$ such that $a_n \in \operatorname{argmin} f_n$ for all $n \in \mathbb{N}$ and $a_n \to P_{\operatorname{argmin} f_0}(x)$ by (a). Put $x_n = Q^{\varphi}_{\lambda_n f_n}(x)$ for all $n \in \mathbb{N}$. Then a_n and x_n are minimizers of the functions f_n and $f_n(\cdot) + \frac{1}{\lambda_n}\varphi(d(\cdot, x))$, respectively. Therefore, we have

$$\begin{aligned} f_n(x_n) + \frac{1}{\lambda_n}\varphi(d(x_n, x)) &\leq f_n(tx_n \oplus (1-t)a_n) + \frac{1}{\lambda_n}\varphi(d(tx_n \oplus (1-t)a_n, x)) \\ &\leq tf_n(x_n) + (1-t)f_n(a_n) + \frac{1}{\lambda_n}\varphi(d(tx_n \oplus (1-t)a_n, x)) \\ &\leq tf_n(x_n) + (1-t)f_n(x_n) + \frac{1}{\lambda_n}\varphi(d(tx_n \oplus (1-t)a_n, x)) \\ &= f_n(x_n) + \frac{1}{\lambda_n}\varphi(d(tx_n \oplus (1-t)a_n, x)) \end{aligned}$$

for all $t \in]0,1[$ and $n \in \mathbb{N}$. Thus, we get $\varphi(d(x_n, x)) \leq \varphi(d(tx_n \oplus (1-t)a_n, x))$, which is equivalent to

$$d(x_n, x) \leq d(tx_n \oplus (1-t)a_n, x).$$

Furthermore, the parallelogram law in CAT(0) spaces implies that

$$\begin{aligned} d(x_n, x)^2 &\leq d(tx_n \oplus (1-t)a_n, x)^2 \\ &\leq td(x_n, x)^2 + (1-t)d(a_n, x)^2 - t(1-t)d(x_n, a_n)^2. \end{aligned}$$

Dividing by $1-t$ and letting $t \to 1$, we obtain

$$d(x_n, x)^2 \leq d(a_n, x)^2 - d(x_n, a_n)^2. \tag{3.1}$$

From this inequality, $d(x_n, x) \leq d(a_n, x)$ holds. Then, since $\{a_n\}$ is convergent, it is bounded and so is $\{x_n\}$. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. We can find a subsequence

$\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δ-convergent to some point x_0 and then $x_0 \in \operatorname{argmin} f_0$ by (b). Thus, we get

$$\begin{aligned} d(P_{\operatorname{argmin} f_0}(x), x) &\leq d(x_0, x) \\ &\leq \liminf_{j\to\infty} d(x_{n_{i_j}}, x) \\ &\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, x) \\ &\leq \limsup_{j\to\infty} d(a_{n_{i_j}}, x) \\ &= d(P_{\operatorname{argmin} f_0}(x), x), \end{aligned}$$

and hence $x_0 = P_{\operatorname{argmin} f_0}(x)$. Since $a_{n_{i_j}} \to P_{\operatorname{argmin} f_0}(x)$ and $x_{n_{i_j}} \xrightarrow{\Delta} x_0$, we have

$$\begin{aligned} 0 &\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, P_{\operatorname{argmin} f_0}(x))^2 \\ &= \limsup_{j\to\infty} d(x_{n_{i_j}}, a_{n_{i_j}})^2 \\ &\leq \limsup_{j\to\infty} (d(a_{n_{i_j}}, x)^2 - d(x_{n_{i_j}}, x)^2) \\ &\leq \limsup_{j\to\infty} d(a_{n_{i_j}}, x)^2 - \liminf_{j\to\infty} d(x_{n_{i_j}}, x)^2 \\ &\leq d(P_{\operatorname{argmin} f_0}(x), x)^2 - d(P_{\operatorname{argmin} f_0}(x), x)^2 = 0 \end{aligned}$$

from (3.1), and thus $x_{n_{i_j}} \to P_{\operatorname{argmin} f_0}(x)$. Consequently, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and $x_{n_{i_j}} \to P_{\operatorname{argmin} f_0}(x)$. It is equivalent to $x_n \to P_{\operatorname{argmin} f_0}(x)$, which is the desired result. □

Furthermore, when $\{\lambda_n\}$ diverges to ∞, the resolvent converges to a minimizer of the limit function under some conditions of $\{f_n\}$.

Theorem 3.4. *Let X be a complete CAT(0) space, $f_0 \colon X \to]-\infty, \infty]$ a proper convex lower semicontinuous function and $x \in X$. Suppose that $\{\lambda_n\}$ diverges to ∞ and $\{f_n\}$ satisfies the following conditions;*

(a) $\emptyset \neq \operatorname{argmin} f_0 \subset \text{d-Li} \operatorname{argmin} f_n$;
(b) For all $b \in X$, there exists $\{b_n\}$ such that $b_n \to b$ and $\limsup_{n\to\infty} f_n(b_n) \leq f(b)$;
(c) For any subsequence $\{f_{n_i}\}$ of $\{f_n\}$ and $\{c_i\}$ which is Δ-convergent to c, it hold that $f(c) \leq \liminf_{i\to\infty} f_{n_i}(c_i)$.

Then

$$Q^{\varphi}_{\lambda_n f_n}(x) \to P_{\operatorname{argmin} f_0}(x).$$

Proof. Put $x_n = Q^{\varphi}_{\lambda_n f_n}(x)$ for $n \in \mathbb{N}$ and let $x_0 \in X$ be a Δ-limit of a Δ-convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$. By (c), it holds that $f(x_0) \leq \liminf_{i\to\infty} f_{n_i}(x_{n_i})$ and by (b), there exists a sequence $\{b_n\}$ such that $b_n \to P_{\operatorname{argmin} f_0}(x)$ and $\limsup_{n\to\infty} f_n(b_n) \leq f(P_{\operatorname{argmin} f_0}(x))$. From the definition of resolvent, we have

$$f_{n_i}(x_{n_i}) + \frac{1}{\lambda_{n_i}} \varphi(d(x_{n_i}, x)) \leq f_{n_i}(b_{n_i}) + \frac{1}{\lambda_{n_i}} \varphi(d(b_{n_i}, x)).$$

Letting $i \to \infty$, we get

$$f(x_0) \leq \liminf_{i\to\infty} f_{n_i}(x_{n_i}) \leq \limsup_{i\to\infty} f_{n_i}(b_{n_i}) \leq f(P_{\operatorname{argmin} f_0}(x)).$$

Hence, $x_0 \in \operatorname{argmin} f_0$ and thus

$$\overline{\Delta}\text{-Ls}\left\{Q^{\varphi}_{\lambda_n f_n}(x)\right\} \subset \operatorname{argmin} f_0.$$

From the previous theorem, we get the desired result. □

This theorem implies that Theorem 3.1 and the next corollary.

Corollary 3.5 ([10], [24])**.** *Let X be a complete CAT(0) space and $\{C_n\}$ be a closed convex subsets of X which is Δ-Mosco convergence to a closed convex subset C_0 of X. Then, For each $x \in X$,*

$$P_{C_n}(x) \to P_{C_0}(x).$$

Proof. Putting $f_n = i_{C_n}$ and applying the previous theorem, we get the desired result. □

We also consider the case that $\{\lambda_n\}$ converges to 0. Assuming the boundedness to the $\{f_n\}$, we can prove the convergence of the resolvent.

Theorem 3.6. *Let X be a complete CAT(0) space and $f_0 \colon X \to]-\infty, \infty]$ a proper convex lower semicontinuous function and $x \in X$. Suppose that $\{\lambda_n\}$ converges to 0 and $\{f_n\}$ satisfies that $\overline{\Delta}$-Ls dom $f_n \subset$ dom f_0 and*

$$\limsup_{n\to\infty} |f_n(P_{\operatorname{cl} \operatorname{dom} f_0}(x)) - f_n(Q^{\varphi}_{\lambda_n f_n}(x))| < \infty.$$

Then

$$Q^{\varphi}_{\lambda_n f_n}(x) \to P_{\operatorname{cl} \operatorname{dom} f_0}(x).$$

Proof. Put $x_n = Q^{\varphi}_{\lambda_n f_n}(x)$ for all $n \in \mathbb{N}$. From the definition of the resolvent, we have

$$\varphi(d(x_n, x)) \leq \lambda_n (f_n(P_{\operatorname{cl} \operatorname{dom} f_0}(x)) - f_n(x_n)) + \varphi(d(P_{\operatorname{cl} \operatorname{dom} f_0}(x), x))$$

for all $n \in \mathbb{N}$. From the assumption of $\{f_n\}$, letting $n \to \infty$, we get

$$\limsup_{n\to\infty} \varphi(d(x_n, x)) \leq \varphi(d(P_{\operatorname{cl} \operatorname{dom} f_0}(x), x)),$$

and hence $\limsup_{n\to\infty} d(x_n, x) \leq d(P_{\operatorname{cl} \operatorname{dom} f_0}(x), x)$. Therefore, $\{x_n\}$ is bounded. For all subsequence $\{x_{n_i}\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δ-convergent to some point $x_0 \in X$ and $\{d(x_{n_{i_j}}, P_{\operatorname{cl} \operatorname{dom} f_0})\}$ converges. Since $\overline{\Delta}$-Ls dom $f_n \subset$ dom f_0, we get $x_0 \in$ dom f_0. Therefore, we have

$$d(P_{\operatorname{cl} \operatorname{dom} f_0}(x), x) \leq d(x_0, x) \leq \liminf_{j\to\infty} d(x_{n_{i_j}}, x) \leq \limsup_{j\to\infty} d(x_{n_{i_j}}, x) \leq d(P_{\operatorname{cl} \operatorname{dom} f_0}(x), x).$$

Thus, $x_0 = P_{\operatorname{cl} \operatorname{dom} f_0}(x)$, and we get

$$\begin{aligned}\lambda_n f_n(x_n) + \varphi(d(x_n, x)) &\leq \lambda_n f_n\left(\frac{1}{2}x_n \oplus \frac{1}{2}P_{\operatorname{cl} \operatorname{dom} f_0}(x)\right) + \varphi\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}P_{\operatorname{cl} \operatorname{dom} f_0}(x), x\right)\right) \\ &\leq \frac{\lambda_n}{2} f_n(x_n) + \frac{\lambda_n}{2} f_n(P_{\operatorname{cl} \operatorname{dom} f_0}(x)) + \varphi\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}P_{\operatorname{cl} \operatorname{dom} f_0}(x), x\right)\right).\end{aligned}$$

Therefore, we have

$$\varphi(d(x_n, x)) \leq \frac{\lambda_n}{2}(f_n(P_{\operatorname{cl}\operatorname{dom} f_0}(x)) - f_n(x_n)) + \varphi(d(P_{\operatorname{cl}\operatorname{dom} f_0}(x), x)).$$

Letting $n \to \infty$, we get

$$\limsup_{n\to\infty} \varphi(d(x_n, x)) \leq \limsup_{n\to\infty} \varphi\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}P_{\operatorname{cl}\operatorname{dom} f_0}(x), x\right)\right),$$

and hence $\limsup_{n\to\infty} d(x_n, x) \leq \limsup_{n\to\infty} d\left(\frac{1}{2}x_n \oplus \frac{1}{2}P_{\operatorname{cl}\operatorname{dom} f_0}(x), x\right)$. From the parallelogram law, we have

$$\begin{aligned}
&d(P_{\operatorname{cl}\operatorname{dom} f_0(x)}, x)^2 \\
&\leq \liminf_{j\to\infty} d(x_{n_{i_j}}, x)^2 \\
&\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, x)^2 \\
&\leq \limsup_{j\to\infty} d\left(\frac{1}{2}x_{n_{i_j}} \oplus \frac{1}{2}P_{\operatorname{cl}\operatorname{dom} f_0}(x), x\right)^2 \\
&\leq \limsup_{j\to\infty}\left(\frac{1}{2}d(x_{n_{i_j}}, x)^2 + \frac{1}{2}d(P_{\operatorname{cl}\operatorname{dom} f_0}(x), x)^2 - \frac{1}{4}d(x_{n_{i_j}}, P_{\operatorname{cl}\operatorname{dom} f_0}(x))^2\right) \\
&\leq \frac{1}{2}\limsup_{j\to\infty} d(x_{n_{i_j}}, x)^2 + \frac{1}{2}d((P_{\operatorname{cl}\operatorname{dom} f_0}(x), x)^2 - \frac{1}{4}\liminf_{j\to\infty} d(x_{n_{i_j}}, P_{\operatorname{cl}\operatorname{dom} f_0}(x))^2 \\
&\leq \frac{1}{2}d(P_{\operatorname{cl}\operatorname{dom} f_0}(x), x)^2 + \frac{1}{2}d(P_{\operatorname{cl}\operatorname{dom} f_0}(x), x)^2 - \frac{1}{4}\lim_{j\to\infty} d(x_{n_{i_j}}, P_{\operatorname{cl}\operatorname{dom} f_0}(x))^2.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
0 &\leq \frac{1}{4}\lim_{j\to\infty} d(x_{n_{i_j}}, P_{\operatorname{cl}\operatorname{dom} f_0}(x))^2 \\
&\leq \frac{1}{2}d(P_{\operatorname{cl}\operatorname{dom} f_0(x)}, x)^2 + \frac{1}{2}d(P_{\operatorname{cl}\operatorname{dom} f_0(x)}, x)^2 - d(P_{\operatorname{cl}\operatorname{dom} f_0(x)}, x)^2 = 0.
\end{aligned}$$

Hence, we get $x_{n_{i_j}} \to P_{\operatorname{cl}\operatorname{dom} f_0}(x)$ and this implies $x_n \to P_{\operatorname{cl}\operatorname{dom} f_0}(x)$, which is the desired result. □

Theorem 3.7. *Let X be a complete CAT(0) space and $f_0 \colon X \to \,]-\infty, \infty]$ a proper convex lower semicontinuous function and $x \in X$. Suppose that $\{\lambda_n\}$ converges to 0, $\overline{\Delta}$-Ls dom $f_n \subset$ dom f_0, and there exists a sequence $\{z_n\}$ such that $z_n \to P_{\operatorname{cl}\operatorname{dom} f_0}(x)$ and that*

$$\limsup_{n\to\infty} |f_n(z_n) - f_n(Q^{\varphi}_{\lambda_n f_n}(x))| < \infty.$$

Then

$$Q^{\varphi}_{\lambda_n f_n}(x) \to P_{\operatorname{cl}\operatorname{dom} f_0}(x).$$

Proof. Put $x_n = Q^{\varphi}_{\lambda_n f_n}(x)$ for all $n \in \mathbb{N}$ and let $\{z_n\}$ be a sequence which satisfies the assumptions of the theorem. Then, from the definition of the resolvent, we get

$$\varphi(d(x_n, x)) \leq \lambda_n(f_n(z_n) - f_n(x_n)) + \varphi(d(z_n, x))$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we have

$$\limsup_{n\to\infty} \varphi(d(x_n, x)) \leq \limsup_{n\to\infty} \varphi(d(z_n, x)) = \varphi(d(P_{\mathrm{cl}\,\mathrm{dom}\, f_0}(x), x)),$$

and hence $\limsup_{n\to\infty} d(x_n, x) \leq d(P_{\mathrm{cl}\,\mathrm{dom}\, f_0}(x), x)$. Consequently, in a similar way to the proof of the previous theorem, we have the desired result.

□

3.2 Resolvents on CAT(1) spaces

We also consider the resolvent of a convex function on complete admissible CAT(1) space. Let X be a complete admissible CAT(1) space and $f\colon X \to \left]-\infty, \infty\right]$ a proper convex lower semicontinuous function, and $\psi\colon \left[0, \frac{\pi}{2}\right[\to [0, \infty[$ satisfying (B1)-(B5). For $\lambda > 0$, $Q^{\psi}_{\lambda f}\colon X \to X$ is the resolvent of the function λf, which is defined by

$$\begin{aligned} Q^{\psi}_{\lambda f}(x) &= \operatorname*{argmin}_{y\in X} \{\lambda f(y) + \psi(d(y, x))\} \\ &= \operatorname*{argmin}_{y\in X} \left\{ f(y) + \frac{1}{\lambda}\psi(d(y, x)) \right\} \end{aligned}$$

for all $x \in X$. Then, the following hold.

Theorem 3.8. *Let X be a complete admissible CAT(1) space, $f\colon X \to \left]-\infty, \infty\right]$ a proper convex lower semicontinuous function and $x \in X$. If there exists a positive real sequence $\{\mu_n\}$ diverging to ∞ and $\sup_{n\in\mathbb{N}} d(R^{d^2}_{\mu_n f}(x), x) < \frac{\pi}{2}$, then $\operatorname{argmin} f \neq \emptyset$ and*

$$\lim_{\lambda\to\infty} Q^{\psi}_{\lambda f}(x) = P_{\operatorname{argmin} f}(x).$$

Theorem 3.9. *Let X be a complete admissible CAT(1) space and $f\colon X \to \left]-\infty, \infty\right]$ a proper convex lower semicontinuous function. Then*

$$\lim_{\lambda\searrow 0} Q^{\psi}_{\lambda f}(x) = P_{\mathrm{cl}\,\mathrm{dom}\, f}(x)$$

and

$$\lim_{\lambda\to\lambda_0} Q^{\psi}_{\lambda f}(x) = Q^{\psi}_{\lambda_0 f}(x),$$

where λ_0 is a positive number.

We define the resolvents of a sequence of convex functions in a CAT(1) space. For a sequence $\{f_n\}$ of proper lower semicontinuous convex functions and a sequence $\{\lambda_n\}$ of positive real numbers, we define the sequence of resolvents $Q^{\psi}_{\lambda_n f_n}: X \to X$ by

$$\begin{aligned} Q^{\psi}_{\lambda_n f_n}(x) &= \operatorname*{argmin}_{y\in X} \{\lambda_n f_n(y) + \psi(d(y, x))\} \\ &= \operatorname*{argmin}_{y\in X} \left\{ f_n(y) + \frac{1}{\lambda_n}\psi(d(y, x)) \right\} \end{aligned}$$

for all $x \in X$. we consider the asymptotic behavior of this resolvent.

Theorem 3.10. *Let X be a complete admissible CAT(1) space and $f_0 \colon X \to]-\infty, \infty]$ a proper lower semicontinuous function and $x \in X$. Suppose that $\{f_n\}$ satisfies the following conditions;*

(a) $\emptyset \neq \operatorname{argmin} f_0 \subset \text{d-Li} \operatorname{argmin} f_n$;

(b) $\overline{\Delta}\text{-Ls} \left\{ Q^{\psi}_{\lambda_n f_n}(x) \right\} \subset \operatorname{argmin} f_0$.

Then

$$Q^{\psi}_{\lambda_n f_n}(x) \to P_{\operatorname{argmin} f_0}(x).$$

Proof. By (a), there exists a sequence $\{a_n\}$ such that $a_n \in \operatorname{argmin} f_n$ for all $n \in \mathbb{N}$ and $a_n \to P_{\operatorname{argmin} f_0}(x)$. Putting $x_n = Q^{\varphi}_{\lambda_n f_n}(x)$, we get

$$\begin{aligned} f_n(x_n) + \frac{1}{\lambda_n}\psi(d(x_n, x)) &\leq f_n\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n\right) + \frac{1}{\lambda_n}\psi\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n, x\right)\right) \\ &\leq \frac{1}{2}f_n(x_n) + \frac{1}{2}f_n(a_n) + \frac{1}{\lambda_n}\psi\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n, x\right)\right) \\ &\leq \frac{1}{2}f_n(x_n) + \frac{1}{2}f_n(x_n) + \frac{1}{\lambda_n}\psi\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n, x\right)\right) \\ &= f_n(x_n) + \frac{1}{\lambda_n}\psi\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n, x\right)\right), \end{aligned}$$

which implies $d(x_n, x) \leq d\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n, x\right)$ for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} -\log\cos(d(x_n, x)) &\leq -\log\cos\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n, x\right)\right) \\ &\leq -\frac{1}{2}\log(\cos d(x_n, x)) - \frac{1}{2}\log(\cos d(a_n, x)) + \log\left(\cos\left(\frac{d(x_n, a_n)}{2}\right)\right) \end{aligned}$$

by the parallelogram law in CAT(1) space. Hence, we get

$$0 \leq -2\log\left(\cos\left(\frac{d(x_n, a_n)}{2}\right)\right) \leq -\log(\cos d(a_n, x)) - \log(\cos d(x_n, x)), \tag{3.2}$$

and thus $d(x_n, x) \leq d(a_n, x)$. Since $\{a_n\}$ is convergent, $\{x_n\}$ is bounded. Thus, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ which is Δ-convergent to some point x_0 and $x_0 \in \operatorname{argmin} f_0$ by (b). Since $d(x_{n_{i_j}}, x) \leq d(a_{n_{i_j}}, x)$ holds for all $j \in \mathbb{N}$, letting $j \to \infty$, we get

$$d(x_0, x) \leq \liminf_{j\to\infty} d(x_{n_{i_j}}, x) \leq \liminf_{j\to\infty} d(a_{n_{i_j}}, x) = d(P_{\operatorname{argmin} f_0}(x), x).$$

Therefore, $x_0 = P_{\operatorname{argmin} f_0}(x)$ and hence $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} P_{\operatorname{argmin} f_0}(x)$. Furthermore, we get

$$\begin{aligned}
0 &\leq \limsup_{j\to\infty} \left(-2\log\left(\cos\left(\frac{d(x_{n_{i_j}}, P_{\operatorname{argmin} f_0}(x))}{2} \right)\right)\right) \\
&= \limsup_{j\to\infty} \left(-2\log\left(\cos\left(\frac{d(x_{n_{i_j}}, a_{n_{i_j}})}{2} \right)\right)\right) \\
&\leq \limsup_{j\to\infty} (-\log(\cos d(a_{n_{i_j}}, x)) - \log(\cos d(x_{n_{i_j}}, x))), \\
&\leq \limsup_{j\to\infty} (-\log(\cos d(a_{n_{i_j}}, x))) - \liminf_{j\to\infty}(-\log(\cos d(a_{n_{i_j}}, x))) \\
&\leq -\log(\cos(d(P_{\operatorname{argmin} f_0}(x), x))) + \log(\cos(d(P_{\operatorname{argmin} f_0}(x), x))) = 0
\end{aligned}$$

from (3.2). Consequently, we get $x_{n_{i_j}} \to P_{\operatorname{argmin} f_0}(x)$, which implies $x_n \to P_{\operatorname{argmin} f_0}(x)$. This is the desired result. □

Theorem 3.11. *Let X be a complete admissible CAT(1) space and $f_0 \colon X \to \,]-\infty, \infty]$ a proper convex lower semicontinuous function and $x \in X$. Suppose that $\{\lambda_n\}$ diverges to ∞ and $\{f_n\}$ satisfies the following conditions;*

(a) $\emptyset \neq \operatorname{argmin} f_0 \subset$ d-Li $\operatorname{argmin} f_n$*;*
(b) For all $b \in X$, there exists $\{b_n\}$ such that $b_n \to b$ and $\limsup_{n\to\infty} f_n(b_n) \leq f(b)$;
(c) For any subsequence $\{f_{n_i}\}$ of $\{f_n\}$ and a sequence $\{c_i\}$ which is Δ-convergent to c, it hold that $f(c) \leq \liminf_{i\to\infty} f_n(c_n)$.

Then

$$Q^{\psi}_{\lambda_n f_n}(x) \to P_{\operatorname{argmin} f_0}(x).$$

Proof. Put $x_n = Q^{\varphi}_{\lambda_n f_n}(x)$ for $n \in \mathbb{N}$. Assume that subsequence $\{x_{n_i}\}$ of $\{x_n\}$ is Δ-convergent to $x_0 \in X$, and $\{b_n\}$ satisfies $b_n \to b$ and $\limsup_{n\to\infty} f_n(b_n) \leq f(b)$. From the definition of resolvent, we get

$$f_{n_i}(x_{n_i}) + \frac{1}{\lambda_{n_i}}\psi(d(x_{n_i}, x)) \leq f_{n_i}(b_{n_i}) + \frac{1}{\lambda_{n_i}}\psi(d(b_{n_i}, x)).$$

and letting $i \to \infty$, we have

$$f_0(x_0) \leq \liminf_{i\to\infty} f_{n_i}(x_{n_i}) \leq \limsup_{i\to\infty} f_{n_i}(b_{n_i}) \leq f_0(P_{\operatorname{argmin} f_0}(x)).$$

Thus, $x_0 \in \operatorname{argmin} f_0$. Hence, from the previous theorem, we get the desired result. □

Theorem 3.12. *Let X be a complete admissible CAT(1) space and $f_0 \colon X \to \,]-\infty, \infty]$ a proper convex lower semicontinuous function and $x \in X$. Suppose that $\{\lambda_n\}$ converges to 0 and $\{f_n\}$ satisfies that $\overline{\Delta}$-Ls $\operatorname{dom} f_n \subset \operatorname{dom} f_0$ and*

$$\limsup_{n\to\infty} |f_n(P_{\operatorname{cl}\operatorname{dom} f_0}(x)) - f_n(Q^{\varphi}_{\lambda_n f_n}(x))| < \infty.$$

Then

$$Q^{\psi}_{\lambda_n f_n}(x) \to P_{\operatorname{cl}\operatorname{dom} f_0}(x).$$

Proof. Put $x_n = Q^{\psi}_{\lambda_n f_n}(x)$ for all $n \in \mathbb{N}$. From the definition of the resolvent, we have

$$\psi(d(x_n, x)) \leq \lambda_n (f_n(P_{\mathrm{cl\,dom}\, f_0}(x)) - f_n(x_n)) + \psi(d(P_{\mathrm{cl\,dom}\, f_0}(x), x))$$

for all $n \in \mathbb{N}$. Thus, letting $n \to \infty$, we get

$$\limsup_{n\to\infty} \psi(d(x_n, x)) \leq \psi(d(P_{\mathrm{cl\,dom}\, f_0}(x), x)),$$

and hence $\limsup_{n\to\infty} d(x_n, x) \leq d(P_{\mathrm{cl\,dom}\, f_0}(x), x)$. Therefore, since $\{x_n\}$ is bounded, for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δ-convergent to $x_0 \in X$ and $\{d(x_{n_{i_j}}, P_{\mathrm{cl\,dom}\, f_0})\}$ converges. Then $x_0 \in \mathrm{dom}\, f_0$ and we have

$$d(P_{\mathrm{cl\,dom}\, f_0}(x), x) \leq d(x_0, x) \leq \liminf_{j\to\infty} d(x_{n_{i_j}}, x) \leq \limsup_{j\to\infty} d(x_{n_{i_j}}, x) \leq d(P_{\mathrm{cl\,dom}\, f_0}(x), x).$$

Hence, we get $x_0 = P_{\mathrm{cl\,dom}\, f_0}(x)$ and we have

$$\begin{aligned}
\lambda_n f_n(x_n) + \psi(d(x_n, x)) &\leq \lambda_n f_n\left(\frac{1}{2}x_n \oplus \frac{1}{2}P_{\mathrm{cl\,dom}\, f_0}(x)\right) + \psi\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}P_{\mathrm{cl\,dom}\, f_0}(x), x\right)\right) \\
&\leq \frac{\lambda_n}{2} f_n\left(x_n\right) + \frac{\lambda_n}{2} f_n\left(P_{\mathrm{cl\,dom}\, f_0}(x)\right) + \psi\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}P_{\mathrm{cl\,dom}\, f_0}(x), x\right)\right).
\end{aligned}$$

Hence, we have

$$\psi(d(x_n, x)) \leq \frac{\lambda_n}{2}(f_n(P_{\mathrm{cl\,dom}\, f_0}(x)) - f_n(x_n)) + \psi(d(P_{\mathrm{cl\,dom}\, f_0}(x), x)).$$

Letting $n \to \infty$, we get

$$\limsup_{n\to\infty} \psi(d(x_n, x)) \leq \limsup_{n\to\infty} \psi\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}P_{\mathrm{cl\,dom}\, f_0}(x), x\right)\right).$$

Thus, $\limsup_{n\to\infty} d(x_n, x) \leq \limsup_{n\to\infty} d\left(\frac{1}{2}x_n \oplus \frac{1}{2}P_{\mathrm{cl\,dom}\, f_0}(x), x\right)$. The parallelogram law implies that

$$\begin{aligned}
&-\log(\cos d(P_{\mathrm{cl\,dom}\, f_0}(x), x)) \\
&\leq \liminf_{j\to\infty}(-\log(\cos d(x_{n_{i_j}}, x))) \\
&\leq \limsup_{j\to\infty}(-\log(\cos d(x_{n_{i_j}}, x))) \\
&\leq \limsup_{j\to\infty} -\log\left(\cos d\left(\frac{1}{2}x_{n_{i_j}} \oplus \frac{1}{2}P_{\mathrm{cl\,dom}\, f_0}(x), x\right)\right) \\
&\leq \limsup_{j\to\infty}\left(-\frac{1}{2}\log(\cos d(x_{n_{i_j}}, x)) + \log\left(\cos\frac{d(x_{n_{i_j}}, P_{\mathrm{cl\,dom}\, f_0}(x))}{2}\right)\right) \\
&\quad - \frac{1}{2}\log(\cos d(P_{\mathrm{cl\,dom}\, f_0}(x), x)) \\
&\leq -\frac{1}{2}\log(\cos d((P_{\mathrm{cl\,dom}\, f_0}(x), x)) + \lim_{n\to\infty}\log\left(\cos\frac{d(x_{n_{i_j}}, P_{\mathrm{cl\,dom}\, f_0}(x))}{2}\right) \\
&\quad - \frac{1}{2}\log(\cos d((P_{\mathrm{cl\,dom}\, f_0}(x), x))
\end{aligned}$$

Therefore, we obtain

$$\lim_{j\to\infty} d(x_{n_{i_j}}, P_{\mathrm{cl\,dom}\,f_0}(x)) \to 0,$$

Hence, for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is convergent to $P_{\mathrm{cl\,dom}\,f_0}(x)$ and this implies

$$x_n \to P_{\mathrm{cl\,dom}\,f_0}(x).$$

It completes the proof. □

Theorem 3.13. *Let X be a complete CAT(0) space and $f_0 \colon X \to]-\infty, \infty]$ a proper convex lower semicontinuous function and $x \in X$. Suppose that $\{\lambda_n\}$ converges to 0, $\overline{\Delta}\text{-Ls}\,\mathrm{dom}\,f_n \subset \mathrm{dom}\,f_0$, and there exists a sequence $\{z_n\}$ such that $z_n \to P_{\mathrm{cl\,dom}\,f_0}(x)$ and that*

$$\limsup_{n\to\infty} |f_n(z_n) - f_n(Q^{\psi}_{\lambda_n f_n}(x))| < \infty.$$

Then

$$Q^{\psi}_{\lambda_n f_n}(x) \to P_{\mathrm{cl\,dom}\,f_0}(x).$$

Proof. Put $x_n = Q^{\psi}_{\lambda_n f_n}(x)$ for all $n \in \mathbb{N}$ Take a sequence $\{z_n\}$ satisfying the assumptions of the theorem. Then, we get

$$\psi(d(x_n, x)) \le \lambda_n(f_n(z_n) - f_n(x_n)) + \psi(d(z_n, x))$$

for all $n \in \mathbb{N}$ and letting $n \to \infty$, we have

$$\limsup_{n\to\infty} \psi(d(z_n, x)) = \psi(d(P_{\mathrm{cl\,dom}\,f_0}(x), x)).$$

Therefore, we get $\limsup_{n\to\infty} d(x_n, x) \le d(P_{\mathrm{cl\,dom}\,f_0}(x), x)$ and we obtain the desired result in a similar way to the proof of the previous theorem. □

Chapter 4

Asymptotic behavior of resolvents for equilibrium problems

In this chapter, we consider the resolvents for equilibrium problems on complete CAT(0) spaces and complete admissible CAT(1) spaces.

4.1 Resolvents on CAT(0) spaces

Let X be a complete CAT(0) space K a closed convex subset of X and $F\colon K \times K \to \mathbb{R}$ bifunction satisfying (E1)-(E4). For positive number λ, the resolvent $R^{d^2}_{\lambda F}\colon X \to K$ is defined as follows;

$$\begin{aligned} R^{d^2}_{\lambda F}(x) &= \left\{ z \in K \mid \inf_{y \in K} \left(\lambda F(z, y) + d(y, x)^2 - d(z, x)^2 \right) \geq 0 \right\} \\ &= \left\{ z \in K \mid \inf_{y \in K} \left(F(z, y) + \frac{1}{\lambda} \left(d(y, x)^2 - d(z, x)^2 \right) \right) \geq 0 \right\} \end{aligned}$$

for all $x \in X$. We consider the increasingness or decreasingness of $\{d(R^{d^2}_{\lambda F}(x), x)\}$ and $f(d(R^{d^2}_{\lambda F}(x))$ in regard to λ.

Lemma 4.1. *Let X be a complete CAT(0) space having the convex hull finite property, K a nonempty closed convex subset of X, and $F\colon K \times K \to \mathbb{R}$ satisfying (E1)-(E4). For $x \in X$ and $t \in [0, 1]$, if $0 < \lambda \leq \mu$, then*

$$d(R^{d^2}_{\lambda F}(x), x) \leq d(tR^{d^2}_{\lambda F}(x) \oplus (1-t)R^{d^2}_{\mu F}(x), x) \leq d(R^{d^2}_{\mu F}(x), x),$$

$$F(R^{d^2}_{\mu F}(x), R^{d^2}_{\lambda F}(x)) \geq 0 \quad \text{and} \quad F(R^{d^2}_{\lambda F}(x), R^{d^2}_{\mu F}(x)) \leq 0.$$

Proof. Put $x_\lambda = R^{d^2}_{\lambda F}(x)$ and $x_\mu = R^{d^2}_{\mu F}(x)$. First we show that $d(x_\lambda, x) \leq d(x_\mu, x)$. By the definition of these resolvents, we have

$$0 \leq F(x_\lambda, x_\mu) + \frac{1}{\lambda} \left(d(x_\mu, x)^2 - d(x_\lambda, x)^2 \right); \tag{4.1}$$

$$0 \leq F(x_\mu, x_\lambda) + \frac{1}{\mu} \left(d(x_\lambda, x)^2 - d(x_\mu, x)^2 \right) \tag{4.2}$$

From these inequalities and (E2), we get

$$\begin{aligned}0 &\leq F(x_\lambda, x_\mu) + F(x_\mu, x_\lambda) + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right)\left(d(x_\mu, x)^2 - d(x_\lambda, x)^2\right) \\ &\leq 0 + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right)\left(d(x_\mu, x)^2 - d(x_\lambda, x)^2\right),\end{aligned}$$

which implies that

$$d(x_\lambda, x) \leq d(x_\mu, x).$$

Then, using (4.2) again, we get

$$0 \leq \frac{1}{\mu}\left(d(x_\mu, x)^2 - d(x_\lambda, x)^2\right) \leq F(x_\mu, x_\lambda).$$

By (E2), we also get

$$F(x_\lambda, x_\mu) \leq -F(x_\mu, x_\lambda) \leq 0.$$

To complete the proof, we show that $d(x_\lambda, x) \leq d(x_\lambda \oplus (1-t)x_\mu, x) \leq d(x_\mu, x)$ for all $t \in]0, 1[$. From (E1), (E2) we get

$$\begin{aligned}0 &\leq \lambda F(x_\lambda, tx_\lambda \oplus (1-t)x_\mu) + d(tx_\lambda \oplus (1-t)x_\mu, x)^2 - d(x_\lambda, x)^2 \\ &\leq \lambda t F(x_\lambda, x_\lambda) + \lambda(1-t)F(x_\lambda, x_\mu) + d(tx_\lambda \oplus (1-t)x_\mu, x)^2 - d(x_\lambda, x)^2 \\ &\leq 0 + 0 + d(tx_\lambda \oplus (1-t)x_\mu, x)^2 - d(x_\lambda, x)^2,\end{aligned}$$

and thus $d(x_\lambda, x) \leq d(tx_\lambda \oplus (1-t)x_\mu, x)$. Further, we get

$$\begin{aligned}d(x_\lambda, x) &\leq d(tx_\lambda \oplus (1-t)x_\mu, x) \\ &\leq td(x_\lambda, x) + (1-t)d(x_\mu, x) \\ &\leq td(x_\mu, x) + (1-t)d(x_\mu, x) = d(x_\mu, x).\end{aligned}$$

It completes the proof. □

Further, the following relation is very important in considering the properties of resolvent with parameter λ.

Lemma 4.2. *Let X be a complete $CAT(0)$ space having the convex hull finite property, K a nonempty closed convex subset of X, and $F\colon K \times K \to \mathbb{R}$ satisfies (E1)-(E4). For $x \in X$, if $0 < \lambda \leq \mu$, then*

$$d(R^{d^2}_{\lambda F}(x), R^{d^2}_{\mu F}(x))^2 \leq d(R^{d^2}_{\lambda F}(x), x)^2 + d(R^{d^2}_{\mu F}(x), x)^2.$$

Proof. Put $x_\lambda = R^{d^2}_{\lambda F}(x)$ and $x_\mu = R^{d^2}_{\mu F}(x)$. From (E1), (E2) and the parallelogram law, we have

$$\begin{aligned}0 &\leq \lambda F(x_\lambda, tx_\lambda \oplus (1-t)x_\mu) + d(tx_\lambda \oplus (1-t)x_\mu, x)^2 - d(x_\lambda, x)^2 \\ &\leq \lambda t F(x_\lambda, x_\lambda) + \lambda(1-t)F(x_\lambda, x_\mu) + d(tx_\lambda \oplus (1-t)x_\mu, x)^2 - d(x_\lambda, x)^2 \\ &\leq 0 + 0 + d(tx_\lambda \oplus (1-t)x_\mu, x)^2 - d(x_\lambda, x)^2 \\ &\leq td(x_\lambda, x)^2 + (1-t)d(x_\mu, x)^2 - t(1-t)d(x_\mu, x_\lambda)^2 - d(x_\lambda, x)^2 \\ &= (1-t)\left(d(x_\lambda, x)^2 - d(x_\mu, x)^2 - td(x_\mu, x_\lambda)^2\right)\end{aligned}$$

for all $t \in \,]0,1[$. Dividing by $1-t$ and letting $t \to 1$, we obtain

$$d(x_\mu, x_\lambda)^2 \le d(x_\lambda, x)^2 - d(x_\mu, x)^2,$$

which is the desired result. □

To show the boundedness of the resolvent, we use the next lemma.

Lemma 4.3. *Let $\tau\colon \mathbb{R} \to \mathbb{R}$ be an increasing function and $\{\lambda_n\}$, $\{\mu_n\}$ real sequences that diverge to ∞. If $\{\tau(\mu_n)\}$ is bounded, then so is $\{\tau(\lambda_n)\}$.*

Proof. Assume $\{\tau(\mu_n)\}$ is bounded and $\{\tau(\lambda_n)\}$ is not. Put $M = \sup_{n\in\mathbb{N}}\{\tau(\mu_n)\}$. Then, there exists $n_0 \in \mathbb{N}$ such that $M < \tau(\lambda_{n_0})$. Further, since $\{\mu_n\}$ is divergent to ∞, there exists $n_1 \in \mathbb{N}$ such that $\lambda_{n_0} < \mu_{n_1}$. As τ is increasing,

$$M < \tau(\lambda_{n_0}) < \tau(\mu_{n_1}) \le M.$$

This is the contradiction, and then $\{\tau(\lambda_n)\}$ is bounded. □

Using these lemmas, we consider the asymptotic behavior of the resolvent. In the case that λ diverges to ∞, the resolvent converges to a solution to the equilibrium problem with some assumptions.

Theorem 4.4. *Let X be a complete CAT(0) space having the convex hull finite property, K a nonempty closed convex subset of X, and $F\colon K \times K \to \mathbb{R}$ satisfying (E1)-(E4). For $x \in X$, if there exists a positive real sequence $\{\mu_n\}$ diverging to ∞ and $\{d(R^{d^2}_{\mu_n f}(x), x)\}$ is bounded, then* Equil $f \neq \emptyset$ *and*

$$\lim_{\lambda\to\infty} R^{d^2}_{\lambda F}(x) = P_{\mathrm{Equil}\, F}(x).$$

Proof. It is sufficient to show that $R^{d^2}_{\lambda_n F}(x) \to P_{\mathrm{Equil}\, F}(x)$ for any positive real sequence $\{\lambda_n\}$ which is increasing and diverges to ∞. Put $x_n = R^{d^2}_{\lambda_n F}(x)$ for $n \in \mathbb{N}$. If there exists a real sequence $\{\mu_n\}$ satisfying the assumption of the theorem, then $\{d(x_n, x)\}$ is bounded by Lemma 4.3. For $n, m \in \mathbb{N}$ with $n \le m$ it hold that

$$d(x_n, x_m) \le d(x_m, x)^2 - d(x_n, x)^2$$

from Lemma 4.2. Therefore, $\{x_n\}$ is a Cauchy sequence on K. Since X is complete and K is closed, $\{x_n\}$ is convergent to some point $x_0 \in K$. By the definition of x_n, it satisfies that

$$0 \le F(x_n, y) + \frac{1}{\lambda_n}\left(d(y,x)^2 - d(x_n,x)^2\right)$$

for all $y \in K$. From (E2) and (E3), letting $n \to \infty$, we get

$$0 \le \limsup_{n\to\infty} F(x_n, y) \le \limsup_{n\to\infty}(-F(y, x_n)) = -\liminf_{n\to\infty} F(y, x_n) = -F(y, x_0)$$

and hence $F(y, x_0) \le 0$ for all $y \in K$. Let $w \in K$ and $t \in \,]0,1[$. Since K is convex, $tw \oplus (1-t)x_0 \in K$ and hence

$$F(tw \oplus (1-t)x_0, x_0) \le 0.$$

From (E1) and (E3), we have

$$\begin{aligned}0 &= F(tw \oplus (1-t)x_0, tw \oplus (1-t)x_0) \\ &\le tF(tw \oplus (1-t)x_0, w) + (1-t)F(tw \oplus (1-t)x_0, x_0) \\ &\le tF(tw \oplus (1-t)x_0, w).\end{aligned}$$

Dividing by t and letting $t \to 0$, we obtain

$$0 \le \limsup_{t\to 0} F(tw \oplus (1-t)x_0, w) = F(x_0, w)$$

for all $w \in K$ by (E4). Therefore, we obtain

$$x_0 \in \operatorname{Equil} F \ne \emptyset.$$

Since $x_n \in K$, we have

$$\begin{aligned}0 &\le \lambda_n F(x_n, z) + d(z, x)^2 - d(x_n, x)^2 \\ &\le -\lambda_n F(z, x_n) + d(z, x)^2 - d(x_n, x)^2 \\ &\le d(z, x)^2 - d(x_n, x)^2\end{aligned}$$

for all $z \in \operatorname{Equil} F$, and thus $d(x_n, x) \le d(z, x)$. Letting $n \to \infty$, we obtain

$$d(x_0, x) \le \liminf_{n\to\infty} d(x_n, x) \le d(z, x)$$

for all $z \in \operatorname{Equil} F$, which implies $x_0 = P_{\operatorname{Equil} F}(x)$. Hence, it holds that

$$\lim_{n\to\infty} x_n = x_0 = P_{\operatorname{Equil} F}(x).$$

It completes the proof. □

We also consider the asymptotic behavior of the resolvent at 0 and the continuity of the resolvent.

Theorem 4.5. *Let X be a complete CAT(0) space having the convex hull finite property, K a nonempty closed convex subset of X, and $F\colon K \times K \to \mathbb{R}$ satisfying (E1)-(E4). Then for $x \in X$,*

$$\lim_{\lambda \searrow 0} R^{d^2}_{\lambda F}(x) = P_K(x).$$

and

$$\lim_{\lambda \to \lambda_0} R^{d^2}_{\lambda F}(x) = Q^{\varphi}_{\lambda_0 f}(x),$$

where λ_0 is a positive real number.

Proof. Let $\{\lambda_n\}$ be an increasing or decreasing sequence of real positive numbers and put $x_n = R^{d^2}_{\lambda_n F}(x)$ for all $n \in \mathbb{N}$. We may show that $\{x_n\}$ converges in X, if $\{\lambda_n\}$ converges to some number in $[0, \infty[$. If $\{\lambda_n\}$ is increasing and convergent to $\lambda_0 > 0$, then $\{d(x_n, x)\}$ is increasing and

$$0 \le d(x_n, x) \le d(Q^{\varphi}_{\lambda_0 f}(x), x)$$

for all $n \in \mathbb{N}$. If $\{\lambda_n\}$ is decreasing,

$$0 \leq d(x_n, x) \leq d(x_1, x)$$

for all $n \in \mathbb{N}$. Therefore, in the both cases, $\{d(x_n, x)\}$ is bounded. Since $\{d(x_n, x)\}$ is also increasing or decreasing, it is convergent. Moreover, we get

$$d(x_n, x_m)^2 \leq |d(x_m, x)^2 - d(x_n, x)^2|.$$

by Lemma 4.2. Thus, $\{x_n\}$ is a Cauchy sequence on K and then $\{x_n\}$ is convergent to some point $x_0 \in K$. From the definition of the resolvent and (E2), we have

$$\begin{aligned}
0 &\leq \lambda_n F(x_n, y) + d(y, x)^2 - d(x_n, x)^2 \\
&\leq -\lambda_n F(y, x_n) + d(y, x)^2 - d(x_n, x)^2
\end{aligned}$$

for all $y \in K$. If $\lambda_n \to \lambda_0 > 0$, letting $n \to \infty$ we get

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \left(-\lambda_n F(y, x_n) + d(y, x)^2 - d(x_n, x)^2\right) \\
&\leq -\liminf_{n\to\infty} \lambda_n F(y, x_n) + d(y, x)^2 - d(x_0, x)^2 \\
&\leq -\lambda_0 F(y, x_0) + d(y, x)^2 - d(x_0, x)^2
\end{aligned}$$

by (E3), and hence $\lambda_0 f(y, x_n) \leq d(x, y)^2 - d(x_n, x)^2$ for all $y \in K$. Let $w \in K$ and $t \in]0, 1[$. From (E1), (E2) and (E3), we have

$$\begin{aligned}
0 &= \lambda_0 F(tw \oplus (1-t)x_0, tw \oplus (1-t)x_0) \\
&\leq \lambda_0 t F(tw \oplus (1-t)x_0, w) + (1-t)(\lambda_0 F(tw \oplus (1-t)x_0, x_0)) \\
&\leq \lambda_0 t F(tw \oplus (1-t)x_0, w) + (1-t)\left(d(tw \oplus (1-t)x_0, x)^2 - d(x_0, x)^2\right) \\
&\leq \lambda_0 t F(tw \oplus (1-t)x_0, w) + (1-t)\left(t d(w, x)^2 + (1-t)d(x_0, x)^2 - d(x_0, x)^2\right) \\
&\leq \lambda_0 t F(tw \oplus (1-t)x_0, w) + t(1-t)\left(d(w, x)^2 - d(x_0, x)^2\right).
\end{aligned}$$

From (E4), dividing by t and letting $t \to 0$, we obtain

$$0 \leq \lambda_0 F(x_0, w) + d(w, x)^2 - d(x_0, x)^2$$

for all $w \in K$, which implies $x_0 = Q^{\varphi}_{\lambda_0 f}(x)$. On the other hand, if $\lambda_n \to 0$, it hold that

$$0 \leq d(y, x)^2 - d(x_0, x)^2$$

for all $y \in K$. Then, $d(x_0, x) \leq d(y, x)$ for all $y \in K$. Since x_0 is a point of K, $x_0 = P_K(x)$. Consequently, the resolvent $R^{d^2}_{\lambda_n F}(x)$ converges to such x_0 for positive real sequence which is convergent. This implies that

$$R^{d^2}_{\lambda F}(x) \xrightarrow[\lambda \searrow 0]{} P_K(x);$$

$$R^{d^2}_{\lambda F}(x) \xrightarrow[\lambda \to \lambda_0]{} R^{d^2}_{\lambda_0 f}(x),$$

which is the desired result. ☐

Next, we consider the resolvents of a sequence of bifunctions $\{F_n\}$. Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of X and $\{F_n\}$ a sequence of bifunctions $F_n\colon K_n \times K_n \to \mathbb{R}$ satisfying (E1)-(E4) for each $n \in \mathbb{N}$. For a sequence $\{\lambda_n\}$ of positive real numbers, $R^{d^2}_{\lambda_n F_n} : X \to K_n$ is defined by

$$\begin{aligned} R^{d^2}_{\lambda_n F_n}(x) &= \left\{ z \in K_n \mid \inf_{y \in K_n} \left(\lambda_n F_n(z,y) + d(y,x)^2 - d(z,x)^2 \right) \geq 0 \right\} \\ &= \left\{ z \in K_n \mid \inf_{y \in K_n} \left(F_n(z,y) + \frac{1}{\lambda_n} \left(d(y,x)^2 - d(z,x)^2 \right) \right) \geq 0 \right\} \end{aligned}$$

for $x \in X$.

Theorem 4.6. *Let X be a complete CAT(0) space having the convex hull finite property, K_0 a nonempty closed convex subset of X, and $F_0\colon K_0 \times K_0 \to \mathbb{R}$ a bifunction satisfying (E1)-(E4) and $x \in X$. Suppose that $\{F_n\}$ satisfies the following conditions;*

(a) $\emptyset \neq \operatorname{Equil} F_0 \subset$ d-Li $\operatorname{Equil} F_n$*;*
(b) $\overline{\Delta}$-Ls $\left\{ R^{d^2}_{\lambda_n F_n}(x) \right\} \subset \operatorname{Equil} F_0$.

Then

$$R^{d^2}_{\lambda_n F_n}(x) \to P_{\operatorname{Equil} F_0}(x).$$

Proof. Put $x_n = R^{d^2}_{\lambda_n F_n}(x)$ for all $n \in \mathbb{N}$. By (a), we can take a sequence $\{a_n\}$ such that $a_n \in \operatorname{Equil} F_n$ for each $n \in \mathbb{N}$ and $a_n \to P_{\operatorname{Equil} f_0}(x)$. Since $x_n \in K_n$, we get $-F_n(a_n, x_n) \leq 0$. From the definition of the resolvent, the parallelogram law, (E1) and (E2), we have

$$\begin{aligned} 0 &\leq F_n(x_n, tx_n \oplus (1-t)a_n) + \frac{1}{\lambda_n} \left(d(tx_n \oplus (1-t)a_n, x)^2 - d(x_n, x)^2 \right) \\ &\leq tF_n(x_n, x_n) + (1-t)F_n(x_n, a_n) + \frac{1}{\lambda_n} \left(d(tx_n \oplus (1-t)a_n, x)^2 - d(x_n, x)^2 \right) \\ &\leq tF_n(x_n, x_n) - (1-t)F_n(a_n, x_n) + \frac{1}{\lambda_n} \left(d(tx_n \oplus (1-t)a_n, x)^2 - d(x_n, x)^2 \right) \\ &\leq 0 + 0 + \frac{1}{\lambda_n} \left(d(tx_n \oplus (1-t)a_n, x)^2 - d(x_n, x)^2 \right) \\ &\leq \frac{1}{\lambda_n} \left(td(x_n, x)^2 + (1-t)d(a_n, x)^2 - t(1-t)d(x_n, a_n)^2 - d(x_n, x)^2 \right) \\ &= \frac{1}{\lambda_n}(1-t) \left(d(a_n, x)^2 - d(x_n, x)^2 - td(x_n, a_n)^2 \right). \end{aligned}$$

Dividing by $\frac{1}{\lambda_n}(1-t)$ and letting $t \to 1$, we get

$$d(x_n, x)^2 \leq d(a_n, x)^2 - d(x_n, a_n)^2,$$

and hence $d(x_n, x) \leq d(a_n, x)$. Since $\{a_n\}$ is convergent, $\{x_n\}$ is bounded. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily and then there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δconvergent to some point $x_0 \in X$. Further, we get $x_0 \in \operatorname{Equil} F_0$ by (b). Therefore,

$$\begin{aligned} d(P_{\operatorname{Equil} f_0}(x), x) &\leq d(x_0, x) \\ &\leq \liminf_{j\to\infty} d(x_{n_{ij}}, x) \\ &\leq \limsup_{j\to\infty} d(x_{n_{ij}}, x) \\ &\leq \limsup_{j\to\infty} d(a_{n_{ij}}, x) \\ &= d(P_{\operatorname{Equil} F_0}(x), x). \end{aligned}$$

Thus, $x_0 = P_{\operatorname{Equil} f_0}(x)$. Moreover, we obtain

$$\begin{aligned} 0 &\leq \limsup_{j\to\infty} d(x_{n_{ij}}, P_{\operatorname{Equil} F_0}(x))^2 \\ &= \limsup_{j\to\infty} d(x_{n_{ij}}, a_{n_{ij}})^2 \\ &\leq \limsup_{j\to\infty} (d(a_{n_{ij}}, x)^2 - d(x_{n_{ij}}, a_{n_{ij}})^2) \\ &\leq \limsup_{j\to\infty} d(a_{n_{ij}}, x)^2 - \liminf_{j\to\infty} d(x_{n_{ij}}, a_{n_{ij}})^2 \\ &\leq d(P_{\operatorname{Equil} f_0}(x), x)^2 - d(P_{\operatorname{Equil} F_0}(x), x)^2 = 0, \end{aligned}$$

and hence $x_{n_{ij}} \to P_{\operatorname{Equil} F_0}(x)$. Therefore, for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$ has a subsequence $\{x_{n_{ij}}\}$ which is convergent to $P_{\operatorname{Equil} F_0}(x)$. It is equivalent to $x_n \to P_{\operatorname{Equil} F_0}(x)$. $\square$

Theorem 4.7. *Let X be a complete CAT(0) space having the convex hull finite property, K_0 a nonempty closed convex subset of X, and $F_0 \colon K_0 \times K_0 \to \mathbb{R}$ a bifunction satisfying (E1)-(E4) and $x \in X$. Suppose $\{\lambda_n\}$ diverges to ∞ and $\{F_n\}$ satisfies the following conditions;*

- *(a) $\emptyset \neq \operatorname{Equil} F_0 \subset$ d-Li $\operatorname{Equil} F_n$;*
- *(b) For all $b \in K_0$, $\{F_{n_i}\}$ of $\{F_n\}$, and $\{c_i\}$ such that $c_i \in K_{n_i}$ and $\{c_i\}$ is Δ-convergent to $c \in X$, there exists $\{b_i\}$ which satisfies $b_i \in K_{n_i}$, $b_i \to b$, and $F_0(b, c) \leq \liminf_{i\to\infty} F_{n_i}(b_i, c_i)$.*

Then

$$R^{d^2}_{\lambda_n F_n}(x) \to P_{\operatorname{Equil} F_0}(x).$$

Proof. Put $x_n = R^{d^2}_{\lambda_n F_n}(x)$ for $n \in \mathbb{N}$. Suppose $b \in X$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ is Δ-convergent to x_0. From the assumption, there exists $\{b_i\}$ such that $b_i \in K_{n_i}$ for each $n \in \mathbb{N}$ and $F_0(b_0, x_0) \leq \liminf_{i\to\infty} F_{n_i}(b_i, x_{n_i})$. From the definition of the resolvent, we have

$$\begin{aligned} 0 &\leq F_{n_i}(x_{n_i}, b_i) + \frac{1}{\lambda_{n_i}} \left(d(b_i, x)^2 - d(x_{n_i}, x)^2 \right) \\ &\leq -F_{n_i}(b_i, x_{n_i}) + \frac{1}{\lambda_{n_i}} \left(d(b_i, x)^2 - d(x_{n_i}, x)^2 \right), \end{aligned}$$

and letting $i \to \infty$, we get

$$0 \leq \limsup_{i\to\infty} F_{n_i}(x_{n_i}, b_i) \leq \limsup_{i\to\infty} (-F_{n_i}(b_i, x_{n_i})) = -\liminf_{i\to\infty} F_{n_i}(b_i, x_{n_i}) \leq -F_0(b_0, x_0).$$

Therefore, by the upper hemicontinuity of $F_0(\cdot, b_0)$, we have $F(x_0, b_0) \geq 0$ for all $b_0 \in K_0$ and thus $x_0 \in \operatorname{Equil} F_0$. From the previous theorem, we get the desired result. $\square$

Theorem 4.8. *Let X be a complete $CAT(0)$ space having the convex hull finite property, K_0 a nonempty closed convex subset of X, $F_0\colon K_0 \times K_0 \to \mathbb{R}$ a bifunction satisfying (E1)-(E4) and $x \in X$. Suppose that $\{K_n\}$ and $\{F_n\}$ satisfies that $\overline{\Delta}\text{-Ls}\, K_n \subset K_0$, and there exists $\{z_n\}$ such that $z_n \in K_n$, $z_n \to P_{K_0}(x)$ and*

$$\limsup_{n\to\infty} F_n(Q^{\varphi}_{\lambda_n f_n}(x), z_n) < \infty.$$

If $\{\lambda_n\}$ converges to 0, then

$$R^{d^2}_{\lambda_n F_n}(x) \to P_{K_0}(x).$$

Proof. Put $x_n = R^{d^2}_{\lambda_n F_n}(x)$ for all $n \in \mathbb{N}$ and let $\{z_n\}$ be a sequence which satisfies the assumptions of the theorem. Then, we have

$$d(x_n, x)^2 \le \lambda_n F(x_n, z_n) + d(z_n, x)^2.$$

Letting $n \to \infty$, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, x)^2 &\le \limsup_{n\to\infty} \left(\lambda_n F(x_n, z_n) + d(z_n, x)^2\right) \\
&\le \limsup_{n\to\infty} \lambda_n F(x_n, z_n) + \limsup_{n\to\infty} d(z_n, x)^2 \\
&\le d(P_{K_0}(x), x)^2.
\end{aligned}$$

and thus $\limsup_{n\to\infty} d(x_n, x) \le d(P_{K_0}(x), x)$. Therefore, $\{x_n\}$ is bounded. Thus, for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δ-convergent to some $x_0 \in X$. Then, $x_0 \in K_0$ by the assumption, and therefore, we get

$$d(P_{K_0}(x), x) \le d(x_0, x) \le \liminf_{j\to\infty} d(x_{n_{i_j}}, x) \le \limsup_{j\to\infty} d(x_{n_{i_j}}, x) \le d(P_{K_0}(x), x).$$

Hence, $x_{n_{i_j}} \to P_{K_0}(x)$. From the parallelogram law, we get

$$\begin{aligned}
d(P_{K_0}(x), x)^2 &\le \liminf_{j\to\infty} d(x_{n_{i_j}}, x)^2 \le \limsup_{j\to\infty} d(x_{n_{i_j}}, x)^2 \\
&\le \limsup_{j\to\infty} d\left(\frac{1}{2}x_{n_{i_j}} \oplus \frac{1}{2}P_{\mathrm{cl\,dom}\, f_0}(x), x\right)^2 \\
&\le \limsup_{j\to\infty} \left(\frac{1}{2}d(x_{n_{i_j}}, x)^2 + \frac{1}{2}d(P_{K_0}(x), x)^2 - \frac{1}{4}d(x_{n_{i_j}}, P_{K_0}(x))^2\right) \\
&\le \frac{1}{2}\limsup_{j\to\infty} d(x_{n_{i_j}}, x)^2 + \frac{1}{2}d((P_{K_0}(x), x)^2 - \frac{1}{4}\liminf_{j\to\infty} d(x_{n_{i_j}}, P_{K_0}(x))^2 \\
&\le \frac{1}{2}d(P_{K_0}(x), x)^2 + \frac{1}{2}d(P_{K_0}(x), x)^2 - \frac{1}{4}\lim_{j\to\infty} d(x_{n_{i_j}}, P_{K_0}(x))^2,
\end{aligned}$$

and hence $\lim_{j\to\infty} d(x_{n_{i_j}}, P_{K_0}(x))^2 = 0$. Thus, for any $\{x_{n_i}\}$, we can find a subsequence $\{x_{n_{i_j}}\}$ which is convergent to $P_{K_0}(x)$. It is equivalent to

$$x_n \to P_{K_0}(x),$$

which is the desired result. □

4.2 Resolvents on CAT(1) spaces

Let X be a complete admissible CAT(1) space, K a closed convex subset of X, and $F\colon K\times K\to\mathbb{R}$ bifunction satisfying (E1)-(E4). For $\lambda>0$, we define the resolvent $R_{\lambda F}^{-\log(\cos d)}\colon X\to K$ by

$$\begin{aligned}R_{\lambda F}^{-\log(\cos d)}(x) &= \left\{z\in K \mid \inf_{y\in K}\left(\lambda f(z,y)-\log(\cos d(y,x))+\log(\cos d(z,x))\right)\geq 0\right\}\\ &= \left\{z\in K \mid \inf_{y\in K}\left(f(z,y)+\frac{1}{\lambda}\left(-\log(\cos d(y,x))+\log(\cos d(z,x))\right)\right)\geq 0\right\}\end{aligned}$$

for all $x\in X$.

To consider the asymptotic behavior of this resolvent, we show some lemmas.

Lemma 4.9. *Let X be a complete admissible CAT(1) space having the convex hull finite property, K a nonempty closed convex subset of X, and $F\colon K\times K\to\mathbb{R}$ satisfying (E1)-(E4). For $x\in X$ and $t\in[0,1]$, if $0<\lambda\leq\mu$, then*

$$d(R_{\lambda F}^{-\log(\cos d)}(x),x)\leq d(tR_{\lambda F}^{-\log(\cos d)}(x)\oplus(1-t)R_{\mu F}^{-\log(\cos d)}(x),x)\leq d(R_{\mu F}^{-\log(\cos d)}(x),x),$$

$$F(R_{\mu F}^{-\log(\cos d)}(x),R_{\lambda F}^{-\log(\cos d)}(x))\geq 0 \quad \textit{and} \quad F(R_{\lambda F}^{-\log(\cos d)}(x),R_{\mu F}^{-\log(\cos d)}(x))\leq 0.$$

Proof. Put $x_\lambda=R_{\lambda F}^{-\log(\cos d)}(x)$ and $x_\mu=R_{\mu F}^{-\log(\cos d)}(x)$ and show that $d(x_\lambda,x)\leq d(x_\mu,x)$. From the definition of x_λ and x_μ, we get

$$0\leq F(x_\lambda,x_\mu)+\frac{1}{\lambda}\left(-\log(\cos d(x_\mu,x))+\log(\cos d(x_\lambda,x))\right); \tag{4.3}$$

$$0\leq F(x_\mu,x_\lambda)+\frac{1}{\mu}\left(-\log(\cos d(x_\lambda,x))+\log(\cos d(x_\mu,x))\right). \tag{4.4}$$

These imply that

$$\begin{aligned}0&\leq F(x_\lambda,x_\mu)+F(x_\mu,x_\lambda)+\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)\left(-\log(\cos d(x_\mu,x))+\log(\cos d(x_\lambda,x))\right)\\ &\leq 0+\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)\left(-\log(\cos d(x_\mu,x))+\log(\cos d(x_\lambda,x))\right)\end{aligned}$$

and thus we get

$$d(x_\lambda,x)\leq d(x_\mu,x).$$

Further, from (E2) and (4.5),

$$0\leq\frac{1}{\mu}\left(-\log(\cos d(x_\mu,x)))+\log(\cos d(x_\lambda,x))\right)\leq F(x_\mu,x_\lambda)\leq -F(x_\lambda,x_\mu)\leq 0.$$

Hence, we obtain $F(x_\mu,x_\lambda)\geq 0$ and $F(x_\lambda,x_\mu)\leq 0$. Next, we show that $d(x_\lambda,x)\leq d(x_\lambda\oplus(1-t)x_\mu,x)\leq d(x_\mu,x)$ for all $t\in\,]0,1[$. From (E1), (E2) we get

$$\begin{aligned}0&\leq\lambda F(x_\lambda,tx_\lambda\oplus(1-t)x_\mu)-\log(\cos d(tx_\lambda\oplus(1-t)x_\mu,x))+\log(\cos d(x_\lambda,x))\\ &\leq\lambda tF(x_\lambda,x_\lambda)+\lambda(1-t)F(x_\lambda,x_\mu)-\log(\cos d(tx_\lambda\oplus(1-t)x_\mu,x))+\log(\cos d(x_\lambda,x))\\ &\leq 0+0-\log(\cos d(tx_\lambda\oplus(1-t)x_\mu,x))+\log(\cos d(x_\lambda,x)),\end{aligned}$$

which implies $-\log(\cos d(x_\lambda, x)) \leq -\log\cos(d(tx_\lambda \oplus x_\mu, x))$. Therefore, we get

$$\begin{aligned}
-\log(\cos d(x_\lambda, x)) &\leq -\log(\cos d(tx_\lambda \oplus (1-t)x_\mu, x)) \\
&\leq -t\log(\cos d(x_\lambda, x)) + (1-t)\log(\cos d(x_\mu, x)) \\
&\leq -t\log(\cos d(x_\mu, x)) - (1-t)\log(\cos d(x_\mu, x)) \\
&= -\log(\cos d(x_\mu, x)).
\end{aligned}$$

Consequently, we get

$$d(x_\lambda, x) \leq d(tx_\lambda \oplus (1-t)x_\mu, x) \leq d(x_\mu, x).$$

It completes the proof. □

Lemma 4.10. *Let X be a complete CAT(0) space having the convex hull finite property, K a nonempty closed convex subset of X, and $F\colon K \times K \to \mathbb{R}$ satisfies (E1)-(E4). For $x \in X$, if $0 < \lambda \leq \mu$, then*

$$\begin{aligned}
&-2\log\left(\cos\frac{d(R_{\lambda F}^{-\log(\cos d)}(x), R_{\mu F}^{-\log(\cos d)}(x))}{2}\right) \\
&\leq -\log(\cos d(R_{\lambda F}^{-\log(\cos d)}(x), x)) + \log(\cos d(R_{\mu F}^{-\log(\cos d)}(x), x)),
\end{aligned}$$

Proof. Put $x_\lambda = R_{\lambda F}^{-\log(\cos d)}(x)$ and $x_\mu = R_{\mu F}^{-\log(\cos d)}(x)$. From (E1), (E2) and the parallelogram law, we have

$$\begin{aligned}
0 &\leq \lambda F\left(x_\lambda, \frac{1}{2}x_\lambda \oplus \frac{1}{2}x_\mu\right) - \log\left(\cos d\left(\frac{1}{2}x_\lambda \oplus \frac{1}{2}x_\mu, x\right)\right) + \log(\cos d(x_\lambda, x)) \\
&\leq \frac{\lambda}{2}F(x_\lambda, x_\lambda) + \frac{\lambda}{2}F(x_\lambda, x_\mu) - \log\left(\cos d\left(\frac{1}{2}x_\lambda \oplus \frac{1}{2}x_\mu, x\right)\right) + \log(\cos d(x_\lambda, x)) \\
&\leq 0 + 0 - \log\left(\cos d\left(\frac{1}{2}x_\lambda \oplus \frac{1}{2}x_\mu, x\right)\right) + \log(\cos d(x_\lambda, x)) \\
&\leq -\frac{1}{2}\log(\cos d(x_\lambda, x)) - \frac{1}{2}\log(\cos d(x_\mu, x)) + \log\left(\cos\frac{d(x_\lambda, x_\mu)}{2}\right) + \log(\cos d(x_\lambda, x)) \\
&= -\frac{1}{2}\log(\cos d(x_\mu, x)) + \frac{1}{2}\log(\cos d(x_\lambda, x))) + \log\left(\cos\frac{d(x_\lambda, x_\mu)}{2}\right).
\end{aligned}$$

Therefore, we get

$$-2\log\left(\cos\frac{d(x_\lambda, x_\mu)}{2}\right) \leq -\log(\cos d(x_\mu, x)) + \log(\cos d(x_\lambda, x)).$$

This is the desired result. □

Lemma 4.11. *Let $\tau\colon \mathbb{R} \to \mathbb{R}$ be an increasing function and $\{\lambda_n\}$, $\{\mu_n\}$ real sequences that diverge to ∞. If $\sup_{n\in\mathbb{N}} \tau(\mu_n) < \frac{\pi}{2}$, then $\sup_{n\in\mathbb{N}} \tau(\lambda_n) < \frac{\pi}{2}$.*

We show the convergence of the resolvent on CAT(1) space as $\lambda \to \infty$.

Theorem 4.12. *Let X be a complete admissible CAT(1) space having the convex hull finite property, K a nonempty closed convex subset of X, and $F\colon K \times K \to \mathbb{R}$ satisfies (E1)-(E4). For $x \in X$, if there exists a positive real sequence $\{\mu_n\}$ diverging to ∞ and $\sup_{n\in\mathbb{N}} d(R_{\mu_n f}^{-\log(\cos d)}(x), x) < \frac{\pi}{2}$, then* $\operatorname{Equil} f \neq \emptyset$ *and*

$$\lim_{\lambda\to\infty} R_{\lambda F}^{-\log(\cos d)}(x) = P_{\operatorname{Equil} F}(x).$$

Proof. We show that $R_{\lambda_n F}^{-\log(\cos d)}(x) \to P_{\operatorname{Equil} F}(x)$ for increasing sequence $\{\lambda_n\}$ diverging to ∞. Put $x_n = R_{\lambda_n F}^{-\log(\cos d)}(x)$ for $n \in \mathbb{N}$. By Lemma 4.11, $\sup_{n\in\mathbb{N}} d(x_n, x) < \frac{\pi}{2}$ and $\{d(x_n, x)\}$ is increasing under the assumptions of the theorem. Hence, $\{d(x_n, x)\}$ is convergent. By Lemma 4.10, we get

$$-2\log\left(\cos\frac{d(x_n, x_m)}{2}\right) \le -\log(\cos d(x_m, x)) + \log(\cos d(x_n, x)).$$

for $n, m \in \mathbb{N}$ with $n \le m$. Then, $\{x_n\}$ is a Cauchy sequence on K and thus $x_n \to x_0 \in K$. By the definition of x_n, we have

$$0 \le F(x_n, y) + \frac{1}{\lambda_n}(-\log(\cos d(y, x)) + \log(\cos d(x_n, x)))$$

for all $y \in K$. Letting $n \to \infty$, we get $F(y, x_0) \le 0$ for all $y \in K$. For $t \in]0, 1[$, we have

$$F(tw \oplus (1-t)x_0, x_0) \le 0$$

for all $w \in K$. From (E1) and (E3), we have

$$\begin{aligned} 0 &= F(tw \oplus (1-t)x_0, tw \oplus (1-t)x_0) \\ &\le tF(tw \oplus (1-t)x_0, w) + (1-t)F(tw \oplus (1-t)x_0, x_0) \\ &\le tF(tw \oplus (1-t)x_0, w). \end{aligned}$$

From (E4), dividing by t and letting $t \to 0$, we obtain

$$x_0 \in \operatorname{Equil} F \neq \emptyset$$

by (E4). From (E2), letting $n \to \infty$, we obtain

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} (\lambda_n F(x_n, z) - \log(\cos d(z, x)) + \log(\cos d(x_n, x))) \\ &\le \limsup_{n\to\infty} (-\lambda_n F(z, x_n) - \log(\cos d(z, x)) + \log(\cos d(x_n, x))) \\ &\le \limsup_{n\to\infty}(-\lambda_n F(z, x_n)) - \log(\cos d(z, x)) + \limsup_{n\to\infty} \log(\cos d(x_n, x)) \\ &\le 0 - \log(\cos d(z, x)) - \liminf_{n\to\infty}(-\log(\cos d(x_n, x))) \\ &\le \limsup_{n\to\infty}(-\log(\cos d(z, x))) - \liminf_{n\to\infty}(-\log(\cos d(x_n, x))) \\ &\le -\log(\cos d(z, x)) + \log(\cos d(x_0, x)) \end{aligned}$$

for all $z \in \operatorname{Equil} F$. Therefore, $x_0 = P_{\operatorname{Equil} F}(x)$ and hence $x_n \to P_{\operatorname{Equil} F}(x)$, which concludes the proof. □

We also show the convergence of the resolvent on CAT(1) space as $\lambda \to \lambda_0 \in [0, \infty[$.

Theorem 4.13. *Let X be a complete admissible $CAT(1)$ space having the convex hull finite property, K a nonempty closed convex subset of X, and $F\colon K \times K \to \mathbb{R}$ satisfying (E1)-(E4). Then for $x \in X$.*

$$\lim_{\lambda \searrow 0} R_{\lambda F}^{-\log(\cos d)}(x) = P_K(x)$$

and

$$\lim_{\lambda \to \lambda_0} R_{\lambda F}^{-\log(\cos d)}(x) = R_{\lambda_0 f}^{-\log(\cos d)}(x),$$

where λ_0 is a positive real number.

Proof. Consider the resolvent $x_n = R_{\lambda_n F}^{-\log(\cos d)}(x)$ for a positive real sequence $\{\lambda_n\}$. Suppose $\{\lambda_n\}$ is increasing or decreasing, and $\lambda_n \to \lambda_0$. Then, $\{d(x_n, x)\}$ is increasing or decreasing, and bounded. Therefore, $\{d(x_n, x)\}$ is convergent. By Lemma 4.10, it hold that

$$-2\log\left(\cos\frac{d(x_n, x_m)}{2}\right) \le |-\log(\cos d(x_m, x)) + \log(\cos d(x_n, x))|$$

for all n, m with $n \le m$. Thus, $\{x_n\}$ is a Cauchy sequence on K and it is convergent to some point $x_0 \in K$. From the definition of the resolvent and (E2), we get

$$\begin{aligned} 0 &\le \lambda_n F(x_n, y) - \log(\cos d(y, x)) + \log(\cos d(x_n, x)) \\ &\le -\lambda_n F(y, x_n) - \log(\cos d(y, x)) + \log(\cos d(x_n, x)) \end{aligned}$$

for all $y \in K$. Letting $n \to \infty$, we have

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} (-\lambda_n F(y, x_n) - \log(\cos d(y, x)) + \log(\cos d(x_n, x))) \\ &\le -\liminf_{n\to\infty} \lambda_n F(y, x_n) - \log(\cos d(y, x)) + \log(\cos d(x_n, x)) \\ &\le -\lambda_0 F(y, x_0) - \log(\cos d(y, x)) + \log(\cos d(x_0, x)) \end{aligned}$$

by (E3). Thus, we get $\lambda_0 F(y, x_n) \le -\log(\cos d(x, y)) + \log(\cos d(x_n, x))$ for all $y \in K$. From (E1), (E2) and (E3), we get

$$\begin{aligned} 0 &= \lambda_0 F(tw \oplus (1-t)x_0, tw \oplus (1-t)x_0) \\ &\le \lambda_0 t F(tw \oplus (1-t)x_0, w) + (1-t)(\lambda_0 F(tw \oplus (1-t)x_0, x_0)) \\ &\le \lambda_0 t F(tw \oplus (1-t)x_0, w) + (1-t)\left(-\log(\cos d(tw \oplus (1-t)x_0, x)) + \log(\cos d(x_0, x))\right) \\ &\le \lambda_0 t F(tw \oplus (1-t)x_0, w) \\ &\quad + (1-t)\left(-t\log(\cos d(w, x)) - (1-t)\log(\cos d(x_0, x)) + \log(\cos d(x_0, x))\right) \\ &\le \lambda_0 t F(tw \oplus (1-t)x_0, w) + t(1-t)\left(-\log(\cos d(w, x)) + \log(\cos d(x_0, x))\right) \end{aligned}$$

for all $w \in K$ and $t \in]0, 1[$. Dividing by t and letting $t \to 0$, we obtain

$$0 \le \lambda_0 F(x_0, w) - \log(\cos d(w, x)) + \log(\cos d(x_0, x))$$

for all $w \in K$. In particular, if $\lambda_0 = 0$, then we have $d(x_0, x) \le d(w, x)$ for all $w \in K$. Hence, it hold that

$$x_0 = \begin{cases} P_K(x) & (\text{if} \quad \lambda_0 = 0); \\ Q_{\lambda_0 f}^{\varphi}(x) & (\text{if} \quad \lambda_0 > 0). \end{cases}$$

Therefore, since $R_{\lambda F}^{-\log(\cos d)}(x)$ converges such a point for convergent sequence $\{\lambda_n\}$, we conclude that

$$R_{\lambda F}^{-\log(\cos d)}(x) \xrightarrow[\lambda \searrow 0]{} P_K(x);$$

$$R_{\lambda F}^{-\log(\cos d)}(x) \xrightarrow[\lambda \to \lambda_0]{} Q_{\lambda_0 f}^{\varphi}(x),$$

which is the desired result. □

Let $\{K_n\}$ be a nonempty closed convex subsets of X and $\{F_n\}$ a sequence of bifunctions $F_n\colon K_n \times K_n \to \mathbb{R}$ satisfying (E1)-(E4) for each $n \in \mathbb{N}$. For $\{\lambda_n\}$ a sequence of positive real numbers, we define $R_{\lambda_n F_n}^{d^2}$ by

$$\begin{aligned} R_{\lambda_n F_n}^{-\log(\cos d)}(x) &= \left\{ z \in K_n \mid \inf_{y \in K_n} (\lambda_n F_n(z,y) - \log(\cos d(y,x)) + \log(\cos d(z,x))) \geq 0 \right\} \\ &= \left\{ z \in K_n \mid \inf_{y \in K_n} \left(F_n(z,y) + \frac{1}{\lambda_n} (-\log\cos(d(y,x)) + \log(\cos d(z,x))) \right) \geq 0 \right\} \end{aligned}$$

for $x \in X$. We next consider the convergence of this sequence of resolvents.

Theorem 4.14. *Let X be a complete admissible $CAT(1)$ space having the convex hull finite property, K_0 a nonempty closed convex subset of X, and $F_0\colon K_0 \times K_0 \to \mathbb{R}$ a bifunction satisfying (E1)-(E4) and $x \in X$. Suppose that $\{F_n\}$ satisfies the following conditions;*

(a) $\emptyset \neq \operatorname{Equil} F_0 \subset$ d-Li $\operatorname{Equil} F_n$*;*
(b) $\overline{\Delta}$-Ls $\left\{ R_{\lambda_n F_n}^{-\log(\cos d)}(x) \right\} \subset \operatorname{Equil} F_0$.

Then

$$R_{\lambda_n F_n}^{-\log(\cos d)}(x) \to P_{\operatorname{Equil} F_0}(x).$$

Proof. Put $x_n = R_{\lambda_n F_n}^{-\log(\cos d)}(x)$ for all $n \in \mathbb{N}$. There exists a sequence $\{a_n\}$ such that $a_n \in \operatorname{Equil} F_n$ for each $n \in \mathbb{N}$ and $a_n \to P_{\operatorname{Equil} f_0}(x)$. From the definition of the resolvent and the parallelogram law, we have

$$\begin{aligned} 0 &\leq F_n\left(x_n, \frac{1}{2}x_n \oplus \frac{1}{2}a_n\right) + \frac{1}{\lambda_n}\left(-\log\left(\cos d\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n, x\right)\right) + \log(\cos d(x_n, x))\right) \\ &\leq \frac{1}{2}F_n(x_n, x_n) + \frac{1}{2}F_n(x_n, a_n) + \frac{1}{\lambda_n}\left(-\log\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n, x\right)\right) + \log(\cos d(x_n, x))\right) \\ &\leq \frac{1}{2}F_n(x_n, x_n) - \frac{1}{2}F_n(a_n, x_n) + \frac{1}{\lambda_n}\left(-\log\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n, x\right)\right) + \log(\cos d(x_n, x))\right) \\ &\leq 0 + 0 + \frac{1}{\lambda_n}\left(-\log\left(d\left(\frac{1}{2}x_n \oplus \frac{1}{2}a_n, x\right)\right) + \log(\cos d(x_n, x))\right) \\ &\leq \frac{1}{\lambda_n}\left(-\frac{1}{2}\log(\cos d(x_n, x)) - \frac{1}{2}\log(\cos d(a_n, x)) - \log\left(\cos\frac{d(x_n, a_n)}{2}\right) + \log(\cos d(x_n, x))\right) \\ &= \frac{1}{\lambda_n}\left(-\frac{1}{2}\log(\cos d(x_n, x)) + \frac{1}{2}\log(\cos d(a_n, x))) + \log\left(\cos\frac{d(x_n, a_n)}{2}\right)\right). \end{aligned}$$

Therefore, we get

$$-2\log\left(\cos\frac{d(x_n, a_n)}{2}\right) \leq -\log(\cos d(a_n, x)) + \log(\cos d(x_n, x))).$$

Thus $d(x_n, x) \le d(a_n, x)$ and so $\sup_{n\in\mathbb{N}} d(x_n, x) < \frac{\pi}{2}$. Hence, for a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δ-convergent to some point $x_0 \in X$. From (b), it holds that $x_0 \in \operatorname{Equil} F_0$ and therefore

$$\begin{aligned} d(P_{\operatorname{Equil} f_0}(x), x) &\le d(x_0, x) \\ &\le \liminf_{j\to\infty} d(x_{n_{i_j}}, x) \\ &\le \liminf_{j\to\infty} d(a_{n_{i_j}}, x) \\ &= d(P_{\operatorname{Equil} F_0}(x), x). \end{aligned}$$

This implies $x_0 = P_{\operatorname{Equil} f_0}(x)$, and we obtain

$$\begin{aligned} 0 &\le \limsup_{j\to\infty} \left(-2\log\left(\cos \frac{d(x_{n_{i_j}}, P_{\operatorname{Equil} F_0}(x))}{2} \right) \right) \\ &\le \limsup_{j\to\infty} \left(-2\log\left(\cos \frac{d(x_{n_{i_j}}, a_{n_{i_j}})}{2} \right) \right) \\ &\le \limsup_{j\to\infty} \left(-\log(\cos d(a_{n_{i_j}}, x)) + \log(\cos d(x_{n_{i_j}}, x)) \right) \\ &\le \limsup_{j\to\infty} (-\log(\cos d(a_{n_{i_j}}, x))) - \liminf_{j\to\infty} (-\log\cos(d(x_{n_{i_j}}, x))) \\ &\le \limsup_{j\to\infty} (-\log(\cos d(P_{\operatorname{Equil} f_0}(x), x))) - (-\log\cos(d(P_{\operatorname{Equil} F_0}(x), x))) = 0. \end{aligned}$$

Hence, we get $x_{n_{i_j}} \to P_{\operatorname{Equil} F_0}(x)$. Therefore, for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$ has a subsequence which is convergent to $P_{\operatorname{Equil} F_0}(x)$ and so we get $x_n \to P_{\operatorname{Equil} F_0}(x)$, which is the desired result. ☐

Theorem 4.15. *Let X be a complete admissible $CAT(1)$ space having the convex hull finite property, K_0 a nonempty closed convex subset of X, and $F_0 \colon K_0 \times K_0 \to \mathbb{R}$ a bifunction satisfies (E1)-(E4) and $x \in X$. Suppose that $\{\lambda_n\}$ diverges to ∞ and $\{F_n\}$ satisfying the following conditions;*

(a) $\emptyset \ne \operatorname{Equil} F_0 \subset \operatorname{d-Li} \operatorname{Equil} F_n$*;*
(b) For all $b \in K_0$, $\{F_{n_i}\}$ of $\{F_n\}$, and $\{c_i\}$ such that $c_i \in K_{n_i}$ and $\{c_i\}$ is Δ-convergent to $c \in X$, there exists $\{b_i\}$ which satisfies $b_i \in K_{n_i}$, $b_i \to b$, and $F_0(b, c) \le \liminf_{i\to\infty} F_{n_i}(b_i, c_i)$.

Then

$$R^{-\log(\cos d)}_{\lambda_n F_n}(x) \to P_{\operatorname{Equil} F_0}(x).$$

Proof. Put $x_n = R^{d^2}_{\lambda_n F_n}(x)$. Let $b \in K_0$ and $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ which is Δ-convergent to $x_0 \in X$. Then, there exists $\{b_i\}$ such that $b_i \in K_{n_i}$ and $F_0(b_0, x_0) \le \liminf_{i\to\infty} F_{n_i}(b_i, x_{n_i})$. From the definition of the resolvent, we get

$$\begin{aligned} 0 &\le F_{n_i}(x_{n_i}, b_i) + \frac{1}{\lambda_{n_i}} \left(-\log(\cos d(b_i, x)) + \log(\cos d(x_{n_i}, x)) \right) \\ &\le -F_{n_i}(b_i, x_{n_i}) + \frac{1}{\lambda_{n_i}} \left(-\log(\cos d(b_i, x)) + \log(\cos d(x_{n_i}, x)) \right), \end{aligned}$$

and letting $i \to \infty$, we have

$$0 \le \limsup_{i\to\infty} F_{n_i}(x_{n_i}, b_i) \le \limsup_{i\to\infty} (-F_{n_i}(b_i, x_{n_i})) = -\liminf_{i\to\infty} F_{n_i}(b_i, x_{n_i}) \le -F_0(b_0, x_0).$$

Thus, we get $F(b_0, x_0) \leq 0$. The upper hemicontinuity of $F_0(\cdot, b_0)$ implies $x_0 \in \operatorname{Equil} F_0$. From the previous theorem, we get the desired result. $\square$

Theorem 4.16. *Let X be a complete admissible $CAT(1)$ space having the convex hull finite property, K_0 a nonempty closed convex subset of X, $F_0\colon K_0 \times K_0 \to \mathbb{R}$ a bifunction satisfying (E1)-(E4) and $x \in X$. Suppose that $\{K_n\}$ and $\{F_n\}$ satisfy that $\overline{\Delta}\text{-}\operatorname{Ls} K_n \subset K_0$, and there exists $\{z_n\}$ such that $z_n \in K_n$, $z_n \to P_{K_0}(x)$ and*

$$\limsup_{n\to\infty} F_n(R^{-\log(\cos d)}_{\lambda_n f_n}(x), z_n) < \infty.$$

If $\{\lambda_n\}$ converges to 0, then

$$R^{-\log(\cos d)}_{\lambda_n F_n}(x) \to P_{K_0}(x).$$

Proof. Put $x_n = R^{-\log(\cos d)}_{\lambda_n F_n}(x)$. Assume $\{z_n\}$ satisfies assumptions of the theorem. From the definition of the resolvent,

$$-\log(\cos d(x_n, x)) \leq \lambda_n F_n(x_n, z_n) - \log(\cos d(z_n, x)).$$

Letting $n \to \infty$, we have

$$\begin{aligned}
\limsup_{n\to\infty}(-\log(\cos d(x_n, x))) &\leq \limsup_{n\to\infty}\left(\lambda_n F_n(x_n, z_n) - \log(\cos d(z_n, x))\right) \\
&\leq \limsup_{n\to\infty} \lambda_n F_n(x_n, z_n) + \limsup_{n\to\infty}(-\log(\cos d(z_n, x))) \\
&\leq -\log(\cos d(P_{K_0}(x), x)).
\end{aligned}$$

and hence $\limsup_{n\to\infty} d(x_n, x) \leq d(P_{K_0}(x), x)$. Therefore, for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δ-convergent to some $x_0 \in X$ and then

$$d(P_{K_0}(x), x) \leq d(x_0, x) \leq \liminf_{j\to\infty} d(x_{n_{i_j}}, x) \leq \limsup_{j\to\infty} d(x_{n_{i_j}}, x) \leq d(P_{K_0}(x), x).$$

Hence, we obtain $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} P_{K_0}(x)$. The parallelogram law implies

$$\begin{aligned}
&-\log(\cos d(P_{K_0}(x), x)) \leq \liminf_{j\to\infty}(-\log(\cos d(x_{n_{i_j}}, x))) \leq \limsup_{j\to\infty}(-\log(\cos d(x_{n_{i_j}}, x))) \\
&\leq \limsup_{j\to\infty} -\log\left(\cos d\left(\frac{1}{2}x_{n_{i_j}} \oplus \frac{1}{2}P_{K_0}(x), x\right)\right) \\
&\leq \limsup_{j\to\infty}\left(-\frac{1}{2}\log(\cos d(x_{n_{i_j}}, x)) + \log\left(\cos\frac{d(x_{n_{i_j}}, P_{K_0}(x))}{2}\right)\right) - \frac{1}{2}\log(\cos d(P_{K_0}(x), x)) \\
&\leq -\frac{1}{2}\log(\cos d((P_{K_0}(x), x)) + \lim_{n\to\infty}\log\left(\cos\frac{d(x_{n_{i_j}}, P_{K_0}(x))}{2}\right) \quad - \frac{1}{2}\log(\cos d((P_{K_0}(x), x)),
\end{aligned}$$

and so $x_{n_{i_j}} \to P_{K_0}(x)$. Consequently, we can prove $x_n \to P_{K_0}(x)$, which is the desired result. $\square$

Chapter 5

Asymptotic behavior of resolvent for monotone operators

In this section, we consider the resolvent with a positive parameter and its asymptotic behavior. For $\lambda > 0$, the resolvent $S_{\lambda A}(x)$ is defined by

$$S_{\lambda A}(x) = \left\{ z \in X \mid \left[\frac{1}{\lambda}\overrightarrow{zx}\right] \in Az \right\}.$$

Lemma 5.1. *Let X be a complete CAT(0) space, X^* its dual, and $A\colon X \rightrightarrows X^*$ a maximal monotone operator satisfying $D(S_{\lambda A}) = X$ for all $\lambda > 0$. For $x \in X$, if $0 \le \lambda \le \mu$, then*

$$d(S_{\lambda A}(x), S_{\mu A}(x))^2 \le d(x, S_{\mu A}(x))^2 - d(x, S_{\lambda A}(x))^2.$$

In particular,

$$d(x, S_{\lambda A}(x)) \le d(x, S_{\mu A}(x)).$$

Proof. From the definition of the resolvent, we have

$$\left[\frac{1}{\lambda}\overrightarrow{S_{\lambda A}(x)x}\right] \in A(S_{\lambda A}(x)) \quad \text{and} \quad \left[\frac{1}{\mu}\overrightarrow{S_{\mu A}(x)x}\right] \in A(S_{\mu A}(x)).$$

The monotonicity of A implies

$$\begin{aligned}
0 &\le \left\langle \left[\frac{1}{\lambda}\overrightarrow{S_{\lambda A}(x)x}\right] - \left[\frac{1}{\mu}\overrightarrow{S_{\mu A}(x)x}\right], \overrightarrow{S_{\mu A}(x)S_{\lambda A}(x).} \right\rangle \\
&\le \frac{1}{\lambda}\left\langle \overrightarrow{S_{\lambda A}(x)x}, \overrightarrow{S_{\mu A}(x)S_{\lambda A}(x)} \right\rangle - \frac{1}{\mu}\left\langle \overrightarrow{S_{\mu A}(x)x}, \overrightarrow{S_{\mu A}(x)S_{\lambda A}(x)} \right\rangle \\
&= \frac{1}{2\lambda}\left\{ d(x, S_{\mu A}(x))^2 - d(S_{\lambda A}(x), S_{\mu A}(x))^2 - d(x, S_{\lambda A}(x))^2 \right\} \\
&\quad - \frac{1}{2\mu}\left\{ d(S_{\lambda A}(x), S_{\mu A}(x))^2 - d(x, S_{\mu A}(x))^2 - d(x, S_{\lambda A}(x))^2 \right\} \\
&= \left(\frac{1}{2\lambda} - \frac{1}{2\mu}\right)\left\{ d(x, S_{\mu A}(x))^2 - d(x, S_{\lambda A}(x))^2 \right\} - \left(\frac{1}{2\lambda} + \frac{1}{2\mu}\right) d(S_{\lambda A}(x), S_{\mu A}(x))^2 \\
&\le \left(\frac{1}{2\lambda} - \frac{1}{2\mu}\right)\left\{ d(x, S_{\mu A}(x))^2 - d(x, S_{\lambda A}(x))^2 - d(S_{\lambda A}(x), S_{\mu A}(x))^2 \right\},
\end{aligned}$$

Hence, we get

$$0 \le d(S_{\lambda A}(x), S_{\mu A}(x))^2 \le d(x, S_{\mu A}(x))^2 - d(x, S_{\lambda A}(x))^2,$$

which implies $d(x, S_{\lambda A}(x)) \le d(x, S_{\mu A}(x))$. It complete the proof.

□

Theorem 5.2. *Let X be a complete $CAT(0)$ space, X^* its dual, and $A\colon X \rightrightarrows X^*$ a maximal monotone operator satisfying $D(S_{\lambda A}) = X$ for all $\lambda > 0$. For $x \in X$, if there exists a positive real sequence $\{\mu_n\}$ divergs to ∞ and $\{d(S_{\mu_n A}(x), x)\}$ is bounded, then* $\operatorname{Equil} f \neq \emptyset$ *and*

$$\lim_{\lambda \to \infty} S_{\lambda_n A}(x) = P_{\operatorname{Equil} F}(x).$$

Proof. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences which is increasing and divergent to ∞. If $\{d(S_{\mu_n A}(x), x)\}$ is bounded, then $\{d(S_{\mu_n A}(x), x)\}$ is also bounded by Lemma 4.3. We put $x_n = S_{\lambda_n A}(x)$ for all $n \in \mathbb{N}$. By Lemma 4.2, $\{d(x_n, x)\}$ is increasing and we have

$$d(x_n, x_m) \le d(x_n, x)^2 - d(x_m, x)^2.$$

Thus, $\{d(x_n, x)\}$ is convergent and hence $\{x_n\}$ is Cauchy sequence on X. From the completeness of X, we can find $x_0 \in X$ such that $x_n \to x_0$. Finally, we show that $x_0 = P_{A^{-1}(\mathbf{0})}(x)$. From the monotonicity of A, we have

$$\begin{aligned} 0 &\le \left\langle \left[\frac{1}{\lambda_n}\overrightarrow{x_n x}\right] - u^*, \overrightarrow{u x_n} \right\rangle \\ &= \frac{1}{\lambda_n} \langle \overrightarrow{x_n x}, \overrightarrow{u x_n} \rangle - \langle u^*, \overrightarrow{u x_n} \rangle \\ &\le \frac{1}{\lambda_n} d(x_n, x) d(u, x_n) - \langle u^*, \overrightarrow{u x_n} \rangle . \end{aligned}$$

for all $(u, u^*) \in A$. Letting $n \to \infty$, we get

$$0 \le -\langle u^*, \overrightarrow{u x_0} \rangle = \langle u^*, \overrightarrow{x_0 u} \rangle = \langle u^* - \mathbf{0}, \overrightarrow{x_0 u} \rangle .$$

Maximality of A implies that $x_0 \in A^{-1}(\mathbf{0})$. Further, we get

$$\begin{aligned} 0 &\le \left\langle \left[\frac{1}{\lambda_n}\overrightarrow{x_n x}\right] - \mathbf{0}, \overrightarrow{z x_n} \right\rangle \\ &= \frac{1}{\lambda_n} \langle \overrightarrow{x_n x}, \overrightarrow{z x_n} \rangle \\ &= \frac{1}{2\lambda_n} \left\{ d(x, z)^2 - d(x_n, z)^2 - d(x, x_n)^2 \right\} . \end{aligned}$$

for all $z \in A^{-1}(\mathbf{0})$. Therefore, we have $d(x_n, z) \le d(x, z)$. Hence, we obtain

$$d(x, x_0) \le \liminf_{n \to \infty} d(x, x_n) \le d(x, z)$$

for all $z \in A^{-1}(\mathbf{0})$, and this implies $x_0 = P_{A^{-1}(\mathbf{0})}(x)$ Consequently, we get

$$S_{\lambda_n A}(x) \to P_{A^{-1}(\mathbf{0})}(x)$$

for all increasing sequence $\{\lambda_n\}$. and we conclude that

$$\lim_{\lambda\to\infty} S_{\lambda A}(x) = P_{A^{-1}(\mathbf{0})}(x),$$

which is desired result.

$\square$

Theorem 5.3. *Let X be a complete $CAT(0)$ space, X^* its dual, and $A\colon X \rightrightarrows X^*$ a maximal monotone operator satisfying $D(S_{\lambda A}) = X$ for all $\lambda > 0$. Then for $x \in X$,*

$$\lim_{\lambda\searrow 0} S_{\lambda A}(x) = P_{\mathrm{cl}\, D(A)}(x)$$

and

$$\lim_{\lambda\to\lambda_0} S_{\lambda A}(x) = S_{\lambda_0 A}(x),$$

where λ_0 is a positive real number.

Proof. Let $\{\lambda_n\}$ be a sequence of positive real numbers and suppose that it is increasing or decreasing, and convergent to $\lambda_0 \in [0,\infty[$. Put $x_n = S_{\lambda_n}(x)$ for $n \in \mathbb{N}$. If $\{\lambda_n\}$ is increasing, then $\{d(x_n, x)\})$ is increasing and

$$0 \le d(x_n, x) \le d(S_{\lambda_0 A}(x), x).$$

On the other hand, if $\{\lambda_n\}$ is decreasing, then $\{d(x_n, x)\})$ is decreasing and

$$0 \le d(x_n, x) \le d(x_1, x).$$

Thus, $\{d(x_n, x)\}$ is convergent. By Lemma 5.1, we get

$$d(x_n, x_m) \le |d(x_m, x)^2 - d(x_n, x)^2|$$

for all $n, m \in \mathbb{N}$, and hence $\{x_n\}$ is a Cauchy sequence on X. By the completeness of X, $\{x_n\}$ is convergent to some $x_0 \in X$. From the monotonicity of A, it hold that

$$\begin{aligned} 0 &\le \langle [\overrightarrow{x_n x}] - \lambda_n u^*, \overrightarrow{u x_n} \rangle \\ &= \langle \overrightarrow{x_n x}, \overrightarrow{u x_n} \rangle - \lambda_n \langle u^*, \overrightarrow{u x_n} \rangle \end{aligned}$$

for $(u, u^*) \in A$. Letting $n \to \infty$, we have

$$0 \le \langle \overrightarrow{x_0 x}, \overrightarrow{u x_0} \rangle - \lambda_0 \langle u^*, \overrightarrow{u x_0} \rangle .$$

If $\lambda_0 > 0$, dividing by λ_0 we get

$$0 \le \left\langle \left[\frac{1}{\lambda_0} \overrightarrow{x_0 x} \right] - u^*, \overrightarrow{u x_0} \right\rangle$$

for all $(u, u^*) \in A$. Maximality of A implies $\left[\frac{1}{\lambda_0}\overrightarrow{x_0 x}\right] \in A$ and thus $x_0 = S_{\lambda A}(x)$. Further, if $\lambda_0 = 0$, we get

$$\begin{aligned} 0 \le \langle \overrightarrow{x_0 x}, \overrightarrow{u x_0} \rangle &= \frac{1}{2}\left(d(x,u)^2 - d(x_0,u)^2 - d(x,x_0)^2 \right) \\ &\le \frac{1}{2}\left(d(u,x)^2 - d(x_0,x)^2 \right). \end{aligned}$$

Hence, $x_0 \in \operatorname{cl} D(A)$ satisfies

$$d(x_0, x) \le \inf_{u \in D(A)} d(u, x) \le \inf_{u \in \operatorname{cl} D(A)} d(u, x).$$

Therefore, if $x \in D(A)$, $x_n = S_{\lambda_n}(x) \to x$. Here, we verify that $\operatorname{cl} D(A)$ is convex. Let $y, z \in \operatorname{cl} D(A)$ and take sequences $\{y_m\}$, $\{z_m\}$ of $D(A)$ such that $y_m \to y$, $z_m \to z$ as $m \to \infty$. From the parallelogram law and the nonexpansiveness of $S_{\lambda_n A}$, we have

$$\begin{aligned}
& d\left(tS_{\lambda_n A}(y_m) \oplus (1-t)S_{\lambda_n A}(z_m), S_{\lambda_n A}\left(ty \oplus (1-t)z\right)\right)^2 \\
& \le td\left(S_{\lambda_n A}(y_m), S_{\lambda_n A}\left(ty \oplus (1-t)z\right)\right)^2 + (1-t)d\left(S_{\lambda_n A}(z_m), S_{\lambda_n A}\left(ty \oplus (1-t)z\right)\right)^2 \\
& \quad - t(1-t)d\left(S_{\lambda_n A}(y_m), S_{\lambda_n A}(z_m)\right)^2 \\
& \le td\left(y_m, ty_m \oplus (1-t)z_m\right)^2 + (1-t)d\left(z_m, ty_m \oplus (1-t)z_m\right)^2 - t(1-t)d\left(S_{\lambda_n A}(y_m), S_{\lambda_n A}(z_m)\right)^2 \\
& \le t(1-t)^2 d(y_m, z_m)^2 + (1-t)t^2 d(y_m, z_m)^2 - t(1-t)d\left(S_{\lambda_n A}(y_m), S_{\lambda_n A}(z_m)\right)^2 \\
& \le t(1-t)d(y_m, z_m)^2 - t(1-t)d\left(S_{\lambda_n A}(y_m), S_{\lambda_n A}(z_m)\right)^2.
\end{aligned}$$

Since $S_{\lambda_n A}(y_m) \to y_m$ and $S_{\lambda_n A}(z_m) \to z_m$ as $n \to \infty$, we get

$$d\left(tS_{\lambda_n A}(y_m) \oplus (1-t)S_{\lambda_n A}(z_m), S_{\lambda_n A}\left(ty_m \oplus (1-t)z_m\right)\right) \to 0.$$

Therefore, we obtain

$$\begin{aligned}
ty_m \oplus (1-t)z_m &= \lim_{n\to\infty} \{tS_{\lambda_n A}(y_m) \oplus (1-t)S_{\lambda_n A}(z_m)\} \\
&= \lim_{n\to\infty} \{S_{\lambda_n A}\left(ty_m \oplus (1-t)z_m\right)\} \in \operatorname{cl} D(A)
\end{aligned}$$

for all $m \in \mathbb{N}$. Letting $m \to \infty$, we get $ty \oplus (1-t)z \in \operatorname{cl} D(A)$. Consequently, $\operatorname{cl} D(A)$ is a nonempty closed convex subset of X. Then, we can define the metric projection $P_{\operatorname{cl} D(A)}$ onto $\operatorname{cl} D(A)$, and then x_0 corresponds to $P_{\operatorname{cl} D(A)}(x)$. It complete the proof. □

For a sequence $\{A_n\}$ of maximal monotone operators satisfying $D(S_{\lambda A_n}) = X$ for $\lambda > 0$, we consider the sequence of resolvents which is defined by

$$S_{A_n}(x) = \{z \in X \mid [\overrightarrow{zx}] \in A_n z\}$$

for $x \in X$.

Theorem 5.4. *Let X be a complete CAT(0) space, X^* its dual, and $A \colon X \rightrightarrows X^*$ a maximal monotone operator satisfying $D(S_{\lambda A}) = X$ for all $\lambda > 0$. For $x \in X$, Suppose $\{A_n\}$ satisfies the following conditions;*

(a) $\emptyset \ne A_0^{-1}(\mathbf{0}) \subset \text{d-Li}\, A_n^{-1}(\mathbf{0})$*;*
(b) $\overline{\Delta}\text{-Ls}\, \{S_{\lambda_n A}(x)\} \subset A_0^{-1}(\mathbf{0})$.

Then

$$S_{\lambda_n A_n}(x) \to P_{A_0^{-1}(\mathbf{0})}(x).$$

Proof. Put $x_n = S_{\lambda_n A}(x)$ for all $n \in \mathbb{N}$. There exists $\{a_n\}$ such that $a_n \in A_n^{-1}(\mathbf{0})$ for each n and $a_n \to P_{A_0^{-1}(\mathbf{0})}$. By the definition of x_n, we have

$$\left[\frac{1}{\lambda_n}\overrightarrow{x_n x}\right] \in A_n x_n$$

for all $n \in \mathbb{N}$. The monotonicity of A_n implies

$$0 \leq \left\langle \left[\frac{1}{\lambda_n}\overrightarrow{x_n x}\right] - \mathbf{0}, \overrightarrow{a_n x_n} \right\rangle = \frac{1}{2\lambda_n}\left\{d(x,a_n)^2 - d(x_n,a_n)^2 - d(x,x_n)^2\right\}.$$

Therefore, we have

$$d(x_n,a_n)^2 \leq d(x,a_n)^2 - d(x,x_n)^2$$

and hence $d(x,x_n) \leq d(x,a_n)$. Since $\{a_n\}$ is bounded, so is $\{x_n\}$. Therefore, each subsequence $\{x_{n_i}\}$ of $\{x_n\}$ has a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δ-convergent to some $x_0 \in X$. Then x_n belongs to $A^{-1}(\mathbf{0})$ by (b). Since $d(x, x_{n_{i_j}}) \leq d(x, a_{n_{i_j}})$ for all $j \in \mathbb{N}$, we have

$$d\left(x, P_{A_0^{-1}(\mathbf{0})}(x)\right) \leq \liminf_{i\to\infty} d(x, a_{n_{i_j}}) \leq \liminf_{i\to\infty} d(x, v_{n_{i_j}}) = d\left(x, P_{A_0^{-1}(\mathbf{0})}(x)\right)$$

as $j \to \infty$. Hence, we have $x_0 = P_{A_0^{-1}(\mathbf{0})}(x)$. Since $a_{n_{i_j}} \to p$ and $x_{n_{i_j}} \xrightarrow{\Delta} x_0$, we obtain

$$\begin{aligned}
\limsup_{j\to\infty} d(P_{A_0^{-1}(\mathbf{0})}(x), x_{n_{i_j}})^2 &= \limsup_{j\to\infty} d(a_{n_{i_j}}, x_{n_{i_j}})^2 \\
&\leq \limsup_{j\to\infty}\{d(x, a_{n_{i_j}})^2 - d(x, x_{n_{i_j}})^2\} \\
&\leq \limsup_{j\to\infty} d(x, a_{n_i})^2 - \liminf_{j\to\infty} d(x, x_{n_{i_j}})^2 \\
&\leq d(x, P_{A_0^{-1}(\mathbf{0})}(x))^2 - d(x, P_{A_0^{-1}(\mathbf{0})}(x))^2 = 0
\end{aligned}$$

by (4) and the lower semicontinuity of $d(x,\cdot)$. Thus, for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, we can find a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ that converges to $P_{A_0^{-1}(\mathbf{0})}(x)$, which is equivalent to $x_n \to P_{A_0^{-1}(\mathbf{0})}(x)$. This is the desired result. □

Theorem 5.5. *Let X be a complete CAT(0) space, X^* its dual, and $A_0\colon X \rightrightarrows X^*$ a maximal monotone operator satisfying $D(S_{\lambda A}) = X$ for all $\lambda > 0$. For $x \in X$, suppose $\{A_n\}$ satisfies the following conditions;*

(a) $\emptyset \neq A_0^{-1}(\mathbf{0}) \subset \text{d-Li}\, A_n^{-1}(\mathbf{0})$;

(b) For all $(b, b^*) \in A_0$, and $\{c_i\}$ such that $c_i \in D(A_{n_i})$ and $\{c_i\}$ is Δ-convergent to $c \in X$, there exists $\{(b_i, b_i^*)\}$ which satisfies $(b_i, b_i^*) \in A_{n_i}$, $b_i \to b$, $b_i^* \to b^*$, and

$$\left\langle b^*, \overrightarrow{bc} \right\rangle \leq \liminf_{i\to\infty} \left\langle b_i^*, \overrightarrow{b_i c_i} \right\rangle.$$

Then

$$S_{\lambda_n A_n}(x) \to P_{A_0^{-1}(\mathbf{0})}(x).$$

Proof. Put $x_n = S_{\lambda_n A_n}(x)$ for all $n \in \mathbb{N}$ and take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is Δ-convergent to some $x_0 \in X$. For all $(b, b^*) \in A_0$, there exists $\{(b_i, b_i^*)\}$ such that $(b_i, b_i^*) \in D(A_{n_i})$, $b_i \to b$, $b_i^* \to b^*$, and

$$\left\langle b^*, \overrightarrow{bx_0} \right\rangle \leq \liminf_{i\to\infty} \left\langle b_i^*, \overrightarrow{b_i x_{n_i}} \right\rangle.$$

From the monotonicity of A_{n_i}, we get

$$\begin{aligned} 0 &\leq \left\langle \left[\frac{1}{\lambda_{n_i}}\overrightarrow{x_{n_i}x}\right] - b_i^*, \overrightarrow{b_i x_{n_i}} \right\rangle \\ &= \frac{1}{\lambda_{n_i}} \left\langle \overrightarrow{x_{n_i}x}, \overrightarrow{b_i x_{n_i}} \right\rangle - \left\langle b_i^*, \overrightarrow{b_i x_{n_i}} \right\rangle \\ &\leq \frac{1}{\lambda_{n_i}} d(x_{n_i}, x) d(b_i, x_{n_i}) - \left\langle b_i^*, \overrightarrow{b_i x_{n_i}} \right\rangle, \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we have

$$0 \leq \limsup_{n\to\infty} \left(- \left\langle b_i^*, \overrightarrow{b_i x_{n_i}} \right\rangle \right) = - \liminf_{i\to\infty} \left\langle b_i^*, \overrightarrow{b_i x_{n_i}} \right\rangle \leq - \left\langle b^*, \overrightarrow{b x_0} \right\rangle = \left\langle b^* - \mathbf{0}, \overrightarrow{x_0 b} \right\rangle$$

for all $(b, b^*) \in A_0$. Maximality of A_0 implies that $x_0 \in A_0^{-1}(\mathbf{0})$. Therefore, we get the desired result. □

Theorem 5.6. *Let X be a complete CAT(0) space, X^* its dual, $A_0 \colon X \rightrightarrows X^*$ a maximal monotone operator satisfying $D(S_{\lambda A}) = X$ for all $\lambda > 0$, and $x \in X$. Suppose that $\{\lambda_n\}$ converges to 0, $\overline{\Delta}\text{-Ls}\, D(A_n) \subset D(A_0)$, and there exists $\{(z_n, z_n^*)\}$ such that $(z_n, z_n^*) \in A_n$, $z_n \to P_{\mathrm{cl}\, D(A_0)}(x)$, and $\limsup_{n\to\infty} \langle z_n^*, \overrightarrow{x_n z_n} \rangle < \infty$. Then*

$$S_{\lambda_n A_n}(x) \to P_{\mathrm{cl}\, D(A_0)}(x).$$

Proof. Put $x_n = R^{d^2}_{\lambda_n F_n}(x)$ for all $n \in \mathbb{N}$ and let $\{z_n\}$ be a sequence which satisfies the assumptions of the theorem. From the definition of the resolvent, we get

$$\begin{aligned} 0 &\leq \left\langle \left[\frac{1}{\lambda_n}\overrightarrow{x_n x}\right] - z_n^*, \overrightarrow{z_n x_n} \right\rangle \\ &= \frac{1}{\lambda_n} \langle \overrightarrow{x_n x}, \overrightarrow{z_n x_n} \rangle - \langle z_n^*, \overrightarrow{z_n x_n} \rangle \\ &= \frac{1}{\lambda_n} \left(d(x, z_n)^2 - d(x_n, z_n)^2 - d(x, x_n)^2 \right) - \langle z_n^*, \overrightarrow{z_n x_n} \rangle \\ &= \frac{1}{\lambda_n} \left(d(x, z_n)^2 - d(x_n, z_n)^2 - d(x, x_n)^2 \right) + \langle z_n^*, \overrightarrow{x_n z_n} \rangle \end{aligned}$$

and thus

$$\begin{aligned} d(x, x_n)^2 &\leq (x, z_n)^2 - d(x_n, z_n)^2 + \lambda_n \langle z_n^*, \overrightarrow{x_n z_n} \rangle \\ &\leq d(x, z_n)^2 + \lambda_n \langle z_n^*, \overrightarrow{x_n z_n} \rangle. \end{aligned}$$

Letting $n \to \infty$, we have

$$\limsup_{n\to\infty} d(x, x_n)^2 \leq \limsup_{n\to\infty} d(x, z_n)^2 = d(x, P_{\mathrm{cl}\, D(A_0)}(x))^2,$$

and hence $\limsup_{n\to\infty} d(x, x_n) \leq d(x, P_{\mathrm{cl}\, D(A_0)}(x))$. Therefore, $\{x_n\}$ is bounded. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. Then, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which is Δ-convergent to some $x_0 \in X$. Since $x_{n_{i_j}} \xrightarrow{\Delta} x_0$, we get

$$d(P_{\mathrm{cl}\, D(A_0)}(x), x) \leq d(x_0, x) \leq \liminf_{j\to\infty} d(x_{n_{i_j}}, x) \leq \limsup_{j\to\infty} d(x_{n_{i_j}}, x) \leq d(P_{\mathrm{cl}\, D(A_0)}(x), x).$$

Hence, $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} P_{\mathrm{cl}\,D(A_0)}(x)$. Moreover, from the parallelogram law, we have

$$\begin{aligned}
d(P_{\mathrm{cl}\,D(A_0)}(x), x)^2 &\leq \liminf_{j\to\infty} d(x_{n_{i_j}}, x)^2 \leq \limsup_{j\to\infty} d(x_{n_{i_j}}, x)^2 \\
&\leq \limsup_{j\to\infty} d\left(\frac{1}{2}x_{n_{i_j}} \oplus \frac{1}{2}P_{\mathrm{cl}\,D(A_0)}(x), x\right)^2 \\
&\leq \limsup_{j\to\infty} \left(\frac{1}{2}d(x_{n_{i_j}}, x)^2 + \frac{1}{2}d(P_{\mathrm{cl}\,D(A_0)}(x), x)^2 - \frac{1}{4}d(x_{n_{i_j}}, P_{\mathrm{cl}\,D(A_0)}(x))^2\right) \\
&\leq \frac{1}{2}\limsup_{j\to\infty} d(x_{n_{i_j}}, x)^2 + \frac{1}{2}d((P_{\mathrm{cl}\,D(A_0)}(x), x)^2 - \frac{1}{4}\liminf_{j\to\infty} d(x_{n_{i_j}}, P_{\mathrm{cl}\,D(A_0)}(x))^2 \\
&\leq \frac{1}{2}d(P_{\mathrm{cl}\,D(A_0)}(x), x)^2 + \frac{1}{2}d(P_{\mathrm{cl}\,D(A_0)}(x), x)^2 - \frac{1}{4}\lim_{j\to\infty} d(x_{n_{i_j}}, P_{\mathrm{cl}\,D(A_0)}(x))^2.
\end{aligned}$$

Thus, we obtain $x_{n_{i_j}} \to P_{\mathrm{cl}\,D(A_0)}(x)$ and hence

$$x_n \to P_{\mathrm{cl}\,D(A_0)}(x),$$

which is the desired result. □

Chapter 6

Conclusion

In this thesis, we consider asymptotic behavior of resolvents for convex functions, equilibrium problems, and monotone operators on complete geodesic spaces. We show the obtained results in a CAT(0) space again.

Let $\varphi\colon [0,\infty[\to [0,\infty[$ be a function satisfying the following conditions;

(A1) $\varphi(0) = 0$;
(A2) φ is increasing;
(A3) φ is continuous;
(A4) $\varphi(d(\cdot, x))$ is strictly convex for all $x \in X$;
(A5) $\varphi(t) - kt \to \infty$ as $t \to \infty$, for all $k \in \mathbb{R}$.

For a proper lower semicontinuous convex function $f\colon X \to]-\infty, \infty]$ on complete CAT(0) space X, define the resolvent $Q_f^\varphi\colon X \to X$ with φ by

$$Q_f^\varphi(x) = \operatorname*{argmin}_{y \in X} \{f(y) + \varphi(d(y, x))\}$$

for all $x \in X$.

Theorem 6.1. *Let X be a complete CAT(0) space, $f_0\colon X \to]-\infty, \infty]$ a proper convex lower semicontinuous function and $x \in X$. Suppose that a sequence $\{\lambda_n\}$ of positive real numbers diverges to ∞ and a sequence $\{f_n\}$ of proper convex lower semicontinuous functions satisfies the following conditions;*

(a) $\emptyset \neq \operatorname{argmin} f_0 \subset \text{d-Li} \operatorname{argmin} f_n$;
(b) For all $b \in X$, there exists $\{b_n\}$ such that $b_n \to b$ and $\limsup_{n\to\infty} f_n(b_n) \leq f(b)$;
(c) For any subsequence $\{f_{n_i}\}$ of $\{f_n\}$ and $\{c_i\}$ which is Δ-convergent to c, it hold that $f(c) \leq \liminf_{i\to\infty} f_{n_i}(c_i)$.

Then

$$Q_{\lambda_n f_n}^\varphi(x) \to P_{\operatorname{argmin} f_0}(x).$$

Theorem 6.2. *Let X be a complete CAT(0) space and $f_0\colon X \to]-\infty, \infty]$ a proper convex lower semicontinuous function and $x \in X$. Suppose that $\{\lambda_n\}$ converges to 0, $\overline{\Delta}\text{-Ls}\, K_n \subset K_0$, and there exists a sequence $\{z_n\}$ such that $z_n \to P_{\operatorname{cl}\operatorname{dom} f_0}(x)$ and that*

$$\limsup_{n\to\infty} |f_n(z_n) - f_n(Q_{\lambda_n f_n}^\psi(x))| < \infty.$$

Then

$$Q_{\lambda_n f_n}^\psi(x) \to P_{\operatorname{cl}\operatorname{dom} f_0}(x).$$

For bifunction F from a closed convex set K to $\mathbb{R}$ satisfying (E1)-(E4), define $R_F^{d^2}: X \to K$ by

$$R_F^{d^2}(x) = \left\{ z \in K \mid \inf_{y \in K} \left(F(z,y) + d(y,x)^2 - d(z,x)^2 \right) \geq 0 \right\}$$

for all $x \in X$.

Theorem 6.3. *Let X be a complete $CAT(0)$ space having the convex hull finite property, K a nonempty closed convex subset of X, and $F\colon K \times K \to \mathbb{R}$ satisfying (E1)-(E4). For $x \in X$, if there exists a positive real sequence $\{\mu_n\}$ diverging to ∞ and $\{d(R_{\mu_n f}^{d^2}(x), x)\}$ is bounded, then* $\operatorname{Equil} f \neq \emptyset$ *and*

$$\lim_{\lambda \to \infty} R_{\lambda F}^{d^2}(x) = P_{\operatorname{Equil} F}(x).$$

Theorem 6.4. *Let X be a complete $CAT(0)$ space having the convex hull finite property, K a nonempty closed convex subset of X, and $F\colon K \times K \to \mathbb{R}$ satisfies (E1)-(E4). Then for $x \in X$,*

$$\lim_{\lambda \searrow 0} R_{\lambda F}^{d^2}(x) = P_K(x).$$

and

$$\lim_{\lambda \to \lambda_0} R_{\lambda F}^{d^2}(x) = Q_{\lambda_0 f}^{\varphi}(x),$$

where λ_0 is a positive real number.

Theorem 6.5. *Let X be a complete $CAT(0)$ space having the convex hull finite property, K_0 a nonempty closed convex subset of X, and $F_0\colon K_0 \times K_0 \to \mathbb{R}$ a bifunction satisfying (E1)-(E4) and $x \in X$. Suppose that a sequence $\{\lambda_n\}$ of positive real numbers diverges to ∞ and a sequence $\{F_n\}$ of bifunctions $F_n\colon K_n \times K_n \to \mathbb{R}$ satisfies the following conditions;*

- *(a)* $\emptyset \neq \operatorname{Equil} F_0 \subset \text{d-Li} \operatorname{Equil} F_n$;
- *(b) For all $b \in K_0$, $\{F_{n_i}\}$ of $\{F_n\}$, and $\{c_i\}$ such that $c_i \in K_{n_i}$ and $\{c_i\}$ is Δ-convergent to $c \in X$, there exists $\{b_i\}$ which satisfies $b_i \in K_{n_i}$, $b_i \to b$, and $F_0(b,c) \leq \liminf_{i \to \infty} F_{n_i}(b_i, c_i)$.*

Then

$$R_{\lambda_n F_n}^{d^2}(x) \to P_{\operatorname{Equil} F_0}(x).$$

Theorem 6.6. *Let X be a complete $CAT(0)$ space having the convex hull finite property, K_0 a nonempty closed convex subset of X, $F_0\colon K_0 \times K_0 \to \mathbb{R}$ a bifunction satisfying (E1)-(E4) and $x \in X$. Suppose that $\{K_n\}$ and $\{F_n\}$ satisfies that $\overline{\Delta}\text{-Ls}\, K_n \subset K_0$, and there exists $\{z_n\}$ such that $z_n \in K_n$, $z_n \to P_{K_0}(x)$ and*

$$\limsup_{n \to \infty} F_n(Q_{\lambda_n f_n}^{\varphi}(x), z_n) < \infty.$$

If $\{\lambda_n\}$ converges to 0, then

$$R_{\lambda_n F_n}^{d^2}(x) \to P_{K_0}(x).$$

For monotone operator $A\colon X \rightrightarrows X^*$, define $S_A\colon X \to X$ by

$$S_A(x) = \{z \in X \mid [\overrightarrow{zx}] \in Az\}$$

for all $x \in X$.

Theorem 6.7. *Let X be a complete CAT(0) space, X^* its dual, and $A\colon X \to X^*$ a maximal monotone operator satisfying $D(S_{\lambda A}) = X$ for all $\lambda > 0$. For $x \in X$, if there exists a positive real sequence $\{\mu_n\}$ divergs to ∞ and $\{d(S_{\mu_n A}(x), x)\}$ is bounded, then* $\operatorname{Equil} f \neq \emptyset$ *and*

$$\lim_{\lambda\to\infty} S_{\lambda A}(x) = P_{\operatorname{Equil} F}(x).$$

Theorem 6.8. *Let X be a complete CAT(0) space, X^* its dual, and $A\colon X \to X^*$ a maximal monotone operator satisfying $D(S_{\lambda A}) = X$ for all $\lambda > 0$. Then for $x \in X$,*

$$\lim_{\lambda\searrow 0} S_{\lambda A}(x) = P_{\operatorname{cl} D(A)}(x).$$

and

$$\lim_{\lambda\to\lambda_0} S_{\lambda A}(x) = S_{\lambda_0 A}(x),$$

where λ_0 is a positive real number.

Theorem 6.9. *Let X be a complete CAT(0) space, X^* its dual, $A_0\colon X \to X^*$ a maximal monotone operator satisfying $D(S_{\lambda A}) = X$ for all $\lambda > 0$. For $x \in X$, suppose a sequence $\{A_n\}$ of maximal monotone operators with $D(S_{A_n}) = X$ satisfies the following conditions;*

- *(a)* $\emptyset \neq A_0^{-1}(\mathbf{0}) \subset \text{d-Li}\, A_n^{-1}(\mathbf{0})$;
- *(b) For all $(b, b^*) \in A_0$, and $\{c_i\}$ such that $c_i \in D(A_{n_i})$ and $\{c_i\}$ is Δ-convergent to $c \in X$, there exists $\{(b_i, b_i^*)\}$ which satisfies $(b_i, b_i^*) \in A_{n_i}$, $b_i \to b$, $b_i^* \to b^*$, and*

$$\left\langle b^*, \overrightarrow{bc} \right\rangle \le \liminf_{i\to\infty} \left\langle b_i^*, \overrightarrow{b_i c_i} \right\rangle.$$

Then

$$S_{\lambda_n A_n}(x) \to P_{A_0^{-1}(\mathbf{0})}(x).$$

Theorem 6.10. *Let X be a complete CAT(0) space, X^* its dual, $A_0\colon X \to X^*$ a maximal monotone operator satisfying $D(S_{\lambda A}) = X$ for all $\lambda > 0$ and $x \in X$ Suppose that $\{\lambda_n\}$ converges to* 0, $\overline{\Delta}\text{-Ls}\, D(A_n) \subset D(A_0)$, *and there exists $\{(z_n, z_n^*)\}$ such that $(z_n, z_n^*) \in A_n$, $z_n \to P_{\operatorname{cl} D(A_0)}(x)$, and $\limsup_{n\to\infty} \langle z_n^*, \overrightarrow{x_n z_n}\rangle < \infty$. Then*

$$S_{\lambda_n A}(x) \to P_{\operatorname{cl} D(A_0)}(x).$$

These are based on results of [14] and [16].

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