Parallelogram laws on geodesic spaces

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Introduction

Recently, some convex optimisation problems are considered in complete geodesic spaces with curvature bounded above. In the 1990s, Jost [10] and Mayer [34] generalised the notion of resolvents of convex functions to Hadamard spaces. Hadamard spaces are complete geodesic spaces with curvature bounded above by 0, namely, they are complete CAT(0) spaces. Mayer defined a resolvent for convex functions in Hadamard spaces as a single-valued mapping, and Bačák [5] showed a Δ-convergence theorem with the proximal point algorithm. After that, Kimura and Kohsaka [19, 21, 22] defined a resolvent for convex functions in CAT(1) spaces, and investigated some properties and convergence theorems. Similarly, Kajimura and Kimura [14] defined a resolvent for convex functions in CAT(-1) spaces and proved a Δ-convergence theorem with the proximal point algorithm. Resolvents for equilibrium problems are also discussed in complete geodesic spaces with curvature bounded above. Kishi and Kimura [18] defined a resolvent for an equilibrium problem in Hadamard spaces, and Kimura [17] also defined it in CAT(1) spaces.

The parallelogram law characterises a part of the structure of Hilbert spaces. To hold the parallelogram law by the norm, reasonably good conditions are required for Banach spaces. Kimura and the author [27] proved the following parallelogram law in a Banach space:

Theorem 1. *Let E be a smooth Banach space with its dual E^*. Then,*

$$\begin{aligned}\phi(tx + (1-t)y, z^*) &= t\phi(x, z^*) + (1-t)\phi(y, z^*)\\ &\quad - t\phi(x, J(tx + (1-t)y)) - (1-t)\phi(y, J(tx + (1-t)y))\end{aligned}$$

for every $x, y \in E$, $z^* \in E^*$ and $t \in \mathbb{R}$, where J is the normalised duality mapping on E. Here, $\phi\colon E \times E^* \to \mathbb{R}$ is defined by $\phi(u, v^*) = \|u\|^2 - 2v^*(u) + \|v^*\|^2$ for each $(u, v^*) \in E \times E^*$.

Furthermore, motivated by this result, Kimura and the author [28] proved the following theorem:

Theorem 2. *Let X be a uniquely D_κ-geodesic space for $\kappa \in \mathbb{R}$. Then, the following propositions are equivalent:*

- *X is a* CAT(κ) *space;*
- *the inequality*

$$\begin{aligned}\phi_\kappa(tx \oplus (1-t)y, z) &\le (t)_l^\kappa \phi_\kappa(x, z) + (1-t)_l^\kappa \phi_\kappa(y, z)\\ &\quad - (t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) - (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y)\end{aligned}$$

  holds for every $x, y, z \in X$ with $d(y, z) + d(z, x) + l < 2D_\kappa$ and $t \in [0, 1]$, where $l = d(x, y)$.

In the theorem above, a function ϕ_κ is defined with the following function proposed by

Kajimura and Kimura [11]:

$$c_\kappa(a) = \frac{1}{2}a^2 + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1}a^{2n}}{(2n)!} = \begin{cases} \frac{1}{\kappa}\left(1 - \cos\left(\sqrt{\kappa}a\right)\right) & (\kappa > 0); \\ \frac{1}{2}a^2 & (\kappa = 0); \\ \frac{1}{-\kappa}\left(\cosh\left(\sqrt{-\kappa}a\right) - 1\right) & (\kappa < 0) \end{cases}$$

for each $a \in \mathbb{R}$. The function c_κ has some good properties. However, for positive κ, the function is bounded, namely, it is not divergent under any circumstances. Because of that, for positive κ, we cannot use c_κ as a perturbation to define resolvent operators.

CAT(κ) spaces are usually defined by using the two-dimensional model space and its comparison triangles. In this thesis, we define a CAT(κ) space as a geodesic space satisfying an inequality corresponding to the parallelogram law. As an example of a CAT(κ) space, in Chapter 2, we introduce the countable infinite dimensional model space, which embedded in the infinite dimensional Euclidean space, that is, the sequence space having an inner product. It is an example of a geodesic space with constant curvature. In Chapter 3, we prove some important propositions for our studies on fixed pint theory in the setting of geodesic spaces. Using those results, we obtain a fixed point existence and an approximation theorem for a class of the mappings called nonspreading mappings in Chapter 4. Further, in Chapter 5, we propose a new type resolvent with c_κ for two problems having solutions. Additionally, we prove an approximation theorem with the proximal point algorithm. In Chapter 6 and 7, we consider another convex combination than the usual one proposed by Kimura and Sasaki [24, 25], and investigate convexity of sets and functions with its combination. Using the combination, we prove fixed point approximation theorems with Mann type and Halpern type iterative scheme, respectively.

Contents

Chapter 1

Preliminaries

In this chapter, we first introduce a function playing important roles in our discussion. We next define CAT(κ) spaces and obtain the fundamental propositions. In this thesis, CAT(κ) spaces are defined with an inequality corresponding to the parallelogram law.

1.1 On a function c_κ

Define a value $D_\kappa \in \,]0, \infty]$ as follows:

$$D_\kappa = \begin{cases} \infty & (\kappa \leq 0); \\ \dfrac{\pi}{\sqrt{\kappa}} & (\kappa > 0). \end{cases}$$

We define a function c_κ from $\mathbb{R}$ into $[0, \infty[$ by

$$c_\kappa(a) = \frac{1}{2}a^2 + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1} a^{2n}}{(2n)!} = \begin{cases} \dfrac{1}{\kappa}\left(1 - \cos\left(\sqrt{\kappa}a\right)\right) & (\kappa > 0); \\ \dfrac{1}{2}a^2 & (\kappa = 0); \\ \dfrac{1}{-\kappa}\left(\cosh\left(\sqrt{-\kappa}a\right) - 1\right) & (\kappa < 0) \end{cases}$$

for $a \in \mathbb{R}$. Then,

$$c'_\kappa(a) = \begin{cases} \dfrac{\sin\left(\sqrt{\kappa}a\right)}{\sqrt{\kappa}} & (\kappa > 0); \\ a & (\kappa = 0); \\ \dfrac{\sinh\left(\sqrt{-\kappa}a\right)}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases} \quad \text{and} \quad c''_\kappa(a) = \begin{cases} \cos\left(\sqrt{\kappa}a\right) & (\kappa > 0); \\ 1 & (\kappa = 0); \\ \cosh\left(\sqrt{-\kappa}a\right) & (\kappa < 0) \end{cases}$$

for $a \in \mathbb{R}$. We get the following properties of functions c_κ, c'_κ and c''_κ:

- c_κ, c'_κ and c''_κ are continuous;
- $c_\kappa \colon [0, D_\kappa[\,\to [0, \infty[$ is strictly increasing;
- $c_\kappa(0) = c'_\kappa(0) = 0$ and $c''_\kappa(0) = 1$;
- $c'_\kappa(-a) = -c'_\kappa(a)$ and $c''_\kappa(-a) = c''_\kappa(a)$ for any $a \in \mathbb{R}$;
- $c''_\kappa(a) > 0$ for any $a \in [0, D_\kappa/2[$.

Fix $a, b \in \mathbb{R}$ arbitrarily. Then, we know the following equations:

$$\begin{aligned} c''_\kappa(a) + \kappa c_\kappa(a) &= 1; \\ c''_\kappa(a)^2 + \kappa c'_\kappa(a)^2 &= 1. \end{aligned}$$

Additionally,

$$\begin{aligned} c'_\kappa(a+b) &= c'_\kappa(a)c''_\kappa(b) + c'_\kappa(b)c''_\kappa(a); \\ c'_\kappa(a-b) &= c'_\kappa(a)c''_\kappa(b) - c'_\kappa(b)c''_\kappa(a); \\ c''_\kappa(a+b) &= c''_\kappa(a)c''_\kappa(b) - \kappa c'_\kappa(a)c'_\kappa(b); \\ c''_\kappa(a-b) &= c''_\kappa(a)c''_\kappa(b) + \kappa c'_\kappa(a)c'_\kappa(b) \end{aligned}$$

and therefore

$$\begin{aligned} c'_\kappa(2a) &= 2c'_\kappa(a)c''_\kappa(a); \\ c''_\kappa(2a) &= c''_\kappa(a)^2 - \kappa c'_\kappa(a)^2 \\ &= 2c''_\kappa(a)^2 - 1 \\ &= 1 - 2\kappa c'_\kappa(a)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} \kappa c'_\kappa \left(\frac{a}{2}\right)^2 &= \frac{1 - c''_\kappa(a)}{2}; \\ c''_\kappa \left(\frac{a}{2}\right)^2 &= \frac{c''_\kappa(a) + 1}{2}. \end{aligned}$$

Further, we obtain the following:

$$\begin{aligned} c'_\kappa(a) + c'_\kappa(b) &= 2c'_\kappa \left(\frac{a+b}{2}\right) c''_\kappa \left(\frac{a-b}{2}\right); \\ c'_\kappa(a) - c'_\kappa(b) &= 2c''_\kappa \left(\frac{a+b}{2}\right) c'_\kappa \left(\frac{a-b}{2}\right); \\ c''_\kappa(a) + c''_\kappa(b) &= 2c''_\kappa \left(\frac{a+b}{2}\right) c''_\kappa \left(\frac{a-b}{2}\right); \\ c''_\kappa(a) - c''_\kappa(b) &= -2\kappa c'_\kappa \left(\frac{a+b}{2}\right) c'_\kappa \left(\frac{a-b}{2}\right) \end{aligned}$$

and

$$\begin{aligned} c'_\kappa(a)c''_\kappa(b) &= \frac{1}{2}\left(c'_\kappa(a+b) + c'_\kappa(a-b)\right); \\ c''_\kappa(a)c'_\kappa(b) &= \frac{1}{2}\left(c'_\kappa(a+b) - c'_\kappa(a-b)\right); \\ -\kappa c'_\kappa(a)c'_\kappa(b) &= \frac{1}{2}\left(c''_\kappa(a+b) - c''_\kappa(a-b)\right); \\ c''_\kappa(a)c''_\kappa(b) &= \frac{1}{2}\left(c''_\kappa(a+b) + c''_\kappa(a-b)\right). \end{aligned}$$

For more details about the function c_κ, see [11]. In this thesis, we often use the following identity: For $a, b \in \mathbb{R}$,

$$(1 - c''_\kappa(a))c_\kappa(b) = (1 - c''_\kappa(b))c_\kappa(a).$$

In what follows, we introduce and investigate a function called an adjuster. We define an adjuster $(\cdot)^\kappa_l$ from $[0, 1]$ onto $[0, 1]$ by

$$(t)^\kappa_l = \lim_{\delta \to l} \frac{c'_\kappa(t\delta)}{c'_\kappa(\delta)} = \begin{cases} \dfrac{c'_\kappa(tl)}{c'_\kappa(l)} & (l \in]0, D_\kappa[); \\ t & (l = 0) \end{cases}$$

for $t \in [0, 1]$.

We obtain the following lemmas:

Lemma 1.1.1. *Let $\kappa \in \mathbb{R}$. Then,*

$$1 - \frac{1}{2(1/2)^\kappa_l} = \kappa c_\kappa \left(\frac{l}{2}\right)$$

for each $l \in [0, D_\kappa[$.

Proof. If $l = 0$, then we have

$$1 - \frac{1}{2(1/2)^\kappa_l} = 1 - \frac{1}{2(1/2)} = 0 = \kappa c_\kappa \left(\frac{l}{2}\right).$$

Suppose that $l \neq 0$. Since $c'_\kappa(2a) = 2c'_\kappa(a)c''_\kappa(a)$ for each $a \in \mathbb{R}$,

$$c'_\kappa(l) = 2c'_\kappa \left(\frac{l}{2}\right) c''_\kappa \left(\frac{l}{2}\right).$$

Then, since $c''_\kappa(a) + \kappa c_\kappa(a) = 1$ for $a \in \mathbb{R}$, we have

$$1 - \frac{1}{2(1/2)^\kappa_l} = 1 - \frac{c'_\kappa(l)}{2c'_\kappa(l/2)} = 1 - \frac{2c'_\kappa(l/2)c''_\kappa(l/2)}{2c'_\kappa(l/2)} = 1 - c''_\kappa \left(\frac{l}{2}\right) = \kappa c_\kappa \left(\frac{l}{2}\right).$$

It completes the proof. □

Lemma 1.1.2. *Let $\kappa \in \mathbb{R}$. Then,*

$$\lim_{t \nearrow 1} \frac{(1-t)^\kappa_l}{1-t} = \frac{l}{c'_\kappa(l)} \text{ and } \lim_{t \nearrow 1} \frac{1-(t)^\kappa_l}{1-t} = \frac{lc''_\kappa(l)}{c'_\kappa(l)}$$

for each $l \in]0, D_\kappa[$.

Proof. Fix $l \in]0, D_\kappa[$. Then, by l'Hospital's rule, we obtain

$$\lim_{t \nearrow 1} \frac{(1-t)^\kappa_l}{1-t} = \lim_{t \nearrow 1} \frac{c'_\kappa((1-t)l)}{(1-t)c'_\kappa(l)} = \lim_{t \nearrow 1} \frac{lc''_\kappa((1-t)l)}{c'_\kappa(l)} = \frac{l}{c'_\kappa(l)}.$$

In the same way, we obtain

$$\lim_{t \nearrow 1} \frac{1-(t)^\kappa_l}{1-t} = \lim_{t \nearrow 1} \frac{c'_\kappa(l) - c'_\kappa(tl)}{(1-t)c'_\kappa(l)} = \lim_{t \nearrow 1} \frac{lc''_\kappa(tl)}{c'_\kappa(l)} = \frac{lc''_\kappa(l)}{c'_\kappa(l)}.$$

It completes the proof. □

Lemma 1.1.3. *Let $\kappa \in \mathbb{R}$. Then,*

$$\lim_{t \searrow 0} \frac{1-(1-t)_l^\kappa}{(t)_l^\kappa} = c_\kappa''(l)$$

for each $l \in [0, D_\kappa[$.

Proof. If $l = 0$, we have

$$\frac{1-(1-t)_l^\kappa}{(t)_l^\kappa} = \frac{1-(1-t)}{t} = 1 = c_\kappa''(l)$$

and this is the desired result. We may suppose that $l \neq 0$. Then, by l'Hospital's rule, we obtain

$$\begin{aligned}\lim_{t \searrow 0} \frac{1-(1-t)_l^\kappa}{(t)_l^\kappa} &= \lim_{t \searrow 0} \frac{c_\kappa'(l) - c_\kappa'((1-t)l)}{c_\kappa'(tl)} = \lim_{t \searrow 0} \frac{lc_\kappa''((1-t)l)}{lc_\kappa''(tl)} \\ &= \lim_{t \searrow 0} \frac{c_\kappa''((1-t)l)}{c_\kappa''(tl)} = c_\kappa''(l).\end{aligned}$$

It completes the proof. □

1.2 Parallelogram laws on geodesic spaces

Let (X, d) be a metric space and $D \in]0, \infty]$. Let $x, y \in X$ and $l = d(x, y)$. We call a mapping γ_{xy} from $[0, l]$ into X a geodesic from x to y if γ_{xy} is isometric, $\gamma_{xy}(0) = x$ and $\gamma_{xy}(l) = y$. Note that γ_{xy} is said to be isometric if

$$d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s - t|$$

for each $s, t \in [0, l]$. Since γ_{xy} is isometric, for $t \in [0, 1]$,

$$d(x, \gamma_{xy}((1-t)l)) = d(\gamma_{xy}(0), \gamma_{xy}((1-t)l)) = (1-t)d(x, y)$$

and

$$d(y, \gamma_{xy}((1-t)l)) = d(\gamma_{xy}(l), \gamma_{xy}((1-t)l)) = td(x, y).$$

X is said to be uniquely D-geodesic if for each $x, y \in X$ with $d(x, y) < D$, there is a unique geodesic from x to y. In particular, we call a uniquely ∞-geodesic space a uniquely geodesic space simply. In a uniquely D-geodesic space X, for $x, y \in X$ with $d(x, y) < D$ and $t \in [0, 1]$, we denote $\gamma_{xy}((1-t)d(x, y))$ by $tx \oplus (1-t)y$, and call it convex combination of x and y. It holds that $tx \oplus (1-t)y \to y$ as $t \searrow 0$.

Remark 1. Let X be a uniquely D-geodesic space for $D \in]0, \infty]$ and $x, y \in X$ with $l = d(x, y) < D$. We define a mapping γ from $[0, l]$ by $\gamma(t) = \gamma_{xy}(l - t)$ for each $t \in [0, l]$, where γ_{xy} is a unique geodesic from x to y. Then, $\gamma(0) = \gamma_{xy}(l) = y$, $\gamma(l) = \gamma_{xy}(0) = x$ and

$$d(\gamma(s), \gamma(t)) = d(\gamma_{xy}(l - s), \gamma_{xy}(l - t)) = |l - s - (l - t)| = |s - t|$$

for each $s, t \in [0, l]$. Therefore, $\gamma_{yx} \equiv \gamma$, where γ_{yx} is a unique geodesic from y to x. Hence,

$$tx \oplus (1-t)y = \gamma_{xy}((1-t)l) = \gamma_{xy}(l - tl) = \gamma_{yx}(tl) = (1-t)y \oplus tx$$

for all $t \in [0, 1]$.

Remark 2. Let X be a metric space and $D \in]0, \infty]$. Let $x, y, z \in X$ with $d(y, z) + d(z, x) + d(x, y) < 2D$. Then,

$$d(x, y) = \frac{d(x, y) + d(x, y)}{2} \leq \frac{d(x, y) + d(x, z) + d(z, y)}{2} < D.$$

Therefore, if X is uniquely D-geodesic, then there exists a unique geodesic from x to y. We notice that $\max\{d(y, z), d(z, x), d(x, y)\} < D$.

For a metric space (X, d) and $\kappa \in \mathbb{R}$, we define a function ϕ_κ from X^2 into $\mathbb{R}$ by

$$\phi_\kappa(x, y) = c_\kappa(d(x, y))$$

for each $x, y \in X$. We know the following properties of ϕ_κ:

- $\phi_\kappa(x, y) \geq 0$ for every $x, y \in X$;
- $\phi_\kappa(x, y) = 0$ if and only if $x = y$, where $d(x, y) < 2D_\kappa$;
- $\phi_\kappa(x, y) = \phi_\kappa(y, x)$ for every $x, y \in X$.

Now, we define a CAT(κ) space. Let $\kappa \in \mathbb{R}$ and X a uniquely D_κ-geodesic space. We call X a CAT(κ) space if

$$\begin{aligned}\phi_\kappa(tx \oplus (1-t)y, z) \leq\ & (t)_l^\kappa \phi_\kappa(x, z) + (1-t)_l^\kappa \phi_\kappa(y, z) \\ & - (t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) - (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y)\end{aligned}$$

for every $x, y, z \in X$ with $d(y, z) + d(z, x) + l < 2D_\kappa$ and $t \in [0, 1]$, where $l = d(x, y)$. In this thesis, we call this inequality the parallelogram law of a CAT(κ) space.

Remark 3. Let $\kappa \neq 0$. Then, for $a \in \mathbb{R}$,

$$c_\kappa(a) = \frac{1}{\kappa}(1 - c''_\kappa(a)) = 2c'_\kappa\left(\frac{a}{2}\right)^2.$$

We remark that this equation also holds if $\kappa = 0$. Therefore, to define a CAT(κ) space, we need only c'_κ, and c_κ is not necessary to be used. Indeed, we define a function s_κ by

$$s_\kappa(a) = c'_\kappa(a) = a + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1} a^{2n-1}}{(2n-1)!} = \begin{cases} \dfrac{\sin\left(\sqrt{\kappa} a\right)}{\sqrt{\kappa}} & (\kappa > 0); \\ a & (\kappa = 0); \\ \dfrac{\sinh\left(\sqrt{-\kappa} a\right)}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases}$$

for $a \in \mathbb{R}$. Let X be a metric space and $\kappa \in \mathbb{R}$. Then,

$$\phi_\kappa(x, y) = 2s_\kappa\left(\frac{d(x, y)}{2}\right)^2$$

for $x, y \in X$, and

$$(t)_l^\kappa = \lim_{\delta \to l} \frac{s_\kappa(t\delta)}{s_\kappa(\delta)}$$

for $t \in [0, 1]$ and $l \in [0, D_\kappa[$.

Remark 4. Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$. Fix $x, y, z \in X$ with $d(y,z)+d(z,x)+d(x,y) < 2D_\kappa$ and $t \in [0,1]$, and set $l = d(x,y)$. Suppose $\kappa = 1$. Then,

$$\begin{aligned}
&(1-\cos d(tx \oplus (1-t)y, z)) \sin l \\
&\leq \sin(tl)(1-\cos d(x,z)) + \sin((1-t)l)(1-\cos d(y,z)) \\
&\quad - \sin(tl)(1-\cos((1-t)l)) - \sin((1-t)l)(1-\cos(tl)) \\
&= -(\cos d(x,z)\sin(tl) + \cos d(y,z)\sin((1-t)l)) \\
&\quad + \sin(tl)\cos((1-t)l) + \sin((1-t)l)\cos(tl) \\
&= -(\cos d(x,z)\sin(tl) + \cos d(y,z)\sin((1-t)l)) + \sin l
\end{aligned}$$

and hence

$$\cos d(tx \oplus (1-t)y, z) \sin l \geq \cos d(x,z)\sin(tl) + \cos d(y,z)\sin((1-t)l).$$

Suppose $\kappa = -1$. Then,

$$\begin{aligned}
&(\cosh d(tx \oplus (1-t)y, z) - 1) \sinh l \\
&\leq \sinh(tl)(\cosh d(x,z) - 1) + \sinh((1-t)l)(\cosh d(y,z) - 1) \\
&\quad - \sinh(tl)(\cosh((1-t)l) - 1) - \sinh((1-t)l)(\cosh(tl) - 1) \\
&= \cosh d(x,z)\sinh(tl) + \cosh d(y,z)\sinh((1-t)l) \\
&\quad - (\sinh(tl)\cosh((1-t)l) + \sinh((1-t)l)\cosh(tl)) \\
&= \cosh d(x,z)\sinh(tl) + \cosh d(y,z)\sinh((1-t)l) - \sinh l
\end{aligned}$$

and hence

$$\cosh d(tx \oplus (1-t)y, z) \sinh l \leq \cosh d(x,z)\sinh(tl) + \cosh d(y,z)\sinh((1-t)l).$$

In general, if $\kappa \neq 0$, then

$$\begin{aligned}
&\frac{c''_\kappa(d(tx \oplus (1-t)y, z))c'_\kappa(d(x,y))}{-\kappa} \\
&\leq \frac{c''_\kappa(d(x,z))c'_\kappa(td(x,y)) + c''_\kappa(d(y,z))c'_\kappa((1-t)d(x,y))}{-\kappa}
\end{aligned}$$

for any $x, y, z \in X$ with $d(y,z) + d(z,x) + d(x,y) < 2D_\kappa$ and $t \in [0,1]$.

Theorem 1.2.1. *Let X be a uniquely geodesic space. Then, the following propositions are equivalent:*

- *X is a* CAT(0) *space;*
- *for every $x, y, z \in X$ and $t \in [0,1]$,*

$$d(tx \oplus (1-t)y, z)^2 \leq td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2.$$

Theorem 1.2.2. *Let X be a uniquely π-geodesic space. Then, the following propositions are equivalent:*

- *X is a* CAT(1) *space;*
- *for every $x, y, z \in X$ with $d(y,z) + d(z,x) + d(x,y) < 2\pi$ and $t \in [0,1]$,*

$$\begin{aligned}
&\cos d(tx \oplus (1-t)y, z) \sin d(x,y) \\
&\geq \cos d(x,z)\sin(td(x,y)) + \cos d(y,z)\sin((1-t)d(x,y)).
\end{aligned}$$

Theorem 1.2.3. *Let X be a uniquely geodesic space. Then, the following propositions are equivalent:*

- *X is a* CAT(-1) *space;*
- *for every $x, y, z \in X$ and $t \in [0,1]$,*

$$\begin{aligned}&\cosh d(tx \oplus (1-t)y, z) \sinh d(x,y)\\ &\leq \cosh d(x,z) \sinh(td(x,y)) + \cosh d(y,z) \sinh((1-t)d(x,y)).\end{aligned}$$

In general, we get the following:

Theorem 1.2.4. *Let X be a uniquely D_κ-geodesic space for $\kappa \in \mathbb{R}$ with $\kappa \neq 0$. Then, the following propositions are equivalent:*

- *X is a* CAT(κ) *space;*
- *for every $x, y, z \in X$ with $d(y,z) + d(z,x) + d(x,y) < 2D_\kappa$ and $t \in [0,1]$,*

$$\begin{aligned}&\frac{c_\kappa''(d(tx \oplus (1-t)y, z)) c_\kappa'(d(x,y))}{-\kappa}\\ &\leq \frac{c_\kappa''(d(x,z)) c_\kappa'(td(x,y)) + c_\kappa''(d(y,z)) c_\kappa'((1-t)d(x,y))}{-\kappa}.\end{aligned}$$

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$. If $d(u,v) < D_\kappa/2$ for any $u, v \in X$, then we say that X is admissible. CAT(κ) spaces are always admissible when $\kappa \leq 0$.

In admissible CAT(κ) spaces, we obtain the following:

Lemma 1.2.5. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$\phi_\kappa(tx \oplus (1-t)y, z) \leq t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z)$$

for each $x, y, z \in X$ and $t \in [0,1]$.

Proof. It obviously holds if $x = y$. Suppose $l = d(x,y) > 0$. If $\kappa = 0$, then we easily obtain the desired inequality. Suppose $\kappa \neq 0$. Then,

$$\frac{c_\kappa''(d(tx \oplus (1-t)y, z))}{-\kappa} \leq \frac{c_\kappa''(d(x,z)) c_\kappa'(tl)}{-\kappa c_\kappa'(l)} + \frac{c_\kappa''(d(y,z)) c_\kappa'((1-t)l)}{-\kappa c_\kappa'(l)}.$$

Since $c_\kappa'(\cdot)/(-\kappa)$ is convex on $[0, D_\kappa/2[$,

$$\frac{c_\kappa'(\tau D)}{-\kappa} = \frac{c_\kappa'(\tau D + (1-\tau)0)}{-\kappa} \leq \tau \cdot \frac{c_\kappa'(D)}{-\kappa} + (1-\tau) \cdot \frac{c_\kappa'(0)}{-\kappa} = \tau \cdot \frac{c_\kappa'(D)}{-\kappa}$$

for any $\tau \in [0,1]$ and $D \in [0, D_\kappa/2[$. Thus, we obtain

$$\frac{c_\kappa''(d(tx \oplus (1-t)y, z))}{-\kappa} \leq \frac{tc_\kappa''(d(x,z))}{-\kappa} + \frac{(1-t)c_\kappa''(d(y,z))}{-\kappa}.$$

Therefore,

$$\frac{1}{\kappa} - \frac{c_\kappa''(d(tx \oplus (1-t)y, z))}{\kappa} \leq \frac{t}{\kappa} - \frac{tc_\kappa''(d(x,z))}{\kappa} + \frac{1-t}{\kappa} - \frac{(1-t)c_\kappa''(d(y,z))}{\kappa}$$

and hence

$$\phi_\kappa(tx \oplus (1-t)y, z) \leq t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z).$$

It completes the proof. □

Theorem 1.2.6. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$\phi_\kappa\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq \frac{1}{2}\phi_\kappa(x,z) + \frac{1}{2}\phi_\kappa(y,z) - c_\kappa''(\delta)c_\kappa\left(\frac{d(x,y)}{2}\right)$$

for every $x, y, z \in X$, where $\delta = d(\gamma_{xy}(d(x,y)/2), z)$.

Proof. Let $x, y, z \in X$, $\delta = d(\gamma_{xy}(d(x,y)/2), z)$ and $l = d(x,y)$. We remark that

$$\gamma_{xy}\left(\frac{d(x,y)}{2}\right) = \frac{1}{2}x \oplus \frac{1}{2}y.$$

From the parallelogram law of X,

$$\begin{aligned}\phi_\kappa\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) &\leq (1/2)_l^\kappa \phi_\kappa(x,z) + (1/2)_l^\kappa \phi_\kappa(y,z) \\ &\quad - (1/2)_l^\kappa \phi_\kappa\left(x, \frac{1}{2}x \oplus \frac{1}{2}y\right) - (1/2)_l^\kappa \phi_\kappa\left(y, \frac{1}{2}x \oplus \frac{1}{2}y\right) \\ &= (1/2)_l^\kappa \phi_\kappa(x,z) + (1/2)_l^\kappa \phi_\kappa(y,z) - 2(1/2)_l^\kappa c_\kappa\left(\frac{l}{2}\right).\end{aligned}$$

Therefore, we obtain

$$\frac{1}{2(1/2)_l^\kappa}\phi_\kappa\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq \frac{1}{2}\phi_\kappa(x,z) + \frac{1}{2}\phi_\kappa(y,z) - c_\kappa\left(\frac{l}{2}\right)$$

and hence

$$\phi_\kappa\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq \frac{1}{2}\phi_\kappa(x,z) + \frac{1}{2}\phi_\kappa(y,z) - c_\kappa\left(\frac{l}{2}\right) + \left(1 - \frac{1}{2(1/2)_l^\kappa}\right)c_\kappa(\delta).$$

Since $1 - 1/(2(1/2)_l^\kappa) = \kappa c_\kappa(l/2)$, we have

$$\begin{aligned}\phi_\kappa\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) &\leq \frac{1}{2}\phi_\kappa(x,z) + \frac{1}{2}\phi_\kappa(y,z) - (1 - \kappa c_\kappa(\delta))\, c_\kappa\left(\frac{l}{2}\right) \\ &= \frac{1}{2}\phi_\kappa(x,z) + \frac{1}{2}\phi_\kappa(y,z) - c_\kappa''(\delta)c_\kappa\left(\frac{l}{2}\right).\end{aligned}$$

This is the desired result. □

1.3 Note on parallelogram laws

In what follows, we consider another type of parallelogram law corresponding to [11]. Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$. Fix $x, y, z \in X$ with $d(y,z) + d(z,x) + d(x,y) < 2D_\kappa$ and $t \in [0,1]$. Set $l = d(x,y)$. Then, from the parallelogram law of X, we have

$$\begin{aligned}\phi_\kappa(tx \oplus (1-t)y, z) &\leq (t)_l^\kappa \phi_\kappa(x,z) + (1-t)_l^\kappa \phi_\kappa(y,z) \\ &\quad - (t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) - (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y).\end{aligned}$$

Further, if $\kappa \neq 0$ and $l \neq 0$, then

$$\begin{aligned}
&(t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) + (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y) \\
&= \frac{c'_\kappa(tl)(1 - c''_\kappa((1-t)l) + c'_\kappa((1-t)l)(1 - c''_\kappa(tl))}{\kappa c'_\kappa(l)} \\
&= \frac{c'_\kappa(tl) + c'_\kappa((1-t)l) - (c'_\kappa(tl)c''_\kappa((1-t)l) + c'_\kappa((1-t)l)c''_\kappa(tl))}{\kappa c'_\kappa(l)} \\
&= \frac{c'_\kappa(tl) + c'_\kappa((1-t)l) - c'_\kappa(l)}{\kappa c'_\kappa(l)}.
\end{aligned}$$

Since

$$\begin{aligned}
c'_\kappa(tl) + c'_\kappa((1-t)l) - c'_\kappa(l) &= 2c'_\kappa\left(\frac{l}{2}\right) c''_\kappa\left(\frac{(2t-1)l}{2}\right) - 2c'_\kappa\left(\frac{l}{2}\right) c''_\kappa\left(\frac{l}{2}\right) \\
&= 2c'_\kappa\left(\frac{l}{2}\right)\left(c''_\kappa\left(\frac{(2t-1)l}{2}\right) - c''_\kappa\left(\frac{l}{2}\right)\right) \\
&= 4\kappa c'_\kappa\left(\frac{l}{2}\right) c'_\kappa\left(\frac{tl}{2}\right) c'_\kappa\left(\frac{(1-t)l}{2}\right),
\end{aligned}$$

we get

$$\begin{aligned}
&(t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) + (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y) \\
&= \frac{4\kappa c'_\kappa(l/2)c'_\kappa(tl/2)c'_\kappa((1-t)l/2)}{\kappa c'_\kappa(l)} = \frac{4c'_\kappa(l/2)^3 c'_\kappa(tl/2)c'_\kappa((1-t)l/2)}{c'_\kappa(l)c'_\kappa(l/2)^2} \\
&= 4(1/2)_l^\kappa (t)_{l/2}^\kappa (1-t)_{l/2}^\kappa c'_\kappa\left(\frac{l}{2}\right)^2 = 2(1/2)_l^\kappa (t)_{l/2}^\kappa (1-t)_{l/2}^\kappa \phi_\kappa(x,y).
\end{aligned}$$

Note that this equation also holds if $\kappa = 0$ or $l = d(x,y) = 0$. From Lemma 1.1.1, we get

$$1 - \frac{1}{2(1/2)_l^\kappa} = \kappa c_\kappa\left(\frac{l}{2}\right) = 1 - c''_\kappa\left(\frac{l}{2}\right)$$

and thus

$$2(1/2)_l^\kappa = \frac{1}{c''_\kappa(l/2)}.$$

Note that $c''_\kappa(l/2) > 0$ since $l = d(x,y) < D_\kappa$. Therefore, we get

$$(t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) + (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y) = \frac{(t)_{l/2}^\kappa (1-t)_{l/2}^\kappa \phi_\kappa(x,y)}{c''_\kappa(l/2)}.$$

In general, for $d \in [0, D_\kappa[$ and $\tau \in [0,1]$,

$$(\tau)_d^\kappa c_\kappa((1-\tau)d) + (1-\tau)_d^\kappa c_\kappa(\tau d) = \frac{(\tau)_{d/2}^\kappa (1-\tau)_{d/2}^\kappa c_\kappa(d)}{c''_\kappa(d/2)}.$$

Consequently, we get the following results:

Theorem 1.3.1. *Let X be a* $\mathrm{CAT}(\kappa)$ *space for $\kappa \in \mathbb{R}$. Then,*

$$\begin{aligned}\phi_\kappa(tx \oplus (1-t)y, z) \leq (t)_l^\kappa \phi_\kappa(x,z) + (1-t)_l^\kappa \phi_\kappa(y,z) \\ - (t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) - (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y)\end{aligned}$$

and

$$\phi_\kappa(tx \oplus (1-t)y, z) \leq (t)_l^\kappa \phi_\kappa(x,z) + (1-t)_l^\kappa \phi_\kappa(y,z) - \frac{(t)_{l/2}^\kappa (1-t)_{l/2}^\kappa \phi_\kappa(x,y)}{c_\kappa''(l/2)}$$

for any $x, y, z \in X$ with $d(y,z) + d(z,x) + l < 2D_\kappa$ and $t \in [0,1]$, where $l = d(x,y)$.

These inequalities are equivalent each other. In the following discussion, it does not matter which inequality is used.

Chapter 2

Infinite dimensional model spaces

In this chapter, we consider examples of infinite dimensional complete CAT(κ) spaces. For examples of finite dimensional ones, refer to [6].

2.1 Hilbert spaces

Let H be a Hilbert space. Note that the distance function d_H is defined by

$$d_H(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}$$

for each $x, y \in H$. Further, the following hold:

(i) For all $x, y \in H$, $|\langle x, y \rangle| \leq \|x\|\|y\|$;
(ii) for all $x, y \in H$ and $t \in \mathbb{R}$,

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2.$$

We call the identity in (ii) the parallelogram law of Hilbert spaces.

Theorem 2.1.1. *A Hilbert space (H, d_H) is a uniquely geodesic space.*

Proof. For $x, y \in H$ with $l = d_H(x, y) > 0$, we define a mapping γ_{xy} from $[0, l]$ into H as

$$\gamma_{xy}(t) = \frac{l - t}{l}x + \frac{t}{l}y$$

for each $t \in [0, l]$. Then, $\gamma_{xy}(0) = x$ and $\gamma_{xy}(l) = y$. Further,

$$\begin{aligned} d_H(\gamma_{xy}(s), \gamma_{xy}(t)) &= \left\| \left(\frac{l - s}{l}x + \frac{s}{l}y \right) - \left(\frac{l - t}{l}x + \frac{t}{l}y \right) \right\| \\ &= \left\| \frac{t - s}{l}x - \frac{t - s}{l}y \right\| = |s - t| \end{aligned}$$

for all $s, t \in [0, l]$. Therefore, γ_{xy} is a geodesic from x to y. Let γ be a geodesic from x to y. Then, for each $t \in [0, l]$, from the parallelogram law of H, we have

$$\begin{aligned} d_H(\gamma_{xy}(t), \gamma(t))^2 &= \left\| \frac{l - t}{l}x + \frac{t}{l}y - \gamma(t) \right\|^2 \\ &= \frac{l - t}{l}\|x - \gamma(t)\|^2 + \frac{t}{l}\|y - \gamma(t)\|^2 - \frac{l - t}{l} \cdot \frac{t}{l}\|x - y\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{l-t}{l} d_H(x, \gamma(t))^2 + \frac{t}{l} d_H(y, \gamma(t))^2 - (l-t)t \\
&= \frac{l-t}{l} d_H(\gamma(0), \gamma(t))^2 + \frac{t}{l} d_H(\gamma(l), \gamma(t))^2 - (l-t)t \\
&= \frac{(l-t)t^2}{l} + \frac{(l-t)^2 t}{l} - (l-t)t \\
&= (l-t)t\left(\frac{t}{l} + \frac{l-t}{l} - 1\right) = 0.
\end{aligned}
$$

It implies that $\gamma_{xy} \equiv \gamma$ and thus H is a uniquely geodesic space. □

Let H be a Hilbert space. Then,

$$
\begin{aligned}
tx \oplus (1-t)y &= \gamma_{xy}((1-t)d_H(x, y)) \\
&= \frac{d_H(x, y) - (1-t)d_H(x, y)}{d_H(x, y)} x + \frac{(1-t)d_H(x, y)}{d_H(x, y)} y \\
&= tx + (1-t)y
\end{aligned}
$$

for each $x, y \in H$ with $x \neq y$ and $t \in [0, 1]$.

Theorem 2.1.2. *A Hilbert space (H, d_H) is a complete* CAT(0) *space.*

Proof. Take $x, y, z \in H$ and $t \in [0, 1]$ arbitrarily. Then, from the parallelogram law of H,

$$
\begin{aligned}
d_H(tx \oplus (1-t)y, z)^2 &= \|tx + (1-t)y - z\|^2 \\
&= t\|x - z\|^2 + (1-t)\|y - z\|^2 - t(1-t)\|x - y\|^2 \\
&= t d_H(x, z)^2 + (1-t)d_H(y, z)^2 - t(1-t)d_H(x, y)^2.
\end{aligned}
$$

Therefore, H is a complete CAT(0) space. □

We define a real linear space $\mathbb{E}^\infty$ by

$$
\mathbb{E}^\infty = \left\{ (x_i) \in \mathbb{R}^\infty \;\middle|\; \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}.
$$

Then, we can define an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{E}^\infty$ as

$$
\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i < \infty
$$

for each $x = (x_i), y = (y_i) \in \mathbb{E}^\infty$. We call the inner product space $\mathbb{E}^\infty$ the infinite dimensional Euclidean space. Define a distance $d_{\mathbb{E}^\infty}$ on $\mathbb{E}^\infty$ by

$$
d_{\mathbb{E}^\infty}(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}
$$

for each $x, y \in \mathbb{E}^\infty$. Then, $(\mathbb{E}^\infty, d_{\mathbb{E}^\infty})$ is a complete metric space, namely, it is a Hilbert space. Therefore, the following holds:

Theorem 2.1.3. *The infinite dimensional Euclidean space $\mathbb{E}^\infty$ is a complete* CAT(0) *space. Moreover,*

$$
d_{\mathbb{E}^\infty}(tx \oplus (1-t)y, z)^2 = t d_{\mathbb{E}^\infty}(x, z)^2 + (1-t)d_{\mathbb{E}^\infty}(y, z)^2 - t(1-t)d_{\mathbb{E}^\infty}(x, y)^2
$$

for any $x, y, z \in \mathbb{E}^\infty$ and $t \in [0, 1]$.

2.2 Spheres

Let $\mathbb{E}^\infty$ be the infinite dimensional Euclidean space and

$$\mathbb{S} = \{x \in \mathbb{E}^\infty \mid \|x\| = 1\} = \{x \in \mathbb{E}^\infty \mid \langle x, x \rangle = 1\}$$

its unit sphere, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{E}^\infty$. Note that $\mathbb{S}$ is a closed set of $\mathbb{E}^\infty$. Then, we know

$$|\langle x, y \rangle| \leq \|x\| \|y\| = 1$$

for any $x, y \in \mathbb{S}$. We define a function $d_\mathbb{S}$ from $\mathbb{S}^2$ to $[0, \pi]$ by

$$d_\mathbb{S}(x, y) = \arccos \langle x, y \rangle$$

for each $x, y \in \mathbb{S}$.

Theorem 2.2.1. $(\mathbb{S}, d_\mathbb{S})$ *is a metric space.*

Proof. Let $x, y \in \mathbb{S}$. If $x = y$, then

$$d_\mathbb{S}(x, y) = \arccos \langle x, y \rangle = \arccos \langle x, x \rangle = \arccos 1 = 0.$$

Conversely, if $d_\mathbb{S}(x, y) = 0$, then $0 = d_\mathbb{S}(x, y) = \arccos \langle x, y \rangle$ and thus $\langle x, y \rangle = 1$. Additionally, we obtain

$$\|x - y\|^2 = \|x\|^2 - 2 \langle x, y \rangle + \|y\|^2 = 2 - 2 = 0$$

and hence $x = y$. Further,

$$d_\mathbb{S}(x, y) = \arccos \langle x, y \rangle = \arccos \langle y, x \rangle = d_\mathbb{S}(y, x).$$

Moreover, $d_\mathbb{S}$ satisfies the triangle inequality. Therefore, $(\mathbb{S}, d_\mathbb{S})$ is a metric space. □

Note that

$$1 - \cos d_\mathbb{S}(x, y) = 1 - \langle x, y \rangle = \frac{1}{2} \langle x - y, x - y \rangle$$

for all $x, y \in \mathbb{S}$.

Theorem 2.2.2. $(\mathbb{S}, d_\mathbb{S})$ *is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence of $\mathbb{S}$. Then, there is a nonnegative real sequence $\{\alpha_n\}$ such that $\lim_{n\to\infty} \alpha_n = 0$ and that $d_\mathbb{S}(x_m, x_n) \leq \alpha_n$ for each $m, n \in \mathbb{N}$ with $m \geq n$. From the definition of $d_\mathbb{S}$,

$$\|x_m - x_n\|^2 = 2 - 2 \cos d_\mathbb{S}(x_m, x_n) \leq 2 - 2 \cos \alpha_n.$$

It means that $\{x_n\}$ is a Cauchy sequence of $\mathbb{E}^\infty$. Since $\mathbb{E}^\infty$ is complete and $\mathbb{S}$ is closed, $\{x_n\}$ converges strongly to some $x_0 \in \mathbb{S}$. Therefore, since

$$d_\mathbb{S}(x_n, x_0) = \arccos \langle x_n, x_0 \rangle \to \arccos \langle x_0, x_0 \rangle = 0,$$

$\{x_n\}$ converges to $x_0 \in \mathbb{S}$ with the distance $d_\mathbb{S}$. Consequently, $(\mathbb{S}, d_\mathbb{S})$ is a complete metric space. □

Lemma 2.2.3. *Let $\mathbb{S}$ be the unit sphere of $\mathbb{E}^\infty$ and K a subset of $\mathbb{S}$. Then, the following hold:*

(i) *K is closed on $\mathbb{E}^\infty$ if it is closed on $\mathbb{S}$;*
(ii) *K is compact on $\mathbb{S}$ if it is compact on $\mathbb{E}^\infty$.*

Proof. They obviously hold if K is empty. Assume that K is nonempty. We first show (i). Suppose that K is closed on $\mathbb{S}$. Take a sequence $\{x_n\}$ of K such that $\|x_n - x_0\| \to 0$ for $x_0 \in \mathbb{E}^\infty$. Since $\mathbb{S}$ is closed on $\mathbb{E}^\infty$, we have $x_0 \in \mathbb{S}$. Further,

$$0 \leq 1 - \cos d_{\mathbb{S}}(x_n, x_0) = \frac{1}{2}\|x_n - x_0\|^2$$

for all $n \in \mathbb{N}$. Therefore, $d_{\mathbb{S}}(x_n, x_0) \to 0$ as $n \to \infty$ and hence $x_0 \in K$ since K is closed on $\mathbb{S}$. Consequently, K is closed on $\mathbb{E}^\infty$. We next show (ii). Suppose that K is compact on $\mathbb{E}^\infty$. Take a sequence $\{y_n\}$ of K. Since K is compact on $\mathbb{E}^\infty$, there exist a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ and $y_0 \in K$ such that $\|y_{n_i} - y_0\| \to 0$. Then, by the same calculation above, we have $d_{\mathbb{S}}(y_{n_i}, y_0) \to 0$ as $i \to \infty$. Consequently, K is compact on $\mathbb{S}$. □

We next prove that the unit sphere $\mathbb{S}$ of $\mathbb{E}^\infty$ is a CAT(1) space.

Lemma 2.2.4. *Let $(\mathbb{S}, d_{\mathbb{S}})$ be the unit sphere of $\mathbb{E}^\infty$. For $x, y \in \mathbb{S}$ with $d_{\mathbb{S}}(x,y) \in \,]0,\pi[$, define a mapping γ_{xy} from $[0, d_{\mathbb{S}}(x,y)]$ into $\mathbb{E}^\infty$ by*

$$\gamma_{xy}(t) = \frac{\sin(d_{\mathbb{S}}(x,y) - t)}{\sin d_{\mathbb{S}}(x,y)} x + \frac{\sin t}{\sin d_{\mathbb{S}}(x,y)} y$$

for each $t \in [0, d_{\mathbb{S}}(x,y)]$. Then,

$$\langle \gamma_{xy}(s), \gamma_{xy}(t) \rangle = \cos |s - t|$$

for all $s, t \in [0, d_{\mathbb{S}}(x,y)]$.

Proof. Let $x, y \in \mathbb{S}$ with $l = d_{\mathbb{S}}(x,y) \in \,]0,\pi[$. From the definition of γ_{xy}, for $s, t \in [0, l]$,

$$\begin{aligned}
\langle \gamma_{xy}(s), \gamma_{xy}(t) \rangle \sin^2 l &= \langle (\sin(l-s))x + (\sin s)y, (\sin(l-t))x + (\sin t)y \rangle \\
&= \langle (\sin(l-s))x, (\sin(l-t))x \rangle + \langle (\sin s)y, (\sin t)y \rangle \\
&\quad + \langle (\sin(l-s))x, (\sin t)y \rangle + \langle (\sin s)y, (\sin(l-t))x \rangle \\
&= \sin(l-s)\sin(l-t) + \sin s \sin t \\
&\quad + \sin(l-s)\sin t \cos l + \sin(l-t)\sin s \cos l.
\end{aligned}$$

Further,

$$\begin{aligned}
&\sin(l-s)\sin t \cos l + \sin(l-t)\sin s \cos l \\
&= \cos l \, (\sin(l-s)\sin t + \sin(l-t)\sin s) \\
&= \cos l \left(-\frac{\cos(l-(s-t)) - \cos(l-s-t)}{2} - \frac{\cos(l+(s-t)) - \cos(l-s-t)}{2} \right) \\
&= \cos l \left(-\frac{\cos(l-(s-t)) + \cos(l+(s-t))}{2} + \cos(l-s-t) \right) \\
&= -\cos^2 l \cos|s-t| + \cos l \cos(l-s-t)
\end{aligned}$$

and

$$\begin{aligned}
&\sin(l-s)\sin(l-t)+\sin s\sin t \\
&= -\frac{\cos(2l-(s+t))-\cos|s-t|}{2}-\frac{\cos(s+t)-\cos|s-t|}{2} \\
&= -\frac{\cos(2l-(s+t))+\cos(s+t)}{2}+\cos|s-t| \\
&= -\cos l\cos(l-s-t)+\cos|s-t|\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle\gamma_{xy}(s),\gamma_{xy}(t)\rangle\sin^2 l &= \cos|s-t|-\cos^2 l\cos|s-t| \\
&= \cos|s-t|\left(1-\cos^2 l\right)=\cos|s-t|\sin^2 l
\end{aligned}$$

and hence $\langle\gamma_{xy}(s),\gamma_{xy}(t)\rangle=\cos|s-t|$ since $l\in\,]0,\pi[$. □

Theorem 2.2.5. $(\mathbb{S},d_{\mathbb{S}})$ *is a uniquely* π*-geodesic space.*

Proof. Let $x,y\in\mathbb{S}$ with $l=d_{\mathbb{S}}(x,y)\in\,]0,\pi[$. Then, $\langle\gamma_{xy}(t),\gamma_{xy}(t)\rangle=\cos|t-t|=1$ for all $t\in[0,l]$. Thus, $\gamma_{xy}(t)\in\mathbb{S}$ for all $t\in[0,l]$. From the definition of γ_{xy}, we have $\gamma_{xy}(0)=x$ and $\gamma_{xy}(l)=y$. Further, for $s,t\in[0,l]$,

$$d_{\mathbb{S}}(\gamma_{xy}(s),\gamma_{xy}(t))=\arccos\langle\gamma_{xy}(s),\gamma_{xy}(t)\rangle=|s-t|\,.$$

It means that γ_{xy} is a geodesic from x to y. Let γ be a geodesic from x to y. Then, for all $t\in[0,l]$,

$$\begin{aligned}
\cos d_{\mathbb{S}}(\gamma_{xy}(t),\gamma(t))\sin l &= \langle(\sin(l-t))x+(\sin t)y,\gamma(t)\rangle \\
&= \langle x,\gamma(t)\rangle\sin(l-t)+\langle y,\gamma(t)\rangle\sin t \\
&= \cos d_{\mathbb{S}}(\gamma(0),\gamma(t))\sin(l-t)+\cos d_{\mathbb{S}}(\gamma(l),\gamma(t))\sin t \\
&= \cos t\sin(l-t)+\cos(l-t)\sin t \\
&= \sin((l-t)+t)=\sin l.
\end{aligned}$$

Since $l\in\,]0,\pi[$, we get $\cos d_{\mathbb{S}}(\gamma_{xy}(t),\gamma(t))=1$. It implies that $\gamma_{xy}\equiv\gamma$. Hence, $(\mathbb{S},d_{\mathbb{S}})$ is a uniquely π-geodesic space. □

Note that for all $x,y\in\mathbb{S}$ with $l=d_{\mathbb{S}}(x,y)\in\,]0,\pi[$ and $t\in[0,1]$,

$$tx\oplus(1-t)y=\gamma_{xy}((1-t)l)=\frac{\sin(tl)}{\sin l}x+\frac{\sin((1-t)l)}{\sin l}y=(t)_l^1x+(1-t)_l^1y,$$

where $(\cdot)_l^1$ is an adjuster.

Theorem 2.2.6. $(\mathbb{S},d_{\mathbb{S}})$ *is a* CAT(1) *space.*

Proof. Fix $x,y,z\in\mathbb{S}$ with $d_{\mathbb{S}}(y,z)+d_{\mathbb{S}}(z,x)+d_{\mathbb{S}}(x,y)<2\pi$ and $t\in[0,1]$ arbitrarily. We remark that $d_{\mathbb{S}}(x,y)<\pi$ and thus there exists a geodesic γ_{xy} from x to y. Set $l=d_{\mathbb{S}}(x,y)$. Then,

$$\cos d_{\mathbb{S}}(tx\oplus(1-t)y,z)\sin l=\langle(\sin(tl))x+(\sin((1-t)l))y,z\rangle$$

$$
\begin{aligned}
&= \langle x, z\rangle \sin(tl) + \langle y, z\rangle \sin((1-t)l) \\
&= \cos d_{\mathbb{S}}(x, z) \sin(tl) + \cos d_{\mathbb{S}}(y, z) \sin((1-t)l).
\end{aligned}
$$

Therefore, $\mathbb{S}$ is a CAT(1) space. □

Consequently, we obtain the following result:

Theorem 2.2.7. *The unit sphere $\mathbb{S}$ of $\mathbb{E}^\infty$ is a complete* CAT(1) *space. Moreover,*

$$
\begin{aligned}
&\cos d_{\mathbb{S}}(tx \oplus (1-t)y, z) \sin d_{\mathbb{S}}(x, y) \\
&= \cos d_{\mathbb{S}}(x, z) \sin(t d_{\mathbb{S}}(x, y)) + \cos d_{\mathbb{S}}(y, z) \sin((1-t) d_{\mathbb{S}}(x, y))
\end{aligned}
$$

for all $x, y, z \in \mathbb{S}$ with $d_{\mathbb{S}}(y, z) + d_{\mathbb{S}}(z, x) + d_{\mathbb{S}}(x, y) < 2\pi$ and $t \in [0, 1]$.

Remark 5. Let $\kappa > 0$ and

$$
\mathbb{S}^\infty_\kappa = \left\{ x \in \mathbb{E}^\infty \;\middle|\; \|x\| = \frac{1}{\sqrt{\kappa}} \right\} = \left\{ x \in \mathbb{E}^\infty \;\middle|\; \langle x, x\rangle = \frac{1}{\kappa} \right\}.
$$

For $x \in \mathbb{S}^\infty_\kappa$, we know that $\sqrt{\kappa}x \in \mathbb{S}^\infty_1 = \mathbb{S}$. Indeed,

$$
\|\sqrt{\kappa}x\|^2 = \kappa\|x\|^2 = 1.
$$

We define a distance $d_{\mathbb{S}^\infty_\kappa}$ on $\mathbb{S}^\infty_\kappa$ as

$$
d_{\mathbb{S}^\infty_\kappa}(x, y) = \frac{\arccos(\kappa \langle x, y\rangle)}{\sqrt{\kappa}} = \frac{\arccos(\langle \sqrt{\kappa}x, \sqrt{\kappa}y\rangle)}{\sqrt{\kappa}} = \frac{d_{\mathbb{S}}(\sqrt{\kappa}x, \sqrt{\kappa}y)}{\sqrt{\kappa}}
$$

for each $x, y \in \mathbb{S}^\infty_\kappa$. Then, $(\mathbb{S}^\infty_\kappa, d_{\mathbb{S}^\infty_\kappa})$ is a complete CAT(κ) space for $\kappa > 0$.

Remark 6. Let

$$
\mathbb{S}_+ = \{x = (x_i) \in \mathbb{S} \mid (x_i) \subset \,]0, \infty[\} \subset \mathbb{S}.
$$

Then, $(\mathbb{S}_+, d_{\mathbb{S}})$ is a metric space. For $x = (x_i), y = (y_i) \in \mathbb{S}_+$ with $l = d_{\mathbb{S}}(x, y) \in \,]0, \pi[$, a geodesic γ_{xy} on $\mathbb{S}$ is also a geodesic on $\mathbb{S}_+$. Indeed, it is isometric, $\gamma_{xy}(0) = x$ and $\gamma_{xy}(l) = y$. Moreover, for all $i \in \mathbb{N}$ and $t \in [0, l]$,

$$
\frac{\sin(l-t)}{\sin l} x_i + \frac{\sin t}{\sin l} y_i > 0.
$$

Thus, $\gamma_{xy}(t) \in \mathbb{S}_+$. We also obtain $\mathbb{S}_+$ is a uniquely π-geodesic space. Since $\mathbb{S}$ is a CAT(1) space, $\mathbb{S}_+$ is also CAT(1) space. Let $u = (u_i), v = (v_i) \in \mathbb{S}_+$. Then,

$$
\cos d_{\mathbb{S}}(u, v) = \langle u, v\rangle = \sum_{i=1}^{\infty} u_i v_i > 0.
$$

Since $d_{\mathbb{S}}(u, v) \in [0, \pi]$, we get $d_{\mathbb{S}}(u, v) < \pi/2$. It means that $\mathbb{S}_+$ is an admissible CAT(1) space.

2.3 Hyperboloids

Let $\mathbb{E}^\infty$ be the infinite dimensional Euclidean space. We define the Minkowski inner product $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{E}^\infty$ by

$$\langle x \,|\, y \rangle = -x_1 y_1 + \sum_{i=2}^{\infty} x_i y_i$$

for each $x = (x_i), y = (y_i) \in \mathbb{E}^\infty$. Then, the following hold:

- $\langle x \,|\, y \rangle = \langle y \,|\, x \rangle$ for any $x, y \in \mathbb{E}^\infty$;
- $\langle \alpha x + \beta y \,|\, z \rangle = \alpha \langle x \,|\, z \rangle + \beta \langle y \,|\, z \rangle$ for any $x, y, z \in \mathbb{E}^\infty$ and $\alpha, \beta \in \mathbb{R}$.

Furthermore, for $x = (x_i), y = (y_i) \in \mathbb{E}^\infty$,

$$\langle x \,|\, y \rangle = -x_1 y_1 + \sum_{i=2}^{\infty} x_i y_i = -2x_1 y_1 + \sum_{i=1}^{\infty} x_i y_i = -2x_1 y_1 + \langle x, y \rangle .$$

It implies that $\langle x \,|\, x \rangle = -2x_1^2 + \|x\|^2$, where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the inner product and the norm defined on $\mathbb{E}^\infty$, respectively.

Lemma 2.3.1. *Let $\mathbb{E}^\infty$ be the infinite dimensional Euclidean space. If sequences $\{x_n\}$ and $\{y_n\}$ of $\mathbb{E}^\infty$ converge strongly to $x, y \in \mathbb{E}^\infty$, respectively, then $\{\langle x_n \,|\, y_n \rangle\}$ converges to $\langle x \,|\, y \rangle$.*

Proof. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ of $\mathbb{E}^\infty$ converge strongly to $x = (x_i), y = (y_i) \in \mathbb{E}^\infty$, respectively. Note that

$$x_n = (x_{n;1}, x_{n;2}, \dots, x_{n;i}, \dots) \text{ and } y_n = (y_{n;1}, y_{n;2}, \dots, y_{n;i}, \dots)$$

for each $n \in \mathbb{N}$. Then,

$$(x_{n;1} - x_1)^2 \leq \sum_{i=1}^{\infty} (x_{n;i} - x_i)^2 = \|x_n - x\|^2 \to 0$$

and hence $x_{n;1} \to x_1$ as $n \to \infty$. Similarly, $y_{n;1} \to y_1$ as $n \to \infty$. Then,

$$\begin{aligned}
|\langle x_n \,|\, y_n \rangle - \langle x \,|\, y \rangle| &= |\langle x_n \,|\, y_n - y + y \rangle - \langle x \,|\, y \rangle| = |\langle x_n \,|\, y_n - y \rangle + \langle x_n \,|\, y \rangle - \langle x \,|\, y \rangle| \\
&= |\langle x_n \,|\, y_n - y \rangle + \langle x_n - x \,|\, y \rangle| \\
&\leq |\langle x_n \,|\, y_n - y \rangle| + |\langle x_n - x \,|\, y \rangle| \\
&= |-2x_{n;1}(y_{n;1} - y_1) + \langle x_n, y_n - y \rangle| + |-2(x_{n;1} - x_1)y_1 + \langle x_n - x, y \rangle| \\
&\leq 2\,|x_{n;1}|\,|y_{n;1} - y_1| + \|x_n\|\|y_n - y\| + 2\,|x_{n;1} - x_1|\,|y_1| + \|x_n - x\|\|y\|.
\end{aligned}$$

Therefore, $\langle x_n \,|\, y_n \rangle \to \langle x \,|\, y \rangle$. □

Let

$$\mathbb{H} = \{x = (x_i) \in \mathbb{E}^\infty \mid \langle x \,|\, x \rangle = -1, x_1 > 0\}$$

and we call it the hyperboloid of $\mathbb{E}^\infty$. Note that for $x = (x_i) \in \mathbb{H}$, $x_1 = 1$ if and only if $x_i = 0$ for all $i \in \mathbb{N} \setminus \{1\}$.

Lemma 2.3.2. *Let $\mathbb{H}$ be the hyperboloid of $\mathbb{E}^\infty$. Then, the following hold:*

(i) *For all $x = (x_i) \in \mathbb{H}$, $x_1 \geq 1$;*
(ii) *for all $x, y \in \mathbb{H}$, $1 \leq x_1 y_1 - \sqrt{(x_1^2 - 1)(y_1^2 - 1)} \leq -\langle x \mid y \rangle$;*
(iii) *for all $x, y \in \mathbb{H}$, $\langle x - y \mid x - y \rangle \geq 0$;*
(iv) *for $x = (x_i), y = (y_i) \in \mathbb{H}$, if $\langle x \mid y \rangle = -1$, then $x_1 = y_1$.*

Proof. We first show (i). Fix $x = (x_i) \in \mathbb{H}$. Then, we get

$$-1 = \langle x \mid x \rangle = -x_1^2 + \sum_{i=2}^{\infty} x_i^2 \geq -x_1^2$$

and thus $x_1^2 \geq 1$. Since $x_1 > 0$, we have $x_1 \geq 1$.

We next show (ii) and (iii). Fix $x = (x_i), y = (y_i) \in \mathbb{H}$ arbitrarily. Then,

$$\langle x \mid y \rangle = -x_1 y_1 + \sum_{i=2}^{\infty} x_i y_i \leq -x_1 y_1 + \sqrt{\sum_{i=2}^{\infty} x_i^2} \sqrt{\sum_{i=2}^{\infty} y_i^2} = -x_1 y_1 + \sqrt{(x_1^2 - 1)(y_1^2 - 1)}.$$

Furthermore, since

$$\begin{aligned}(x_1 y_1 - 1)^2 - (x_1^2 - 1)(y_1^2 - 1) &= x_1^2 y_1^2 - 2x_1 y_1 + 1 - (x_1^2 y_1^2 - x_1^2 - y_1^2 + 1) \\ &= x_1^2 - 2x_1 y_1 + y_1^2 = (x_1 - y_1)^2 \geq 0,\end{aligned}$$

we obtain

$$x_1 y_1 - 1 \geq \sqrt{(x_1^2 - 1)(y_1^2 - 1)}.$$

It implies that

$$\langle x \mid y \rangle \leq -x_1 y_1 + \sqrt{(x_1^2 - 1)(y_1^2 - 1)} \leq -1$$

and thus we obtain the desired inequality. Further,

$$\langle x - y \mid x - y \rangle = \langle x \mid x \rangle - 2 \langle x \mid y \rangle + \langle y \mid y \rangle = -2 - 2 \langle x \mid y \rangle \geq 0.$$

Hence, we obtain (ii) and (iii).

We finally prove (iv). For $x = (x_i), y = (y_i) \in \mathbb{H}$, assume that $\langle x \mid y \rangle = -1$. Then,

$$-1 = \langle x \mid y \rangle \leq -x_1 y_1 + \sqrt{(x_1^2 - 1)(y_1^2 - 1)} \leq -1.$$

It implies that $x_1 y_1 - 1 = \sqrt{(x_1^2 - 1)(y_1^2 - 1)}$. Since $(x_1 y_1 - 1)^2 = (x_1^2 - 1)(y_1^2 - 1)$, we have

$$x_1^2 y_1^2 - 2x_1 y_1 + 1 = (x_1 y_1 - 1)^2 = (x_1^2 - 1)(y_1^2 - 1) = x_1^2 y_1^2 - x_1^2 - y_1^2 + 1$$

and hence

$$(x_1 - y_1)^2 = x_1^2 - 2x_1 y_1 - y_1^2 = 0$$

and thus $x_1 = y_1$. Consequently, if $\langle x \mid y \rangle = -1$, then $x_1 = y_1$. □

Lemma 2.3.3. *$\mathbb{H}$ is a closed subset of $\mathbb{E}^\infty$.*

Proof. Take a strong convergent sequence $\{x_n\}$ of $\mathbb{H}$ with a limit $x = (x_i) \in \mathbb{E}^\infty$. Note that

$$x_n = (x_{n;1}, x_{n;2}, \dots, x_{n;i}, \dots)$$

for each $n \in \mathbb{N}$. Then, a real sequence $\{\langle x_n \,|\, x_n \rangle\}$ converges to $\langle x \,|\, x \rangle$. On the other hand, $\langle x_n \,|\, x_n \rangle = -1$ for all $n \in \mathbb{N}$. Therefore, $\langle x \,|\, x \rangle = -1$. Since $\|x_n - x\| \to 0$, we have $x_{n;1} \to x_1$. Since $x_{n;1} \geq 1$ for all $n \in \mathbb{N}$, we get $x_1 \geq 1 > 0$. It implies that $x \in \mathbb{H}$ and therefore $\mathbb{H}$ is closed on $\mathbb{E}^\infty$. ☐

We define a function $d_{\mathbb{H}}$ from $\mathbb{H}^2$ into $[0, \infty[$ as

$$d_{\mathbb{H}}(x, y) = \operatorname{arcosh}(- \langle x \,|\, y \rangle)$$

for each $x, y \in \mathbb{H}$, where $\operatorname{arcosh}\colon\ [1, \infty[\to [0, \infty[$ is the inverse function of $\cosh|_{[0,\infty[}$. Here, $\cosh|_{[0,\infty[}$ is a restriction of the hyperbolic cosine function to $[0, \infty[$.

Theorem 2.3.4. *$(\mathbb{H}, d_{\mathbb{H}})$ is a metric space.*

Proof. Let $x = (x_i), y = (y_i) \in \mathbb{H}$. If $x = y$, then

$$d_{\mathbb{H}}(x, y) = \operatorname{arcosh}(- \langle x \,|\, y \rangle) = \operatorname{arcosh}(- \langle x \,|\, x \rangle) = \operatorname{arcosh} 1 = 0.$$

Inversely, if $d_{\mathbb{H}}(x, y) = 0$, then $1 = \cosh d_{\mathbb{H}}(x, y) = - \langle x \,|\, y \rangle$ and hence $x_1 = y_1$. Thus,

$$\begin{aligned} \|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle &= \langle x \,|\, x \rangle + 2x_1^2 + \langle y \,|\, y \rangle + 2y_1^2 - 2\langle x \,|\, y \rangle - 4x_1 y_1 \\ &= -1 + 2x_1^2 - 1 + 2y_1^2 + 2 - 4x_1 y_1 = 2(x_1 - y_1)^2 = 0 \end{aligned}$$

which implies that $x = y$. Further, $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(y, x)$ and $d_{\mathbb{H}}$ satisfies the triangle inequality. Therefore, $(\mathbb{H}, d_{\mathbb{H}})$ is a metric space. ☐

Note that

$$\cosh d_{\mathbb{H}}(x, y) - 1 = - \langle x \,|\, y \rangle - 1 = \frac{1}{2} \langle x - y \,|\, x - y \rangle$$

for all $x, y \in \mathbb{H}$.

Theorem 2.3.5 (Burger–Iozzi–Monod [7])**.** *$(\mathbb{H}, d_{\mathbb{H}})$ is complete.*

Lemma 2.3.6. *Let $\mathbb{H}$ be the hyperboloid of $\mathbb{E}^\infty$ and K a subset of $\mathbb{H}$. Then, the following hold:*

(i) *K is closed on $\mathbb{E}^\infty$ if it is closed on $\mathbb{H}$;*
(ii) *K is compact on $\mathbb{H}$ if it is compact on $\mathbb{E}^\infty$.*

Proof. They obviously hold if K is empty. Assume that K is nonempty. We first show (i). Suppose that K is closed on $\mathbb{H}$. Take a sequence $\{x_n\}$ of K such that $\|x_n - x_0\| \to 0$ for $x_0 \in \mathbb{E}^\infty$. Note that $\langle x_n, x_0 \rangle \to \langle x_0, x_0 \rangle$ since $\{x_n\}$ converges weakly to x_0. Since $\mathbb{H}$ is closed on $\mathbb{E}^\infty$, we have $x_0 \in \mathbb{H}$. Moreover,

$$0 \leq (x_{n;1} - x_{0;1})^2 \leq \sum_{i=1}^{\infty} (x_{n;i} - x_{0;i})^2 = \|x_n - x_0\|^2 \to 0$$

and thus $x_{n;1} \to x_{0;1}$ as $n \to \infty$. Since

$$\cosh d_{\mathbb{H}}(x_n, x_0) = -\langle x_n \mid x_0 \rangle = 2x_{n;1}x_{0;1} - \langle x_n, x_0 \rangle ,$$

we obtain

$$1 \leq \cosh d_{\mathbb{H}}(x_n, x_0) = 2x_{n;1}x_{0;1} - \langle x_n, x_0 \rangle \to 2x_{0;1}^2 - \langle x_0, x_0 \rangle = -\langle x_0 \mid x_0 \rangle = 1.$$

Therefore, $d_{\mathbb{H}}(x_n, x_0) \to 0$ as $n \to \infty$ and hence $x_0 \in K$ since K is closed on $\mathbb{H}$. Consequently, K is closed on $\mathbb{E}^\infty$. We next show (ii). Suppose that K is compact on $\mathbb{E}^\infty$. Take a sequence $\{y_n\}$ of K. Since K is compact on $\mathbb{E}^\infty$, there exist a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ and $y_0 \in K$ such that $\|y_{n_i} - y_0\| \to 0$. Then, by the same calculation above, we have $d_{\mathbb{H}}(y_{n_i}, y_0) \to 0$ as $i \to \infty$. Consequently, K is compact on $\mathbb{H}$. □

We next prove that the hyperboloid $\mathbb{H}$ on $\mathbb{E}^\infty$ is a CAT(-1) space.

Lemma 2.3.7. *Let $(\mathbb{H}, d_{\mathbb{H}})$ be the hyperboloid of $\mathbb{E}^\infty$. For $x, y \in \mathbb{H}$ with $d_{\mathbb{H}}(x, y) > 0$, define a mapping γ_{xy} from $[0, d_{\mathbb{H}}(x, y)]$ into $\mathbb{E}^\infty$ as*

$$\gamma_{xy}(t) = \frac{\sinh(d_{\mathbb{H}}(x, y) - t)}{\sinh d_{\mathbb{H}}(x, y)} x + \frac{\sinh t}{\sinh d_{\mathbb{H}}(x, y)} y$$

for each $t \in [0, d_{\mathbb{H}}(x, y)]$. Then,

$$\langle \gamma_{xy}(s) \mid \gamma_{xy}(t) \rangle = -\cosh |s - t|$$

for all $s, t \in [0, d_{\mathbb{H}}(x, y)]$.

Proof. Let $x, y \in \mathbb{H}$ with $l = d_{\mathbb{H}}(x, y) > 0$. For $s, t \in [0, l]$,

$$\begin{aligned}
-\langle \gamma_{xy}(s) \mid \gamma_{xy}(t) \rangle \sinh^2 l &= -\langle (\sinh(l - s))x + (\sinh s)y \mid (\sinh(l - t))x + (\sinh t)y \rangle \\
&= -\langle (\sinh(l - s))x \mid (\sinh(l - t))x \rangle - \langle (\sinh s)y \mid (\sinh t)y \rangle \\
&\quad - \langle (\sinh(l - s))x \mid (\sinh t)y \rangle - \langle (\sinh s)y \mid (\sinh(l - t))x \rangle \\
&= \sinh(l - s)\sinh(l - t) + \sinh s \sinh t \\
&\quad + \sinh(l - s)\sinh t \cosh l + \sinh(l - t)\sinh s \cosh l.
\end{aligned}$$

Further,

$$\begin{aligned}
&\sinh(l - s)\sinh t \cosh l + \sinh(l - t)\sinh s \cosh l \\
&= \cosh l \left(\sinh(l - s)\sinh t + \sinh(l - t)\sinh s\right) \\
&= \cosh l \left(\frac{\cosh(l - (s - t)) - \cosh(l - s - t)}{2} + \frac{\cosh(l + (s - t)) - \cosh(l - s - t)}{2} \right) \\
&= \cosh l \left(\frac{\cosh(l - (s - t)) + \cosh(l + (s - t))}{2} - \cosh(l - s - t) \right) \\
&= \cosh^2 l \cosh |s - t| - \cosh l \cosh(l - s - t)
\end{aligned}$$

and

$$\begin{aligned}
&\sinh(l - s)\sinh(l - t) + \sinh s \sinh t \\
&= \frac{\cosh(2l - (s + t)) - \cosh |s - t|}{2} + \frac{\cosh(s + t) - \cosh |s - t|}{2}
\end{aligned}$$

$$= \frac{\cosh(2l - (s+t)) + \cosh(s+t)}{2} - \cosh|s-t|$$
$$= \cosh l \cosh(l - s - t) - \cosh|s-t|.$$

Consequently,

$$-\langle \gamma_{xy}(s) \mid \gamma_{xy}(t) \rangle \sinh^2 l = -\cosh|s-t| + \cosh^2 l \cosh|s-t|$$
$$= \cosh|s-t| \left(\cosh^2 l - 1\right) = \cosh|s-t| \sinh^2 l$$

and hence $\langle \gamma_{xy}(s) \mid \gamma_{xy}(t) \rangle = -\cosh|s-t|$ since $l > 0$. □

Theorem 2.3.8. $(\mathbb{H}, d_{\mathbb{H}})$ *is a uniquely geodesic space.*

Proof. Let $x, y \in \mathbb{H}$ with $l = d_{\mathbb{H}}(x,y) > 0$. From the definition of γ_{xy}, we get

$$\frac{\sinh(l-t)}{\sinh l} x_1 + \frac{\sinh t}{\sinh l} y_1 > 0.$$

Further, $\langle \gamma_{xy}(t) \mid \gamma_{xy}(t) \rangle = -\cosh|t-t| = -1$ for all $t \in [0,l]$. It means that $\gamma_{xy}(t) \in \mathbb{H}$ for all $t \in [0,l]$. From the definition of γ_{xy}, we have $\gamma_{xy}(0) = x$ and $\gamma_{xy}(l) = y$. Additionally,

$$d_{\mathbb{H}}(\gamma_{xy}(s), \gamma_{xy}(t)) = \operatorname{arcosh}(-\langle \gamma_{xy}(s) \mid \gamma_{xy}(t) \rangle) = |s-t|$$

for all $s, t \in [0,l]$. It means that γ_{xy} is a geodesic from x to y. Let γ be a geodesic from x to y. Then, for all $t \in [0,l]$,

$$\cosh d_{\mathbb{H}}(\gamma_{xy}(t), \gamma(t)) \sinh l = -\langle (\sinh(l-t))x + (\sinh t)y \mid \gamma(t) \rangle$$
$$= -\langle x \mid \gamma(t) \rangle \sinh(l-t) - \langle y \mid \gamma(t) \rangle \sinh t$$
$$= \cosh d_{\mathbb{H}}(\gamma(0), \gamma(t)) \sinh(l-t) + \cosh d_{\mathbb{H}}(\gamma(l), \gamma(t)) \sinh t$$
$$= \cosh t \sinh(l-t) + \cosh(l-t) \sinh t$$
$$= \sinh((l-t)+t) = \sinh l.$$

Since $l > 0$, we get $\cosh d_{\mathbb{H}}(\gamma_{xy}(t), \gamma(t)) = 1$. It implies that $\gamma_{xy} \equiv \gamma$. Hence, $(\mathbb{H}, d_{\mathbb{H}})$ is a uniquely geodesic space. □

Note that for all $x, y \in \mathbb{H}$ with $l = d_{\mathbb{H}}(x,y) > 0$ and $t \in [0,1]$,

$$tx \oplus (1-t)y = \gamma_{xy}((1-t)l) = \frac{\sinh(tl)}{\sinh l} x + \frac{\sinh((1-t)l)}{\sinh l} y = (t)_l^{-1} x + (1-t)_l^{-1} y,$$

where $(\cdot)_l^{-1}$ is an adjuster.

Theorem 2.3.9. $(\mathbb{H}, d_{\mathbb{H}})$ *is a* CAT(-1) *space.*

Proof. Fix $x, y, z \in \mathbb{H}$ and $t \in [0,1]$ arbitrarily. Set $l = d_{\mathbb{H}}(x,y)$. Then,

$$\cosh d_{\mathbb{H}}(tx \oplus (1-t)y, z) \sinh l = -\langle (\sinh(tl))x + (\sinh((1-t)l))y \mid z \rangle$$
$$= -\langle x \mid z \rangle \sinh(tl) - \langle y \mid z \rangle \sinh((1-t)l)$$
$$= \cosh d_{\mathbb{H}}(x,z) \sinh(tl) + \cosh d_{\mathbb{H}}(y,z) \sinh((1-t)l).$$

Therefore, $\mathbb{H}$ is a CAT(-1) space. □

Consequently, we have the following result:

Theorem 2.3.10. *The hyperboloid* $\mathbb{H}$ *of* $\mathbb{E}^\infty$ *is a complete* CAT(-1) *space. Moreover,*

$$\begin{aligned}&\cosh d_{\mathbb{H}}(tx \oplus (1-t)y, z) \sinh d_{\mathbb{H}}(x,y) \\ &= \cosh d_{\mathbb{H}}(x,z) \sinh(t d_{\mathbb{H}}(x,y)) + \cosh d_{\mathbb{H}}(y,z) \sinh((1-t) d_{\mathbb{H}}(x,y))\end{aligned}$$

for all $x, y, z \in \mathbb{H}$ *and* $t \in [0,1]$.

Remark 7. Let $\kappa < 0$ and

$$\mathbb{H}^\infty_\kappa = \left\{ x = (x_i) \in \mathbb{E}^\infty \;\middle|\; \langle x \mid x \rangle = \frac{1}{\kappa}, x_1 > 0 \right\}.$$

For $x = (x_i) \in \mathbb{H}^\infty_\kappa$, we know $\sqrt{-\kappa} x \in \mathbb{H}^\infty_{-1} = \mathbb{H}$. Indeed,

$$\left\langle \sqrt{-\kappa} x \;\middle|\; \sqrt{-\kappa} x \right\rangle = -\kappa \langle x \mid x \rangle = -1$$

and $\sqrt{-\kappa} x_1 > 0$. We define a distance $d_{\mathbb{H}^\infty_\kappa}$ on $\mathbb{H}^\infty_\kappa$ as

$$d_{\mathbb{H}^\infty_\kappa}(x,y) = \frac{\operatorname{arcosh}(\kappa \langle x \mid y \rangle)}{\sqrt{-\kappa}} = \frac{\operatorname{arcosh}(-\left\langle \sqrt{-\kappa} x \;\middle|\; \sqrt{-\kappa} y \right\rangle)}{\sqrt{-\kappa}} = \frac{d_{\mathbb{H}}(\sqrt{-\kappa} x, \sqrt{-\kappa} y)}{\sqrt{-\kappa}}$$

for each $x, y \in \mathbb{H}^\infty_\kappa$. Then, $(\mathbb{H}^\infty_\kappa, d_{\mathbb{H}^\infty_\kappa})$ is a complete CAT(κ) space for $\kappa < 0$.

2.4 Model spaces

For $\kappa \in \mathbb{R}$, we define the infinite dimensional model space $(M^\infty_\kappa, d_{M^\infty_\kappa})$ as

$$M^\infty_\kappa = \begin{cases} \mathbb{S}^\infty_\kappa & (\kappa > 0); \\ \mathbb{E}^\infty & (\kappa = 0); \\ \mathbb{H}^\infty_\kappa & (\kappa < 0) \end{cases}$$

with a distance function defined by

$$d_{M^\infty_\kappa}(x,y) = \begin{cases} \dfrac{\arccos(\kappa \langle x, y \rangle)}{\sqrt{\kappa}} & (\kappa > 0); \\ \sqrt{\langle x-y, x-y \rangle} & (\kappa = 0); \\ \dfrac{\operatorname{arcosh}(\kappa \langle x \mid y \rangle)}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases}$$

for each $x, y \in M^\infty_\kappa$. Then, M^∞_κ is a complete CAT(κ) space for $\kappa \in \mathbb{R}$. Additionally, for any $x, y \in M^\infty_\kappa$ with $l = d_{M^\infty_\kappa}(x,y) \in \,]0, D_\kappa[$, a unique geodesic γ_{xy} from x to y is defined by

$$\gamma_{xy}(t) = \frac{c'_\kappa(l-t)}{c'_\kappa(l)} x + \frac{c'_\kappa(t)}{c'_\kappa(l)} y$$

for each $t \in [0, l]$. Moreover,

$$D_\kappa = \operatorname{diam} M^\infty_\kappa = \sup_{u,v \in M^\infty_\kappa} d_{M^\infty_\kappa}(u,v)$$

and

$$tx \oplus (1-t)y = (t)_l^\kappa x + (1-t)_l^\kappa y$$

for all $x, y \in M_\kappa^\infty$ and $t \in [0,1]$, where $l = d_{M_\kappa^\infty}(x,y) < D_\kappa$.

Theorem 2.4.1. *Let M_κ^∞ be the infinite dimensional model space for $\kappa \in \mathbb{R}$. Then,*

$$\begin{aligned}\phi_\kappa(tx \oplus (1-t)y, z) &= (t)_l^\kappa \phi_\kappa(x,z) + (1-t)_l^\kappa \phi_\kappa(y,z) \\ &\quad - (t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) - (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y)\end{aligned}$$

and

$$\phi_\kappa(tx \oplus (1-t)y, z) = (t)_l^\kappa \phi_\kappa(x,z) + (1-t)_l^\kappa \phi_\kappa(y,z) - \frac{(t)_{l/2}^\kappa (1-t)_{l/2}^\kappa \phi_\kappa(x,y)}{c''_\kappa(l/2)}$$

for every $x, y, z \in M_\kappa^\infty$ with $d_{M_\kappa^\infty}(y,z) + d_{M_\kappa^\infty}(z,x) + l < 2D_\kappa$ and $t \in [0,1]$, where $l = d_{M_\kappa^\infty}(x,y)$.

Lemma 2.4.2. *Let M_κ^∞ be the infinite dimensional model space for $\kappa \in \mathbb{R}$ and K a subset of M_κ^∞. Then, the following hold:*

(i) *K is closed on $\mathbb{E}^\infty$ if it is closed on M_κ^∞;*
(ii) *K is compact on M_κ^∞ if it is compact on $\mathbb{E}^\infty$.*

In what follows, we consider the finite dimensional model space.

Fix $n \in \mathbb{N}$. We define a subset E^n of the infinite dimensional Euclidean space $\mathbb{E}^\infty$ as follows: For $x = (x_i) \in \mathbb{E}^\infty$, $x \in E^n$ if and only if $x_m = 0$ for all $m \geq n+1$. Let $\mathbb{E}^n$ be the n-dimensional Euclidean space. We define a mapping ι from $\mathbb{E}^n$ to E^n by

$$\iota x = (x_1, x_2, \dots, x_n, 0, 0, \dots) \in E^n$$

for each $x = (x_1, x_2, \dots, x_n) \in \mathbb{E}^n$. Then, for all $x = (x_1, \dots, x_n) \in \mathbb{E}^n$,

$$\|\iota x\|_{\mathbb{E}^\infty}^2 = \sum_{i=1}^\infty x_i^2 = \sum_{i=1}^n x_i^2 = \|x\|_{\mathbb{E}^n}^2.$$

Further, for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{E}^n$ and $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}\iota(\alpha x + \beta y) &= (\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n, 0, \dots) \\ &= \alpha(x_1, \dots, x_n, 0, \dots) + \beta(y_1, \dots, y_n, 0, \dots) = \alpha \iota x + \beta \iota y.\end{aligned}$$

Moreover, ι is surjective. Consequently, ι is a surjective isometric mapping and thus E^n and $\mathbb{E}^n$ are isometric. Henceforth, we denote $\mathbb{E}^n$ as E^n. That is, $\mathbb{E}^n$ is a linear subspace of $\mathbb{E}^\infty$.

Let M_κ^∞ be the infinite dimensional model space for $\kappa \neq 0$. Fix $n \in \mathbb{N}$ arbitrarily. We define a subset M_κ^n of M_κ^∞ by

$$M_\kappa^n = M_\kappa^\infty \cap \mathbb{E}^{n+1}.$$

We define a distance $d_{M_\kappa^n}$ on M_κ^n by

$$d_{M_\kappa^n}(x,y) = d_{M_\kappa^\infty}(x,y)$$

for each $x, y \in M_\kappa^n$. Then, $(M_\kappa^n, d_{M_\kappa^n})$ is a complete metric space. Fix $x, y \in M_\kappa^n$ with $l = d_{M_\kappa^n}(x, y) \in]0, D_\kappa[$. Then, there exists a unique geodesic γ_{xy} on M_κ^∞. We notice that for fixed $t \in [0, l]$,

$$\gamma_{xy}(t) = \frac{c'_\kappa(l-t)}{c'_\kappa(l)}x + \frac{c'_\kappa(t)}{c'_\kappa(l)}y.$$

We remark that $x = (x_1, \ldots, x_n, x_{n+1}, 0, \ldots)$ and $y = (y_1, \ldots, y_n, y_{n+1}, 0, \ldots)$. Thus,

$$\gamma_{xy}(t) = \left(\frac{c'_\kappa(l-t)}{c'_\kappa(l)}x_1 + \frac{c'_\kappa(t)}{c'_\kappa(l)}y_1, \ldots, \frac{c'_\kappa(l-t)}{c'_\kappa(l)}x_{n+1} + \frac{c'_\kappa(t)}{c'_\kappa(l)}y_{n+1}, 0, \ldots\right) \in \mathbb{E}^{n+1}.$$

Since $\gamma_{xy}(t) \in M_\kappa^\infty$, we have $\gamma_{xy}(t) \in M_\kappa^n = M_\kappa^\infty \cap \mathbb{E}^{n+1}$. Consequently, M_κ^n is a uniquely D_κ-geodesic space and therefore it is a complete CAT(κ) space. Set $M_0^n = \mathbb{E}^n$. For $\kappa \in \mathbb{R}$ and $n \in \mathbb{N}$, we call M_κ^n the n-dimensional model space. Hence, each finite dimensional model space is included in the infinite dimensional model space.

Chapter 3

Convex sets and delta-convergence

In this chapter, we first consider convexity of sets of a geodesic space. After that, we define metric projections from a geodesic space onto a nonempty closed convex subset. We next consider a notion of convergence, which is weaker than convergence with a distance, and investigate some useful properties. At the end of the chapter, we prove a lemma corresponding to the KKM lemma in the setting of CAT(κ) spaces.

3.1 Convex sets

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and S a subset of X. We say that S is convex if

$$tx \oplus (1-t)y \in S$$

for every $x, y \in S$ and $t \in [0,1]$. For a family of convex subsets $\{S_i \mid i \in I\}$ of X, its intersection $\bigcap_{i \in I} S_i$ is also convex. A convex subset of an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ is also an admissible CAT(κ) space. Namely, A closed convex subset of an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ is an admissible complete CAT(κ) space. The convex hull $\operatorname{co} S$ of S is defined by

$$\operatorname{co} S = \bigcup_{j=1}^{\infty} W_j,$$

where $W_1 = S$ and $W_{j+1} = \{tu_j \oplus (1-t)v_j \mid u_j, v_j \in W_j, t \in [0,1]\}$ for $j \in \mathbb{N}$.

Lemma 3.1.1. *Let X be an admissible* CAT(κ) *space and S a subset of X. Then, $\operatorname{co} S$ is convex. Further, for a convex subset C of X which contains S, the convex hull $\operatorname{co} S$ is included in C.*

Proof. If S is empty, then $\operatorname{co} S$ is also empty. Thus, $\operatorname{co} S$ is a convex set. Suppose that S is nonempty. We remark that $\operatorname{co} S = \bigcup_{j=1}^{\infty} W_j$, where $W_1 = S$ and

$$W_{j+1} = \{tu_j \oplus (1-t)v_j \mid u_j, v_j \in W_j, t \in [0,1]\}$$

for $j \in \mathbb{N}$. Note that $W_j \subset W_{j+1}$ for every $j \in \mathbb{N}$. Let $x, y \in \operatorname{co} S$ and $s \in [0,1]$. Then, there exist $j_x, j_y \in \mathbb{N}$ such that $x \in W_{j_x}$ and $y \in W_{j_y}$. Let $j_0 = j_x + j_y$. Then, $x, y \in W_{j_0}$. Therefore, we obtain $sx \oplus (1-s)y \in W_{j_0+1} \subset \operatorname{co} S$. It means that $\operatorname{co} S$ is convex.

We next show $\operatorname{co} S \subset C$ for a convex subset C of X which satisfies that $S \subset C$. We prove the statement by induction. We know $W_1 = S \subset C$. Suppose that $W_j \subset C$ for fixed $j \in \mathbb{N}$.

Take $x \in W_{j+1}$ arbitrarily. Then, there exist $u, v \in W_j$ and $t \in [0,1]$ such that $x = tu \oplus (1-t)v$. Since $u, v \in W_j \subset C$ and C is convex, we have $x \in C$ and hence $W_{j+1} \subset C$. Therefore, we have $W_j \subset C$ for each $j \in \mathbb{N}$, which implies that $\operatorname{co} S = \bigcup_{j=1}^{\infty} W_j \subset C$. ☐

Remark 8. Let X be an admissible CAT(κ) space and S a subset of X. Let $\mathcal{C}_S$ be a family of all convex subsets of X which contain S. That is, $C \in \mathcal{C}_S$ if and only if C is a convex subset of X satisfying that $S \subset C$. Since $\bigcap_{C \in \mathcal{C}_S} C$ is convex and S is included in $\bigcap_{C \in \mathcal{C}_S} C$, from Lemma 3.1.1, $\operatorname{co} S$ is included in $\bigcap_{C \in \mathcal{C}_S} C$ and $\operatorname{co} S \in \mathcal{C}_S$. Therefore, $\operatorname{co} S = \bigcap_{C \in \mathcal{C}_S} C$. Hence, the convex hull of S can be defined by the intersection of all convex sets containing S.

Lemma 3.1.2. *Let X be an admissible* CAT(κ) *space and C a convex set of X. Then, $\operatorname{cl} C$ is convex, where $\operatorname{cl} C$ is the closure of C.*

Proof. If C is empty, then $\operatorname{cl} C$ is also empty. Thus, $\operatorname{cl} C$ is a convex set. Suppose C is nonempty. Let $x, y \in \operatorname{cl} C$, $t \in [0,1]$ and $z = tx \oplus (1-t)y$. Then, there are sequences $\{x_n\}$ and $\{y_n\}$ of C such that $x_n \to x$ and $y_n \to y$. Set $z_n = tx_n \oplus (1-t)y_n \in C$ for each $n \in \mathbb{N}$. We show that $\{z_n\}$ converges to z. Set $l = d(x,y)$ and $l_n = d(x_n, y_n)$. Note that $l_n \to l$ and $(\tau)_{l_n}^{\kappa} \to (\tau)_l^{\kappa}$ for each $\tau \in [0,1]$. It holds from the parallelogram law of X that

$$\phi_\kappa(z, z_n) \le (t)_{l_n}^{\kappa} \phi_\kappa(z, x_n) + (1-t)_{l_n}^{\kappa} \phi_\kappa(z, y_n) - (t)_{l_n}^{\kappa} c_\kappa((1-t)l_n) - (1-t)_{l_n}^{\kappa} c_\kappa(tl_n)$$

and hence

$$\begin{aligned}\limsup_{n\to\infty} \phi_\kappa(z, z_n) &\le (t)_l^{\kappa} \phi_\kappa(z, x) + (1-t)_l^{\kappa} \phi_\kappa(z, y) - (t)_l^{\kappa} c_\kappa((1-t)l) - (1-t)_l^{\kappa} c_\kappa(tl) \\ &= (t)_l^{\kappa} c_\kappa((1-t)l) + (1-t)_l^{\kappa} c_\kappa(tl) - (t)_l^{\kappa} c_\kappa((1-t)l) - (1-t)_l^{\kappa} c_\kappa(tl) = 0.\end{aligned}$$

It implies that $z_n \to z$ as $n \to \infty$. Since $\{z_n\}$ is a sequence of C, we obtain $z \in \operatorname{cl} C$. Therefore, $\operatorname{cl} C$ is convex. ☐

Remark 9. Let X be an admissible CAT(κ) space and S a subset of X. Let $\overline{\mathcal{C}}_S$ be a family of all closed convex subsets of X which contain S. That is, $C \in \overline{\mathcal{C}}_S$ if and only if C is a closed convex subset of X such that $S \subset C$. Note that $\bigcap_{C \in \overline{\mathcal{C}}_S} C$ is closed and convex, and $S \subset \bigcap_{C \in \overline{\mathcal{C}}_S} C$. Since $\bigcap_{C \in \overline{\mathcal{C}}_S} C$ is convex, $\operatorname{co} S \subset \bigcap_{C \in \overline{\mathcal{C}}_S} C$ and thus $\operatorname{cl} \operatorname{co} S \subset \bigcap_{C \in \overline{\mathcal{C}}_S} C$. Further, from Lemma 3.1.2, we have $\bigcap_{C \in \overline{\mathcal{C}}_S} C \subset \operatorname{cl} \operatorname{co} S$. Hence, $\operatorname{cl} \operatorname{co} S = \bigcap_{C \in \overline{\mathcal{C}}_S} C$. It means that the closed convex hull of S can be a defined by the intersection of all closed convex sets containing S.

Let X be a metric space and T a mapping from X into itself. We denote the set of fixed points of T by $\operatorname{Fix} T$, that is,

$$\operatorname{Fix} T = \{x \in X \mid x = Tx\}.$$

Further, T is called quasinonexpansive if $\operatorname{Fix} T$ is nonempty and $d(u, Tx) \le d(u, x)$ for every $u \in \operatorname{Fix} T$ and $x \in X$.

Lemma 3.1.3. *Let X be an admissible* CAT(κ) *space and T a quasinonexpansive mapping from X into itself. Then, $\operatorname{Fix} T$ is closed and convex.*

Proof. We first show that $\operatorname{Fix} T$ is closed. Take a sequence $\{x_n\}$ of $\operatorname{Fix} T$ which converges to a point $x_0 \in X$. Then, for any $n \in \mathbb{N}$,

$$d(x_0, Tx_0) \le d(x_0, x_n) + d(x_n, Tx_0) \le 2d(x_0, x_n).$$

Letting $n \to \infty$, we get $d(x_0, Tx_0) = 0$. Hence, $x_0 \in \operatorname{Fix} T$ and thus $\operatorname{Fix} T$ is closed. We next show that $\operatorname{Fix} T$ is convex. Let $u, v \in \operatorname{Fix} T$, $t \in [0,1]$ and $l = d(u,v)$. Put $w = tu \oplus (1-t)v$. Then, from the parallelogram law of X, we have

$$\begin{aligned}\phi_\kappa(w, Tw) &\le (t)_l^\kappa \phi_\kappa(u, Tw) + (1-t)_l^\kappa \phi_\kappa(v, Tw) - (t)_l^\kappa \phi_\kappa(u, w) - (1-t)_l^\kappa \phi_\kappa(v, w) \\ &\le (t)_l^\kappa \phi_\kappa(u, w) + (1-t)_l^\kappa \phi_\kappa(v, w) - (t)_l^\kappa \phi_\kappa(u, w) - (1-t)_l^\kappa \phi_\kappa(v, w) \\ &= 0,\end{aligned}$$

which implies that $tu \oplus (1-t)v = w \in \operatorname{Fix} T$. Therefore, $\operatorname{Fix} T$ is convex. □

In what follows, we consider metric projections on CAT(κ) spaces.

For $\kappa \in \mathbb{R}$, we define $A_\kappa \in]0, \infty]$ as

$$A_\kappa = \lim_{\delta \to D_\kappa/2} c_\kappa(\delta) = \begin{cases} \dfrac{1}{\kappa} & (\kappa > 0); \\ \infty & (\kappa \le 0). \end{cases}$$

We know the following:

- $c_\kappa \colon [0, D_\kappa/2[\to [0, A_\kappa[$ is continuous, convex, strictly increasing and bijective;
- $c_\kappa^{-1} \colon [0, A_\kappa[\to [0, D_\kappa/2[$ is continuous and strictly increasing.

Note that $1 - \kappa M > 0$ for any $\kappa \in \mathbb{R}$ and $M \in [0, A_\kappa[$.

Let S be a nonempty set and f a function from S into $]-\infty, \infty]$. We denote the set of minimisers of f by $\operatorname{Min} f$ or $\operatorname{Argmin}_{x \in S} f(x)$, that is,

$$\operatorname{Min} f = \operatorname*{Argmin}_{x \in S} f(x) = \left\{ x \in S \;\middle|\; f(x) = \inf_{y \in S} f(y) \right\}.$$

Theorem 3.1.4. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and C a nonempty closed convex subset of X. Let f be a continuous function from C into $[0, A_\kappa]$ such that $\inf_{y \in C} f(y) < A_\kappa$ and that*

$$f\left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) \le (1/2)_l^\kappa f(y_1) + (1/2)_l^\kappa f(y_2) - 2(1/2)_l^\kappa c_\kappa\left(\frac{l}{2}\right)$$

for each $y_1, y_2 \in C$, where $l = d(y_1, y_2)$. Then, $\operatorname{Min} f$ *consists of exactly one point.*

Proof. Let $M = \inf_{p \in C} f(p) < A_\kappa$. Then, we can take a sequence $\{p_n\}$ of C such that

$$M \le f(p_n) \le M + \frac{1}{n} < \infty$$

for each $n \in \mathbb{N}$. Fix $m, n \in \mathbb{N}$ with $m \ge n$ and put $l = d(p_m, p_n)$. Since C is convex, we get

$$\frac{1}{2}p_m \oplus \frac{1}{2}p_n \in C.$$

From the assumption of f,

$$M \le f\left(\frac{1}{2}p_m \oplus \frac{1}{2}p_n\right) \le (1/2)_l^\kappa f(p_m) + (1/2)_l^\kappa f(p_n) - 2(1/2)_l^\kappa c_\kappa\left(\frac{l}{2}\right)$$

$$\leq (1/2)_l^\kappa \left(M + \frac{1}{m}\right) + (1/2)_l^\kappa \left(M + \frac{1}{n}\right) - 2(1/2)_l^\kappa c_\kappa \left(\frac{l}{2}\right)$$
$$\leq 2(1/2)_l^\kappa \left(M + \frac{1}{n}\right) - 2(1/2)_l^\kappa c_\kappa \left(\frac{l}{2}\right)$$

and hence

$$c_\kappa \left(\frac{l}{2}\right) \leq M + \frac{1}{n} - \frac{1}{2(1/2)_l^\kappa} M = \left(1 - \frac{1}{2(1/2)_l^\kappa}\right) M + \frac{1}{n} = \kappa M c_\kappa \left(\frac{l}{2}\right) + \frac{1}{n}.$$

It means that

$$(1 - \kappa M)\, c_\kappa \left(\frac{l}{2}\right) \leq \frac{1}{n}.$$

Since $1 - \kappa M > 0$ and $l = d(p_m, p_n)$, we obtain

$$d(p_m, p_n) \leq 2 c_\kappa^{-1} \left(\frac{1}{(1 - \kappa M)\, n}\right)$$

for sufficiently large $n \in \mathbb{N}$, which implies that $\{p_n\}$ is a Cauchy sequence. Namely, $\{p_n\}$ converges to some point $p_0 \in C$. Since f is continuous and $f(p_n) \to M$ as $n \to \infty$, we get $f(p_0) = M$. Therefore, $\operatorname{Min} f$ is a nonempty set. Let $p_0, p_0' \in \operatorname{Min} f$ and $l_0 = d(p_0, p_0')$. Then, we have

$$M \leq f\left(\frac{1}{2} p_0 \oplus \frac{1}{2} p_0'\right) \leq 2(1/2)_{l_0}^\kappa M - 2(1/2)_{l_0}^\kappa c_\kappa \left(\frac{l_0}{2}\right)$$

and thus

$$c_\kappa \left(\frac{l_0}{2}\right) \leq M - \frac{1}{2(1/2)_{l_0}^\kappa} M = \left(1 - \frac{1}{2(1/2)_{l_0}^\kappa}\right) M = \kappa M c_\kappa \left(\frac{l_0}{2}\right).$$

Therefore, we obtain

$$(1 - \kappa M) c_\kappa \left(\frac{l_0}{2}\right) \leq 0.$$

It implies that $l_0 = d(p_0, p_0') = 0$. Therefore, $\operatorname{Min} f$ consists of exactly one point. □

Corollary 3.1.5. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and C a nonempty closed convex subset of X. Then, for each $x \in X$, a subset*

$$\operatorname*{Argmin}_{p \in C} d(p, x) = \operatorname*{Argmin}_{p \in C} \phi_\kappa(p, x)$$

of C consists of exactly one point.

Proof. Fix $x \in X$, and define a function f from C into $[0, A_\kappa[$ by

$$f(y) = \phi_\kappa(y, x)$$

for $y \in C$. Since X is admissible, $\inf_{y \in C} f(y) < A_\kappa$. Further, from the parallelogram law of X, for $y_1, y_2 \in C$ with $l = d(y_1, y_2)$,

$$\begin{aligned} f\left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) &= \phi_\kappa\left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2, x\right) \\ &\le (1/2)_l^\kappa \phi_\kappa(y_1, x) + (1/2)_l^\kappa \phi_\kappa(y_2, x) - 2(1/2)_l^\kappa c_\kappa\left(\frac{l}{2}\right) \\ &= (1/2)_l^\kappa f(y_1) + (1/2)_l^\kappa f(y_2) - 2(1/2)_l^\kappa c_\kappa\left(\frac{l}{2}\right). \end{aligned}$$

Therefore, from Theorem 3.1.4, $\operatorname{Min} f$ consists of one point. □

Let C be a nonempty closed convex subset of an admissible complete CAT(κ) space X for $\kappa \in \mathbb{R}$. Then for $x \in X$, there exists a unique point $p_x \in C$ such that

$$\{p_x\} = \operatorname*{Argmin}_{p \in C} d(p, x).$$

We call such a mapping $P_C \colon x \mapsto p_x$ a metric projection from X onto C. Note that $\operatorname{Fix} P_C = C$.

Now, we can prove the following lemma:

Lemma 3.1.6. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and C a nonempty closed convex subset of X. Then, for $x \in X$ and $u \in C$,*

$$\phi_\kappa(u, P_C x) \le \phi_\kappa(u, x) - c_\kappa''(d(u, P_C x))\phi_\kappa(P_C x, x),$$

where P_C is a metric projection from X onto C.

Proof. Let $x \in X$, $u \in C$ and $l = d(u, P_C x)$. From the parallelogram law of X, for $\tau \in \,]0, 1[$,

$$\begin{aligned} 0 &\le \phi_\kappa(\tau u \oplus (1-\tau)P_C x, x) - \phi_\kappa(P_C x, x) \\ &\le (\tau)_l^\kappa \phi_\kappa(u, x) - \left(1 - (1-\tau)_l^\kappa\right)\phi_\kappa(P_C x, x) - (\tau)_l^\kappa \phi_\kappa(u, \tau u \oplus (1-\tau)P_C x). \end{aligned}$$

Dividing both sides by $(\tau)_l^\kappa > 0$ and letting $\tau \searrow 0$, from Lemma 1.1.3, we obtain

$$0 \le \phi_\kappa(u, x) - c_\kappa''(l)\phi_\kappa(P_C x, x) - \phi_\kappa(u, P_C x).$$

Therefore, we obtain the desired result. □

We have the following corollary:

Corollary 3.1.7. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and C a nonempty closed convex subset of X. Then,*

$$d(u, P_C x) \le d(u, x)$$

for every $x \in X$ and $u \in C$, where P_C is a metric projection from X onto C. Namely, P_C is a quasinonexpansive mapping.

3.2 Delta-convergence

In this section, we consider a notion of convergence, which is weaker than convergence with a distance.

Let X be a metric space and $\{x_n\}$ a bounded sequence of X. An asymptotic centre $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathrm{AC}(\{x_n\}) = \operatorname*{Argmin}_{u \in X} \left(\limsup_{n\to\infty} d(u, x_n) \right) = \operatorname*{Argmin}_{u \in X} \left(\limsup_{n\to\infty} \phi_\kappa(u, x_n) \right).$$

For $\kappa \in \mathbb{R}$, we say that $\{x_n\}$ is κ-bounded if

$$\inf_{y \in X} \limsup_{n\to\infty} d(y, x_n) < \frac{D_\kappa}{2}.$$

Every κ-bounded sequence is bounded with the distance d.

Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ and $\{x_n\}$ a κ-bounded sequence of X. From Theorem 3.1.4, its asymptotic centre $\mathrm{AC}(\{x_n\})$ consists of one point.

We have the following fact:

Lemma 3.2.1 (Bačák [5], Espínola–Fernández-León [8], Kirk–Panyanak [30])**.** *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and C a nonempty closed convex subset of X. Let $\{x_n\}$ be a κ-bounded sequence of C. Then,* $\mathrm{AC}(\{x_n\})$ *is included in C.*

Proof. Let $\{x_0\} = \mathrm{AC}(\{x_n\})$. We show that $x_0 \in C$. We know $d(P_C x_0, x_n) \le d(x_0, x_n)$ for each $n \in \mathbb{N}$, where P_C is a metric projection onto C. Therefore, since P_C is quasinonexpansive,

$$\limsup_{n\to\infty} d(P_C x_0, x_n) \le \limsup_{n\to\infty} d(x_0, x_n).$$

Since x_0 is a unique asymptotic centre of $\{x_n\}$, we get $x_0 = P_C x_0 \in C$. □

Let $\{x_n\}$ be a bounded sequence of a metric space X and $x_0 \in X$. We say that $\{x_n\}$ Δ-converges to a Δ-limit x_0 if $\{x_0\}$ is a unique asymptotic centre for any subsequence of $\{x_n\}$. Namely, $\{x_n\}$ Δ-converges to x_0 if and only if $\{x_0\} = \mathrm{AC}(\{x_{n_i}\})$ for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$. We denote it by $x_n \stackrel{\Delta}{\to} x_0$.

Theorem 3.2.2. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $\{x_n\}$ a sequence converging to $x_0 \in X$. Then, $\{x_n\}$ Δ-converges to x_0.*

Proof. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. We remark that $\{x_{n_i}\}$ also converges to x_0 and that it is κ-bounded. Let $\{z\} = \mathrm{AC}(\{x_{n_i}\})$. Then,

$$\limsup_{i\to\infty} d(x_0, x_{n_i}) = 0 \le \limsup_{i\to\infty} d(z, x_{n_i}).$$

Since z is a unique asymptotic centre of $\{x_{n_i}\}$, we get $x_0 = z$ and thus $\{x_n\}$ Δ-converges to x_0. □

From the theorem above, Δ-convergence is weaker than convergence with a distance of a CAT(κ) space. However, in CAT(κ) spaces, every κ-bounded sequence has a Δ-convergent subsequence. The fact is effective and important for our discussions.

Lemma 3.2.3 (Bačák [5], Espínola–Fernández-León [8], Kirk–Panyanak [30]). *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $\{x_n\}$ a κ-bounded sequence of X. Then, $\{x_n\}$ has a Δ-convergent subsequence.*

Proof. Let $\{x_n\}$ be a κ-bounded sequence. Let

$$r_1 = \inf \left\{ \inf_{x \in X} \limsup_{n \to \infty} d(x, u_n) \;\middle|\; \{u_n\} \subset \{x_n\} \right\}.$$

We take a subsequence $\{x_n^1\}$ of $\{x_n\}$ as

$$\inf_{x \in X} \limsup_{n \to \infty} d(x, x_n^1) \leq r_1 + \frac{1}{1}.$$

Further, let

$$r_2 = \inf \left\{ \inf_{x \in X} \limsup_{n \to \infty} d(x, u_n^1) \;\middle|\; \{u_n^1\} \subset \{x_n^1\} \right\}.$$

Then, we can take a subsequence $\{x_n^2\}$ of $\{x_n^1\}$ as

$$\inf_{x \in X} \limsup_{n \to \infty} d(x, x_n^2) \leq r_2 + \frac{1}{2}.$$

In this way, for a sequence $\{x_n^k\}$, let

$$r_{k+1} = \inf \left\{ \inf_{x \in X} \limsup_{n \to \infty} d(x, u_n^k) \;\middle|\; \{u_n^k\} \subset \{x_n^k\} \right\}$$

and take a subsequence $\{x_n^{k+1}\}$ of $\{x_n^k\}$ as

$$\inf_{x \in X} \limsup_{n \to \infty} d(x, x_n^{k+1}) \leq r_{k+1} + \frac{1}{k+1}.$$

Then, $\{r_k\}$ is increasing and bounded above. Therefore, $\{r_k\}$ converges to some real number r. Now, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as $x_{n_k} = x_k^k$ for each $k \in \mathbb{N}$. Fix $i \in \mathbb{N}$ arbitrarily. For sufficiently large $k \in \mathbb{N}$, $\{x_{n_k}\}$ is a subsequence of $\{x_n^i\}$. Thus,

$$r_{i+1} \leq \inf_{x \in X} \limsup_{k \to \infty} d(x, x_{n_k}).$$

Moreover, since

$$\inf_{x \in X} \limsup_{k \to \infty} d(x, x_{n_k}) \leq r_{i+1} + \frac{1}{i+1},$$

letting $i \to \infty$, we obtain

$$\inf_{x \in X} \limsup_{k \to \infty} d(x, x_{n_k}) = r.$$

In the same way, for any subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$, we get

$$\inf_{x \in X} \limsup_{l \to \infty} d(x, x_{n_{k_l}}) = r.$$

Let $\{x_0\} = \mathrm{AC}(\{x_{n_k}\})$. Then, for any subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$, we have

$$\begin{aligned}\limsup_{l\to\infty} d(x_0, x_{n_{k_l}}) \le \limsup_{k\to\infty} d(x_0, x_{n_k}) &= \inf_{x\in X} \limsup_{k\to\infty} d(x, x_{n_k}) \\ &= r = \inf_{x\in X} \limsup_{l\to\infty} d(x, x_{n_{k_l}})\end{aligned}$$

and thus $\{x_0\} = \mathrm{AC}(\{x_{n_{k_l}}\})$. It means that $\{x_{n_k}\}$ Δ-converges to x_0. □

In what follows, we consider the Δ-Kadec–Klee property of a CAT(κ) space. We first obtain the following lemma:

Lemma 3.2.4 (He–Fang–López–Li [9]). *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then, for any $z \in X$,*

$$d(z, x_0) \le \liminf_{n\to\infty} d(z, x_n)$$

whenever a κ-bounded sequence $\{x_n\}$ of X is Δ-convergent to $x_0 \in X$.

Proof. Let $z \in X$ and $l = d(z, x_0)$. Then, we can take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \phi_\kappa(z, x_{n_i}) = \liminf_{n\to\infty} \phi_\kappa(z, x_n).$$

For $\tau \in \,]0,1[$, we have

$$\begin{aligned}&\limsup_{i\to\infty} \phi_\kappa(x_0, x_{n_i}) \\ &\le \limsup_{i\to\infty} \phi_\kappa(\tau z \oplus (1-\tau)x_0, x_{n_i}) \\ &\le (\tau)_l^\kappa \limsup_{i\to\infty} \phi_\kappa(z, x_{n_i}) + (1-\tau)_l^\kappa \limsup_{i\to\infty} \phi_\kappa(x_0, x_{n_i}) - (\tau)_l^\kappa \phi_\kappa(z, \tau z \oplus (1-\tau)x_0)\end{aligned}$$

and hence

$$\begin{aligned}0 &\le (1 - (1-\tau)_l^\kappa) \limsup_{i\to\infty} \phi_\kappa(x_0, x_{n_i}) \\ &\le (\tau)_l^\kappa \limsup_{i\to\infty} \phi_\kappa(z, x_{n_i}) - (\tau)_l^\kappa \phi_\kappa(z, \tau z \oplus (1-\tau)x_0).\end{aligned}$$

Dividing both sides by $(\tau)_l^\kappa > 0$ and letting $\tau \searrow 0$, we obtain

$$\begin{aligned}0 &\le \limsup_{i\to\infty} \phi_\kappa(z, x_{n_i}) - \phi_\kappa(z, x_0) = \lim_{i\to\infty} \phi_\kappa(z, x_{n_i}) - \phi_\kappa(z, x_0) \\ &= \liminf_{n\to\infty} \phi_\kappa(z, x_n) - \phi_\kappa(z, x_0).\end{aligned}$$

Consequently,

$$d(z, x_0) \le \liminf_{n\to\infty} d(z, x_n).$$

This is the desired result. □

Now, we can prove the following lemma called the Δ-Kadec–Klee property:

Lemma 3.2.5 (Bačák [5], Kimura [16], Kimura–Satô [26]). *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $\{x_n\}$ a sequence of X. If $\{x_n\}$ Δ-converges to $x_0 \in X$ and $\{d(x_n, z)\}$ converges to $d(x_0, z)$ for some $z \in X$, then $\{x_n\}$ converges to x_0.*

Proof. Suppose that a sequence $\{x_n\}$ of X Δ-converges to $x_0 \in X$ and that $\{d(x_n, z)\}$ converges to $d(x_0, z)$ for $z \in X$. Then, $\{x_n\}$ is κ-bounded. Set

$$y_n = \frac{1}{2}x_n \oplus \frac{1}{2}x_0$$

for each $n \in \mathbb{N}$. We first show that $\{y_n\}$ Δ-converges to x_0. We remark that $\{y_n\}$ is also κ-bounded. Take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ arbitrarily and let $\{y\} = \mathrm{AC}(\{y_{n_i}\})$. Then,

$$\begin{aligned}
\limsup_{i\to\infty} d(y, x_{n_i}) &\le \limsup_{i\to\infty} d(y, y_{n_i}) + \limsup_{i\to\infty} d(y_{n_i}, x_{n_i}) \\
&\le \limsup_{i\to\infty} d(x_0, y_{n_i}) + \limsup_{i\to\infty} d(y_{n_i}, x_{n_i}) \\
&= \frac{1}{2}\limsup_{i\to\infty} d(x_0, x_{n_i}) + \frac{1}{2}\limsup_{i\to\infty} d(x_0, x_{n_i}) = \limsup_{i\to\infty} d(x_0, x_{n_i}).
\end{aligned}$$

Since x_0 is a unique asymptotic centre of $\{x_{n_i}\}$, we have $y = x_0$ and hence $\{y_n\}$ also Δ-converges to x_0.

We finally show that $\{x_n\}$ converges to x_0. Since $\{y_n\}$ Δ-converges to x_0, it holds form Lemma 3.2.4 that

$$\begin{aligned}
\phi_\kappa(z, x_0) &\le \liminf_{n\to\infty} \phi_\kappa(z, y_n) \le \limsup_{n\to\infty} \phi_\kappa(z, y_n) \\
&\le \frac{1}{2}\limsup_{n\to\infty} \phi_\kappa(z, x_n) + \frac{1}{2}\phi_\kappa(z, x_0) = \phi_\kappa(z, x_0),
\end{aligned}$$

which implies that $\{d(y_n, z)\}$ also converges to $d(x_0, z)$. Then, from Theorem 1.2.6,

$$\phi_\kappa(y_n, z) \le \frac{1}{2}\phi_\kappa(x_n, z) + \frac{1}{2}\phi_\kappa(x_0, z) - c''_\kappa(d(y_n, z))c_\kappa\left(\frac{d(x_n, x_0)}{2}\right)$$

for each $n \in \mathbb{N}$. Therefore,

$$c''_\kappa(d(y_n, z))c_\kappa\left(\frac{d(x_n, x_0)}{2}\right) \le \frac{1}{2}\phi_\kappa(x_n, z) + \frac{1}{2}\phi_\kappa(x_0, z) - \phi_\kappa(y_n, z).$$

Letting $n \to \infty$, we have

$$c''_\kappa(d(x_0, z))c_\kappa\left(\frac{\limsup_{n\to\infty} d(x_n, x_0)}{2}\right) \le 0.$$

It implies that $d(x_n, x_0) \to 0$ as $n \to \infty$ and we complete the proof. □

We next prove the following useful lemma:

Lemma 3.2.6. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $\{x_n\}$ a κ-bounded sequence of X. Then, $\{x_n\}$ Δ-converges to $x_0 \in X$ if and only if x_0 is a Δ-limit of every Δ-convergent subsequence of $\{x_n\}$.*

Proof. Since the "only if" part is obvious, we prove the "if" part. Suppose that x_0 is a Δ-limit of every Δ-converging subsequence of $\{x_n\}$. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily and let $\{z\} = \mathrm{AC}(\{x_{n_i}\})$. Here, we take a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} d(x_0, x_{n_{i_j}}) = \limsup_{i\to\infty} d(x_0, x_{n_i})$$

and that $\{x_{n_{i_j}}\}$ is a Δ-convergent sequence. From the assumption, x_0 is its Δ-limit. Then, we obtain

$$\limsup_{i\to\infty} d(x_0, x_{n_i}) = \lim_{j\to\infty} d(x_0, x_{n_{i_j}}) \le \limsup_{j\to\infty} d(z, x_{n_{i_j}}) \le \limsup_{i\to\infty} d(z, x_{n_i}).$$

Since z is a unique asymptotic centre of $\{x_{n_i}\}$, we obtain $z = x_0$ and thus $\{x_n\}$ Δ-converges to x_0. □

3.3 KKM lemma

In this section, we obtain a lemma corresponding to the KKM lemma in the setting of CAT(κ) spaces.

Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ such that $\operatorname{rad} X < D_\kappa/2$ and $\{x_\lambda\}$ a net of X with a directed index set Λ. Here,

$$\operatorname{rad} X = \inf_{y\in X} \sup_{x\in X} d(y, x).$$

From Theorem 3.1.4, an asymptotic centre

$$\mathrm{AC}(\{x_\lambda\}) = \operatorname*{Argmin}_{u\in X} \left(\limsup_{\lambda\in\Lambda} d(u, x_\lambda) \right)$$

consists of exactly one point. Further, in the same way as Lemma 3.2.1, for a closed convex subset C of X, an asymptotic centre $\mathrm{AC}(\{x_\lambda\})$ is included in C whenever $\{x_\lambda\}$ is a sequence of C. We say that $\{x_\lambda\}$ Δ-converges to a Δ-limit $x_0 \in X$ if $\{x_0\} = \mathrm{AC}(\{x_{\lambda_\mu}\})$ for any subnet $\{x_{\lambda_\mu}\}$ of $\{x_\lambda\}$.

Lemma 3.3.1 (Kirk–Massa [29]). *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $\operatorname{rad} X < D_\kappa/2$ and $\{x_\lambda\}$ a net of X. Then, $\{x_\lambda\}$ has a Δ-convergent subnet.*

In [29], this result has shown on Banach spaces. However, using the same method, we obtain the lemma above.

Let X be a metric space and S a nonempty set of X. For $x \in X$, denote

$$d(x, S) = \inf_{y\in S} d(x, y).$$

If a sequence $\{x_n\}$ of X converges to $x_0 \in X$, then $\{d(x_n, S)\}$ converges to $d(x_0, S)$. Indeed, since

$$d(x_n, S) \le d(x_n, x_0) + d(x_0, S) \text{ and } d(x_0, S) \le d(x_0, x_n) + d(x_n, S),$$

we get $|d(x_n, S) - d(x_0, S)| \le d(x_n, x_0)$.

We first prove the following lemma:

Lemma 3.3.2. *Let X be an admissible complete* $\mathrm{CAT}(\kappa)$ *space for $\kappa \in \mathbb{R}$ such that* $\operatorname{rad} X < D_\kappa/2$ *and $E = \{w_1, \dots, w_N\}$ a finite subset of X. Let $\{G_1, \dots, G_N\}$ be a finite family of nonempty closed convex subsets of X such that $\bigcap_{i=1}^N G_i$ is empty. For given $x \in \operatorname{cl}\operatorname{co} E$, define a function h_x from* $\operatorname{cl}\operatorname{co} E$ *into $[0, \infty[$ as follows:*

$$h_x(z) = \frac{1}{\sum_{j=1}^N d(x, G_j)} \sum_{i=1}^N d(x, G_i)\phi_\kappa(w_i, z)$$

for each $z \in \operatorname{cl}\operatorname{co} E$. Then, its minimiser set $\operatorname{Min} h_x$ consists of one point. Further, a single-valued mapping T from $\operatorname{cl}\operatorname{co} E$ into itself defined by

$$Tx = \operatorname{Min} h_x = \operatorname*{Argmin}_{z \in \operatorname{cl}\operatorname{co} E} h_x(z)$$

for each $x \in \operatorname{cl}\operatorname{co} E$ is continuous.

Proof. Fix $x \in \operatorname{cl}\operatorname{co} E$ arbitrarily. Let $\zeta(x) = \sum_{j=1}^N d(x, G_j)$. Since $\bigcap_{i=1}^N G_i$ is empty, there exists $i_0 \in \{1, \dots, N\}$ such that $d(x, G_{i_0}) > 0$ and hence $\zeta(x) > 0$. From Theorem 3.1.4, $\operatorname{Min} h_x$ consists of exactly one point. We define the single-valued mapping T from $\operatorname{cl}\operatorname{co} E$ into itself as

$$Tx = \operatorname{Min} h_x = \operatorname*{Argmin}_{z \in \operatorname{cl}\operatorname{co} E} h_x(z)$$

for each $x \in \operatorname{cl}\operatorname{co} E$. We show the continuity of T. Let $x, y \in \operatorname{cl}\operatorname{co} E$, $l = d(Tx, Ty)$ and $\tau \in]0, 1[$. It holds from the parallelogram law of X that

$$\begin{aligned} h_x(Tx) &\le h_x(\tau Ty \oplus (1-\tau)Tx) \\ &\le (\tau)_l^\kappa h_x(Ty) + (1-\tau)_l^\kappa h_x(Tx) - (\tau)_l^\kappa \phi_\kappa(Ty, \tau Ty \oplus (1-\tau)Tx) \end{aligned}$$

and hence

$$0 \le (\tau)_l^\kappa h_x(Ty) - \left(1 - (1-\tau)_l^\kappa\right) h_x(Tx) - (\tau)_l^\kappa \phi_\kappa(Ty, \tau Ty \oplus (1-\tau)Tx).$$

Dividing both sides by $(\tau)_l^\kappa$ and letting $\tau \searrow 0$, we obtain

$$0 \le h_x(Ty) - c_\kappa''(l) h_x(Tx) - \phi_\kappa(Ty, Tx).$$

Additionally,

$$\begin{aligned} 0 &\le h_x(Ty) - c_\kappa''(l) h_x(Tx) - \phi_\kappa(Ty, Tx) \\ &= h_x(Ty) - h_x(Tx) + (1 - c_\kappa''(l)) h_x(Tx) - \phi_\kappa(Ty, Tx) \\ &= h_x(Ty) - h_x(Tx) + (\kappa h_x(Tx) - 1)\phi_\kappa(Ty, Tx). \end{aligned}$$

Similarly, we have

$$0 \le h_y(Tx) - h_y(Ty) + (\kappa h_y(Ty) - 1)\phi_\kappa(Tx, Ty).$$

Adding their both sides, we get

$$(2 - \kappa h_x(Tx) - \kappa h_y(Ty))\phi_\kappa(Tx, Ty) \le h_x(Ty) - h_x(Tx) + h_y(Tx) - h_y(Ty).$$

Then, since $\kappa c_\kappa(a) = 1 - c''_\kappa(a)$ for all $a \in \mathbb{R}$,

$$\begin{aligned}
&2 - \kappa h_x(Tx) - \kappa h_y(Ty) \\
&= 2 - \kappa \frac{1}{\zeta(x)} \sum_{i=1}^{N} d(x, G_i)\phi_\kappa(w_i, Tx) - \kappa \frac{1}{\zeta(y)} \sum_{i=1}^{N} d(y, G_i)\phi_\kappa(w_i, Ty) \\
&= \sum_{i=1}^{N} \frac{d(x, G_i)}{\zeta(x)} c''_\kappa(d(w_i, Tx)) + \sum_{i=1}^{N} \frac{d(y, G_i)}{\zeta(y)} c''_\kappa(d(w_i, Ty)).
\end{aligned}$$

Moreover,

$$h_x(Ty) - h_x(Tx) = \sum_{i=1}^{N} \frac{d(x, G_i)}{\zeta(x)} \left(\phi_\kappa(w_i, Ty) - \phi_\kappa(w_i, Tx)\right)$$

and

$$h_y(Tx) - h_y(Ty) = \sum_{i=1}^{N} \frac{d(y, G_i)}{\zeta(y)} \left(\phi_\kappa(w_i, Tx) - \phi_\kappa(w_i, Ty)\right).$$

Consequently, we have

$$\begin{aligned}
&\left(\sum_{i=1}^{N} \frac{d(x, G_i)}{\zeta(x)} c''_\kappa(d(w_i, Tx)) + \sum_{i=1}^{N} \frac{d(y, G_i)}{\zeta(y)} c''_\kappa(d(w_i, Ty)) \right) \phi_\kappa(Tx, Ty) \\
&\leq \sum_{i=1}^{N} \left(\frac{d(x, G_i)}{\zeta(x)} - \frac{d(y, G_i)}{\zeta(y)} \right) \left(\phi_\kappa(w_i, Ty) - \phi_\kappa(w_i, Tx)\right) \\
&\leq \sum_{i=1}^{N} \left| \frac{d(x, G_i)}{\zeta(x)} - \frac{d(y, G_i)}{\zeta(y)} \right| \cdot \left|\phi_\kappa(w_i, Ty) - \phi_\kappa(w_i, Tx)\right|.
\end{aligned}$$

Now, we can prove the continuity of T. Let $\{x_n\}$ be a sequence of $\operatorname{cl}\operatorname{co} E$ which converges to $x_0 \in \operatorname{cl}\operatorname{co} E$. Set

$$Z = \frac{\sum_{i=1}^{N} d(x_0, G_i) c''_\kappa(d(w_i, Tx_0))}{\zeta(x_0)} > 0.$$

Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&Z\phi_\kappa(Tx_n, Tx_0) \\
&\leq \left(\sum_{i=1}^{N} \frac{d(x_n, G_i)}{\zeta(x_n)} c''_\kappa(d(w_i, Tx_n)) + \sum_{i=1}^{N} \frac{d(x_0, G_i)}{\zeta(x_0)} c''_\kappa(d(w_i, Tx_0)) \right) \phi_\kappa(Tx_n, Tx_0) \\
&\leq \sum_{i=1}^{N} \left| \frac{d(x_n, G_i)}{\zeta(x_n)} - \frac{d(x_0, G_i)}{\zeta(x_0)} \right| \cdot \left|\phi_\kappa(w_i, Tx_0) - \phi_\kappa(w_i, Tx_n)\right|.
\end{aligned}$$

Since $\operatorname{rad} X < D_\kappa/2$, for each $i \in \{1, \dots, N\}$, a real sequence $\{\phi_\kappa(w_i, Tx_n)\}$ is bounded and thus there is $B \geq 0$ such that for all $n \in \mathbb{N}$,

$$\left|\phi_\kappa(w_i, Tx_0) - \phi_\kappa(w_i, Tx_n)\right| \leq B.$$

Hence,

$$\phi_\kappa(Tx_0, Tx_n) \leq \frac{B}{Z} \sum_{i=1}^{N} \left| \frac{d(x_n, G_i)}{\zeta(x_n)} - \frac{d(x_0, G_i)}{\zeta(x_0)} \right|.$$

Since $d(x_n, G_i) \to d(x_0, G_i)$ for each $i \in \{1, \dots, N\}$, we have $\zeta(x_n) \to \zeta(x_0)$ as $n \to \infty$. Consequently, we obtain

$$0 \leq \phi_\kappa(Tx_0, Tx_n) \leq \frac{B}{Z} \sum_{i=1}^{N} \left| \frac{d(x_n, G_i)}{\zeta(x_n)} - \frac{d(x_0, G_i)}{\zeta(x_0)} \right| \to 0$$

as $n \to \infty$, which implies that $\{Tx_n\}$ converges to Tx_0. It completes the proof. □

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$. We say that X has the convex hull finite property if every continuous self-mapping on $\operatorname{cl}\operatorname{co} E$ has a fixed point for every finite subset E of X, where $\operatorname{cl}\operatorname{co} E$ is closure of $\operatorname{co} E$.

We next obtain the following lemma:

Lemma 3.3.3. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ having the convex hull finite property. Suppose that $\operatorname{rad} X < D_\kappa/2$. Let G be a set-valued mapping from X to a nonempty closed convex subset of X such that $\operatorname{co} E \subset \bigcup_{x\in E} G(x)$ for every finite subset E of X. Then, $\{G(x) \mid x \in X\}$ has the finite intersection property. Namely, $\bigcap_{i=1}^{N} G_i$ is nonempty for every finite family $\{G_1, \dots, G_N\}$ of $\{G(x) \mid x \in X\}$.*

Proof. We show the statement by contradiction. Suppose that there exists a finite subset $E = \{w_1, \dots, w_N\}$ of X such that $\bigcap_{i=1}^{N} G(w_i)$ is empty. We define a set-valued mapping T on $\operatorname{cl}\operatorname{co} E$ as follows:

$$Tx = \operatorname*{Argmin}_{z\in \operatorname{cl}\operatorname{co} E} \left(\frac{1}{\sum_{j=1}^{N} d(x, G(w_j))} \sum_{i=1}^{N} d(x, G(w_i))\phi_\kappa(w_i, z) \right)$$

for each $x \in \operatorname{cl}\operatorname{co} E$. Then, from Lemma 3.3.2, T is single-valued and continuous. From the convex hull finite property of X, the mapping T has a fixed point $x_0 \in \operatorname{cl}\operatorname{co} E$. Since $\bigcap_{i=1}^{N} G(w_i)$ is empty, $d(x_0, G(w_i)) > 0$ for some $i \in \{1, \dots, N\}$. Without loss of generality, we can assume that $d(x_0, G(w_i)) > 0$ for $i \in \{1, \dots, N_0\}$; $d(x_0, G(w_i)) = 0$ for $i \in \{N_0+1, \dots, N\}$. Then, we know $x_0 \notin \bigcup_{i=1}^{N_0} G(w_i)$. Let $E_0 = \{w_1, \dots, w_{N_0}\}$ and P a metric projection from $\operatorname{cl}\operatorname{co} E$ onto $\operatorname{cl}\operatorname{co} E_0$. Note that $\operatorname{cl}\operatorname{co} E_0 \subset \bigcup_{i=1}^{N_0} G(w_i)$ since $G(w_i)$ is closed for each $i \in \{1, \dots, N_0\}$. Since P is quasinonexpansive, $\phi_\kappa(Px_0, w_i) \leq \phi_\kappa(x_0, w_i)$ for each $i \in \{1, \dots, N_0\}$. Then, we obtain

$$\begin{aligned}
\sum_{i=1}^{N} \frac{d(x_0, G(w_i))}{\sum_{j=1}^{N} d(x_0, G(w_j))}\phi_\kappa(w_i, Px_0) &= \sum_{i=1}^{N_0} \frac{d(x_0, G(w_i))}{\sum_{j=1}^{N} d(x_0, G(w_j))}\phi_\kappa(w_i, Px_0) \\
&\leq \sum_{i=1}^{N_0} \frac{d(x_0, G(w_i))}{\sum_{j=1}^{N} d(x_0, G(w_j))}\phi_\kappa(w_i, x_0) \\
&= \sum_{i=1}^{N} \frac{d(x_0, G(w_i))}{\sum_{j=1}^{N} d(x_0, G(w_j))}\phi_\kappa(w_i, x_0)
\end{aligned}$$

and thus $Px_0 = Tx_0 = x_0$. Therefore,

$$x_0 = Px_0 \in \operatorname{cl} \operatorname{co} E_0 \subset \bigcup_{i=1}^{N_0} G(w_i).$$

This is a contradiction and it completes the proof. $\square$

Now, we can prove the KKM lemma in the setting of CAT(κ) spaces:

Lemma 3.3.4 (Kimura [17], Kimura–Kishi [18], Niculescu–Rovenţa [33]). *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and C a nonempty closed convex subset of X. Let G be a set-valued mapping from C to a subset of C. Suppose the following conditions:*

- *C has the convex hull finite property and $\operatorname{rad} C < D_\kappa/2$;*
- *$G(x)$ is nonempty, closed and convex for every $x \in C$;*
- *$\operatorname{co} E \subset \bigcup_{x\in E} G(x)$ for every finite subset E of C.*

Then, $\bigcap_{x\in C} G(x)$ is nonempty.

Proof. We know that C is an admissible complete CAT(κ) space since it is nonempty, closed and convex. From the assumptions, $\{G(x) \mid x \in C\}$ has the finite intersection property. Let Λ be the family of all finite subsets of C and define a binary relation $\preceq$ on Λ as follows: $\lambda \preceq \mu$ if and only if $\lambda \subset \mu$ for $\lambda, \mu \in \Lambda$. Then, $(\Lambda, \preceq)$ is a directed set. For fixed $\lambda = \{x_1, \dots, x_N\} \in \Lambda$, a set $\bigcap_{i=1}^{N} G(x_i)$ is nonempty and we can take a point

$$x_\lambda \in \bigcap_{i=1}^{N} G(x_i) = \bigcap_{x\in\lambda} G(x).$$

Therefore, we obtain a net $\{x_\lambda\}$ of C with the index set Λ. From the assumption of C, $\{x_\lambda\}$ has a Δ-convergent subnet $\{x_{\lambda_\mu}\}$ with the directed index set $(\Lambda', \preceq')$. Let x_0 be its Δ-limit.

We show $x_0 \in \bigcap_{x\in C} G(x)$. Fix $z \in C$ arbitrarily and let $\lambda_z = \{z\} \in \Lambda$. From the definition of subnet, there exists $\mu_0 \in \Lambda'$ such that $\lambda_z \preceq \lambda_\mu$ whenever $\mu \in \Lambda'$ satisfies $\mu_0 \preceq' \mu$. Then, we have

$$x_{\lambda_\mu} \in \bigcap_{x\in\lambda_\mu} G(x) \subset \bigcap_{x\in\lambda_z} G(x) = G(z)$$

for any $\mu \in \Lambda'$ with $\mu_0 \preceq' \mu$. Since $G(z)$ is closed and convex,

$$\{x_0\} = \operatorname{AC}(\{x_{\lambda_\mu}\}) \subset G(z)$$

and hence $x_0 \in \bigcap_{x\in C} G(x)$. $\square$

3.4 Note on the convex hull finite property

Ariza-Ruiz–Li–López-Acedo [4] proved the Schauder fixed point theorem in the setting of CAT(κ) spaces.

Theorem 3.4.1 (Ariza-Ruiz–Li–López-Acedo [4]). *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $\sup_{u,v\in X} d(u,v) < D_\kappa/2$. Let T be a continuous mapping from X into itself such that $\operatorname{cl} T(X)$ is compact. Then, T has a fixed point.*

As a direct consequence of the theorem above, a nonempty compact convex subset of an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ satisfies the convex hull finite property. Moreover, we obtain the following fact:

Theorem 3.4.2. *Let $(M_\kappa^\infty, d_{M_\kappa^\infty})$ be the infinite dimensional model space for $\kappa \in \mathbb{R}$ and X an admissible closed convex subset of M_κ^∞. Then, X has the convex hull finite property.*

Proof. Note that X is an admissible complete CAT(κ) space. Fix a finite subset E of X arbitrarily. It is sufficient to show that $\operatorname{cl}\operatorname{co} E$ is compact. Indeed, if $\operatorname{cl}\operatorname{co} E$ is compact, then from the theorem above, X has the convex hull finite property. We remark that $\operatorname{co} E$ is defined by $\operatorname{co} E = \bigcup_{j=1}^\infty W_j$, where $W_1 = E$ and

$$\begin{aligned} W_{j+1} &= \{tu_j \oplus (1-t)v_j \mid u_j, v_j \in W_j, t \in [0,1]\} \\ &= \left\{(t)_l^\kappa u_j + (1-t)_l^\kappa v_j \in \mathbb{E}^\infty \;\middle|\; u_j, v_j \in W_j, l = d_{M_\kappa^\infty}(u_j, v_j), t \in [0,1]\right\} \end{aligned}$$

for $j \in \mathbb{N}$. Therefore, W_j is included in $\operatorname{span} E$ for all $j \in \mathbb{N}$, where $\operatorname{span} E$ is the linear span of E. Note that $\operatorname{cl}\operatorname{co} E$ is a closed set with respect to the norm on $\mathbb{E}^\infty$. Therefore, $\operatorname{cl}\operatorname{co} E$ is included in $\operatorname{cl}\operatorname{span} E$. Since E is a finite subset of $\mathbb{E}^\infty$, $\operatorname{cl}\operatorname{span} E$ is a finite dimensional subspace of $\mathbb{E}^\infty$. Then, $\operatorname{cl}\operatorname{co} E$ is compact on $\mathbb{E}^\infty$ since it is bounded and closed on $\operatorname{cl}\operatorname{span} E$. It follows that $\operatorname{cl}\operatorname{co} E$ is also compact with respect to the distance on X. Therefore, every continuous self-mapping on $\operatorname{cl}\operatorname{co} E$ has a fixed point. Consequently, X has the convex hull finite property. □

Chapter 4

Fixed point theory

In CAT(κ) spaces, there are mappings called nonspreading mappings. In this chapter, we consider fixed point existence and approximation theorem of nonspreading mappings. A notion of nonspreading mappings is first introduced by Kohsaka and Takahashi [32] on smooth Banach spaces. In a smooth Banach space E, a mapping T on E is said to be nonspreading if

$$\phi(Tx, JTy) + \phi(Ty, JTx) \leq \phi(Tx, Jy) + \phi(Ty, Jx)$$

for each $x, y \in E$, where J is the normalised duality mapping on E defined by

$$Jz = \left\{z^* \in E^* \;\middle|\; z^*(z) = \|z\|^2 = \|z^*\|^2\right\}$$

for each $z \in E$, and ϕ is a function defined by

$$\phi(u, v^*) = \|u\|^2 - 2v^*(u) + \|v^*\|^2$$

for each $(u, v^*) \in E \times E^*$.

4.1 Geodesically nonspreading mappings

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and T a mapping from X into itself. T is said to be firmly geodesically nonspreading if

$$\begin{aligned}
&\phi_\kappa(Tx, Ty) + \phi_\kappa(Ty, Tx) \\
&\leq \phi_\kappa(Tx, y) + \phi_\kappa(Ty, x) - c''_\kappa(d(Tx, Ty))\phi_\kappa(Tx, x) - c''_\kappa(d(Tx, Ty))\phi_\kappa(Ty, y)
\end{aligned}$$

for each $x, y \in X$. Further, T is said to be geodesically nonspreading if

$$\phi_\kappa(Tx, Ty) + \phi_\kappa(Ty, Tx) \leq \phi_\kappa(Tx, y) + \phi_\kappa(Ty, x)$$

for each $x, y \in X$. Note that a firmly geodesically nonspreading mapping is always geodesically nonspreading. Suppose that X is complete. From Lemma 3.1.6, a metric projection onto a nonempty closed convex subset of X is firmly geodesically nonspreading.

Remark 10. Let X be a CAT(0) space and T a firmly geodesically nonspreading mapping from X into itself. Fix $x, y \in X$ arbitrarily. Then,

$$2d(Tx, Ty)^2 \leq d(Tx, y)^2 + d(Ty, x)^2 - d(Tx, x)^2 - d(Ty, y)^2.$$

We call a mapping satisfying the inequality a firmly nonexpansive mapping. Further, it holds from the parallelogram law of X that

$$\begin{aligned}
0 &\leq 4d\left(\frac{1}{2}Ty \oplus \frac{1}{2}x, \frac{1}{2}Tx \oplus \frac{1}{2}y\right)^2 \\
&\leq 2d\left(Ty, \frac{1}{2}Tx \oplus \frac{1}{2}y\right)^2 + 2d\left(x, \frac{1}{2}Tx \oplus \frac{1}{2}y\right)^2 - d(Ty,x)^2 \\
&\leq d(Ty,Tx)^2 + d(Ty,y)^2 - \frac{1}{2}d(Tx,y)^2 + d(x,Tx)^2 + d(x,y)^2 - \frac{1}{2}d(Tx,y)^2 - d(Ty,x)^2 \\
&= d(Ty,Tx)^2 + d(x,y)^2 + d(Tx,x)^2 + d(Ty,y)^2 - d(Tx,y)^2 - d(Ty,x)^2
\end{aligned}$$

and hence

$$d(Tx,y)^2 + d(Ty,x)^2 - d(Tx,x)^2 - d(Ty,y)^2 \leq d(Ty,Tx)^2 + d(x,y)^2.$$

Therefore, we have

$$2d(Tx,Ty)^2 \leq d(Tx,y)^2 + d(Ty,x)^2 - d(Tx,x)^2 - d(Ty,y)^2 \leq d(Ty,Tx)^2 + d(x,y)^2$$

and thus

$$d(Tx,Ty) \leq d(x,y).$$

It means that T is a nonexpansive mapping.

Remark 11. Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and T a geodesically nonspreading mapping from X into itself. Let $x, y \in X$. If $\kappa = 0$, then

$$2d(Tx,Ty)^2 \leq d(Tx,y)^2 + d(Ty,x)^2.$$

We call such a mapping a metrically nonspreading mapping [31]. If $\kappa = 1$, then

$$1 - \cos d(Tx,Ty) + 1 - \cos d(Ty,Tx) \leq 1 - \cos d(Tx,y) + 1 - \cos d(Ty,x)$$

and hence

$$2\cos d(Tx,Ty) \geq \cos d(Tx,y) + \cos d(Ty,x).$$

We call such a mapping a spherically nonspreading mapping of sum type [12]. Further, the previous inequality implies that

$$\begin{aligned}
\cos^2 d(Tx,Ty) &\geq \left(\frac{\cos d(Tx,y) + \cos d(Ty,x)}{2}\right)^2 \\
&\geq \left(\frac{2\sqrt{\cos d(Tx,y)\cos d(Ty,x)}}{2}\right)^2 = \cos d(Tx,y)\cos d(Ty,x).
\end{aligned}$$

It means that T is spherically nonspreading of product type [17, 19]. Similarly, if $\kappa = -1$, then

$$2\cosh d(Tx,Ty) \leq \cosh d(Tx,y) + \cosh d(Ty,x).$$

We call such a mapping a hyperbolically nonspreading mapping [13].

Lemma 4.1.1. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a geodesically nonspreading mapping from X into itself. If* $\operatorname{Fix} T$ *is nonempty, then T is quasinonexpansive.*

Proof. Let $x \in X$ and $p \in \operatorname{Fix} T$. Then,

$$\phi_\kappa(Tx, p) + \phi_\kappa(p, Tx) \leq \phi_\kappa(Tx, p) + \phi_\kappa(p, x)$$

and hence

$$\phi_\kappa(p, Tx) \leq \phi_\kappa(p, x).$$

It implies that T is quasinonexpansive. □

Corollary 4.1.2. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a geodesically nonspreading mapping from X into itself. Then,* $\operatorname{Fix} T$ *is closed and convex.*

Lemma 4.1.3. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a geodesically nonspreading mapping from X into itself. Then, $x_0 \in \operatorname{Fix} T$ whenever a κ-bounded sequence $\{x_n\}$ of X is Δ-convergent to $x_0 \in X$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. Let $\{x_n\}$ be a κ-bounded sequence which Δ-converges to a point $x_0 \in X$. Suppose that $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Then, for fixed $y \in X$ and $n \in \mathbb{N}$,

$$d(y, x_n) \leq d(y, Tx_n) + d(Tx_n, x_n) \leq d(y, x_n) + 2d(Tx_n, x_n)$$

and thus

$$\limsup_{n\to\infty} d(y, x_n) = \limsup_{n\to\infty} d(y, Tx_n) < \infty,$$

or equivalently

$$\limsup_{n\to\infty} \phi_\kappa(y, x_n) = \limsup_{n\to\infty} \phi_\kappa(y, Tx_n) < \infty.$$

Since T is geodesically nonspreading, for all $n \in \mathbb{N}$,

$$2\phi_\kappa(Tx_n, Tx_0) \leq \phi_\kappa(Tx_n, x_0) + \phi_\kappa(Tx_0, x_n).$$

Letting $n \to \infty$, we have

$$\begin{aligned} 2\limsup_{n\to\infty} \phi_\kappa(x_n, Tx_0) &= 2\limsup_{n\to\infty} \phi_\kappa(Tx_n, Tx_0) \\ &\leq \limsup_{n\to\infty} \phi_\kappa(Tx_n, x_0) + \limsup_{n\to\infty} \phi_\kappa(Tx_0, x_n) \\ &= \limsup_{n\to\infty} \phi_\kappa(x_n, x_0) + \limsup_{n\to\infty} \phi_\kappa(Tx_0, x_n). \end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \phi_\kappa(x_n, Tx_0) \leq \limsup_{n\to\infty} \phi_\kappa(x_n, x_0).$$

Since x_0 is a unique asymptotic centre of $\{x_n\}$, we obtain $x_0 = Tx_0$. □

4.2 Picard type iterative scheme

In this section, we prove a fixed point approximation theorem. Before that, we obtain the following facts:

Lemma 4.2.1. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $\{T_n\}$ a sequence of quasinonexpansive mappings from X into itself such that $\bigcap_{n=1}^{\infty} \operatorname{Fix} T_n$ is nonempty. For a given initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = T_n x_n$$

for each $n \in \mathbb{N}$. Then, $\{Px_n\}$ converges to a common fixed point $x_0 \in \bigcap_{n=1}^{\infty} \operatorname{Fix} T_n$, where P is a metric projection onto $\bigcap_{n=1}^{\infty} \operatorname{Fix} T_n$.

Proof. From the definition of a metric projection P and quasinonexpansiveness of T_n, we have

$$d(Px_{n+1}, x_{n+1}) \leq d(Px_n, x_{n+1}) = d(Px_n, T_n x_n) \leq d(Px_n, x_n)$$

for any $n \in \mathbb{N}$ and thus $\{\phi_\kappa(Px_n, x_n)\}$ is convergent. Note that there exists a nonnegative real sequence $\{\beta_n\}$ such that $\lim_{n\to\infty} \beta_n = 0$ and that

$$\phi_\kappa(Px_n, x_n) - \phi_\kappa(Px_m, x_m) = |\phi_\kappa(Px_n, x_n) - \phi_\kappa(Px_m, x_m)| \leq \beta_n$$

for any $m, n \in \mathbb{N}$ with $m \geq n$. Moreover, there exists $L > 0$ such that

$$L \leq \inf_{n\in\mathbb{N}} c''_\kappa(d(Px_n, x_n)).$$

Fix $m, n \in \mathbb{N}$ with $m \geq n$ arbitrarily. From Lemma 3.1.6, we have

$$0 \leq \phi_\kappa(Px_n, x_m) - c''_\kappa(l)\phi_\kappa(Px_m, x_m) - \phi_\kappa(Px_n, Px_m),$$

where $l = d(Px_n, Px_m)$. Then,

$$\begin{aligned}
0 &\leq \phi_\kappa(Px_n, x_m) - c''_\kappa(l)\phi_\kappa(Px_m, x_m) - \phi_\kappa(Px_n, Px_m) \\
&= \phi_\kappa(Px_n, x_m) - \phi_\kappa(Px_m, x_m) + (1 - c''_\kappa(l))\,\phi_\kappa(Px_m, x_m) - \phi_\kappa(Px_n, Px_m) \\
&= \phi_\kappa(Px_n, x_m) - \phi_\kappa(Px_m, x_m) - c''_\kappa(d(Px_m, x_m))\phi_\kappa(Px_n, Px_m)
\end{aligned}$$

and hence

$$\phi_\kappa(Px_n, Px_m) \leq \frac{\phi_\kappa(Px_n, x_m) - \phi_\kappa(Px_m, x_m)}{c''_\kappa(d(Px_m, x_m))} \leq \frac{\phi_\kappa(Px_n, x_n) - \phi_\kappa(Px_m, x_m)}{L} \leq \frac{\beta_n}{L}.$$

It means that $\{Px_n\}$ is a Cauchy sequence, namely, $\{Px_n\}$ converges to some point $x_0 \in \bigcap_{n=1}^{\infty} \operatorname{Fix} T_n$. □

Corollary 4.2.2. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a quasinonexpansive mapping from X into itself such that* Fix T *is nonempty. For a given initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = Tx_n$$

for each $n \in \mathbb{N}$*. Then,* $\{P_{\operatorname{Fix} T} x_n\}$ *converges to a fixed point* $x_0 \in \operatorname{Fix} T$*, where* $P_{\operatorname{Fix} T}$ *is a metric projection onto* $\operatorname{Fix} T$*.*

Now, we can prove the following approximation theorem with the Picard type iterative scheme:

Theorem 4.2.3. *Let* X *be an admissible complete* $\mathrm{CAT}(\kappa)$ *space for* $\kappa \in \mathbb{R}$ *and* T *a geodesically nonspreading mapping from* X *into itself. For a given initial point* $x_1 \in X$*, generate a sequence* $\{x_n\}$ *of* X *as follows:*

$$x_{n+1} = Tx_n$$

for each $n \in \mathbb{N}$*. Then, the following hold:*

(i) $\operatorname{Fix} T$ *is nonempty if and only if* $\{x_n\}$ *is* κ*-bounded;*
(ii) *if* $\operatorname{Fix} T$ *is nonempty and* T *is firmly geodesically nonspreading, then* $\{x_n\}$ Δ*-converges to a fixed point* $x_0 \in \operatorname{Fix} T$ *which is a limit of* $\{P_{\operatorname{Fix} T} x_n\}$*, where* $P_{\operatorname{Fix} T}$ *is a metric projection onto* $\operatorname{Fix} T$*.*

Proof. We first prove (i). We first show that the "only if" part. Since $\operatorname{Fix} T$ is nonempty, we can take $p \in \operatorname{Fix} T$. Then, since T is quasinonexpansive,

$$d(p, x_{n+1}) = d(p, Tx_n) \leq d(p, x_n)$$

and thus $d(p, x_n) \leq d(p, x_1)$ for all $n \in \mathbb{N}$. It implies that

$$\limsup_{n\to\infty} d(p, x_n) \leq d(p, x_1) < \frac{D_\kappa}{2},$$

which means that $\{x_n\}$ is κ-bounded. To prove the "if" part of (i), we suppose that $\{x_n\}$ is κ-bounded. Since $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, for any $y \in X$,

$$\limsup_{n\to\infty} \phi_\kappa(x_n, y) = \limsup_{n\to\infty} \phi_\kappa(x_{n+1}, y) = \limsup_{n\to\infty} \phi_\kappa(Tx_n, y).$$

Let $\{p\} = \mathrm{AC}(\{x_n\})$. Since T is geodesically nonspreading, for fixed $n \in \mathbb{N}$,

$$2\phi_\kappa(Tx_n, Tp) \leq \phi_\kappa(Tx_n, p) + \phi_\kappa(Tp, x_n)$$

and thus

$$\begin{aligned} 2\limsup_{n\to\infty} \phi_\kappa(x_n, Tp) &= 2\limsup_{n\to\infty} \phi_\kappa(Tx_n, Tp) \\ &\leq \limsup_{n\to\infty} \phi_\kappa(Tx_n, p) + \limsup_{n\to\infty} \phi_\kappa(Tp, x_n) \\ &= \limsup_{n\to\infty} \phi_\kappa(x_n, p) + \limsup_{n\to\infty} \phi_\kappa(Tp, x_n). \end{aligned}$$

It implies that

$$\limsup_{n\to\infty} \phi_\kappa(x_n, Tp) \leq \limsup_{n\to\infty} \phi_\kappa(x_n, p).$$

Since p is a unique asymptotic centre of $\{x_n\}$, we have $Tp = p$ and hence $p \in \operatorname{Fix} T$. Therefore, $\operatorname{Fix} T$ is nonempty.

We next prove (ii). Assume that $\operatorname{Fix} T$ is nonempty and T is firmly geodesically nonspreading. We already know that $\{x_n\}$ is κ-bounded. Note that a sequence $\{P_{\operatorname{Fix} T} x_n\}$ converges to a point $x_0 \in \operatorname{Fix} T$ from Corollary 4.2.2. We prove that $\{x_n\}$ Δ-converges to x_0. Since T is quasinonexpansive,

$$\phi_\kappa(x_0, x_{n+1}) = \phi_\kappa(x_0, Tx_n) \le \phi_\kappa(x_0, x_n)$$

for each $n \in \mathbb{N}$. Therefore, there exists a limit of $\{\phi_\kappa(x_0, x_n)\}$. Note that there exists $L > 0$ such that $L \le \inf_{n\in\mathbb{N}} c''_\kappa(d(x_0, x_n))$. Since T is firmly geodesically nonspreading,

$$\begin{aligned} 2\phi_\kappa(Tx_n, Tx_0) &\le \phi_\kappa(Tx_n, x_0) + \phi_\kappa(Tx_0, x_n) \\ &\quad - c''_\kappa(d(Tx_n, Tx_0))\phi_\kappa(Tx_n, x_n) - c''_\kappa(d(Tx_n, Tx_0))\phi_\kappa(Tx_0, x_0) \end{aligned}$$

and hence

$$\phi_\kappa(Tx_n, x_0) \le \phi_\kappa(x_n, x_0) - c''_\kappa(d(x_{n+1}, x_0))\phi_\kappa(Tx_n, x_n).$$

Then, we get

$$\phi_\kappa(Tx_n, x_n) \le \frac{\phi_\kappa(x_n, x_0) - \phi_\kappa(x_{n+1}, x_0)}{L}.$$

Letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \phi_\kappa(Tx_n, x_n) \le 0,$$

which implies that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Take a Δ-convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily and let $z \in X$ be its Δ-limit. From Lemma 4.1.3, we obtain $z \in \operatorname{Fix} T$. Then, we have

$$\begin{aligned} \limsup_{i\to\infty} d(x_0, x_{n_i}) &\le \limsup_{i\to\infty} \left(d(x_0, P_{\operatorname{Fix} T} x_{n_i}) + d(P_{\operatorname{Fix} T} x_{n_i}, x_{n_i})\right) \\ &= \limsup_{i\to\infty} d(P_{\operatorname{Fix} T} x_{n_i}, x_{n_i}) \le \limsup_{i\to\infty} d(z, x_{n_i}), \end{aligned}$$

which implies that $x_0 = z$ since z is a unique asymptotic centre of $\{x_{n_i}\}$. From Lemma 3.2.6, $\{x_n\}$ Δ-converges to $x_0 \in \operatorname{Fix} T$, which completes the proof. □

As a direct consequence of the previous theorem, we obtain the following:

Theorem 4.2.4. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a geodesically nonspreading mapping from X into itself. Suppose that $\operatorname{rad} X < D_\kappa/2$. Then, $\operatorname{Fix} T$ is nonempty. Here, $\operatorname{rad} X$ is defined by $\inf_{y\in X} \sup_{x\in X} d(y, x)$.*

Chapter 5

Convex minimisation problems and equilibrium problems

Resolvent operators play an important role on convex minimisation problems and equilibrium problems. In this chapter, we consider resolvent operators for both problems. We finally obtain solution approximation theorems with the proximal point algorithm.

5.1 Convex functions

In this section, we consider convex functions on a geodesic space.

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and f a function from X into $]-\infty, \infty]$. f is said to be proper if the effective domain of f is nonempty, that is,

$$\operatorname{dom} f = \{z \in X \mid f(z) \in \mathbb{R}\} \neq \varnothing.$$

Further, f is said to be lower semicontinuous if

$$f(x) \leq \liminf_{n\to\infty} f(x_n)$$

whenever $\{x_n\}$ converges to $x \in X$. Moreover, f is said to be convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$$

for any $x, y \in X$ and $t \in\,]0,1[$. We define a level set $\operatorname{Lev}_a f$ for f at $a \in \mathbb{R}$ by

$$\operatorname{Lev}_a f = \{x \in X \mid f(x) \leq a\}.$$

A function $\phi_\kappa(\cdot, z)\colon X \to \mathbb{R}$ is continuous and convex for every $z \in X$ from Lemma 1.2.5.

Remark 12. Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$. Let $\{f_i \mid i \in I\}$ be a family of lower semicontinuous convex functions from X into $]-\infty, \infty]$. Then, $\sup_{i\in I} f_i$ is also lower semicontinuous and convex.

Remark 13. Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. For $\lambda > 0$, a function λf from X into $]-\infty, \infty]$ is also proper, lower semicontinuous and convex. Additionally, $\operatorname{Min} f = \operatorname{Min} \lambda f$.

Lemma 5.1.1. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a function from X into $]-\infty, \infty]$. Then, the following hold:*

(i) *If f is lower semicontinuous, then* $\operatorname{Lev}_a f$ *is closed for all* $a \in \mathbb{R}$*;*
(ii) *if f is convex, then* $\operatorname{Lev}_a f$ *is convex for all* $a \in \mathbb{R}$*.*

Proof. Fix $a \in \mathbb{R}$ arbitrarily. If $\operatorname{Lev}_a f$ is empty, then it is obviously closed and convex. We assume that $\operatorname{Lev}_a f$ is nonempty. We first show (i). Take a convergent sequence $\{x_n\}$ of $\operatorname{Lev}_a f$ arbitrarily and let $x \in X$ be its limit. Then, since f is lower semicontinuous,

$$f(x) \leq \liminf_{n\to\infty} f(x_n) \leq a$$

and thus $x \in \operatorname{Lev}_a f$. Therefore, the level set $\operatorname{Lev}_a f$ is closed. We next show $\operatorname{Lev}_a f$ is convex. Fix $u, v \in \operatorname{Lev}_a f$ and $t \in \,]0,1[$ arbitrarily. Since f is convex,

$$f(tu \oplus (1-t)v) \leq tf(u) + (1-t)f(v) \leq a$$

and hence the level set $\operatorname{Lev}_a f$ is convex. □

Corollary 5.1.2. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a function from X into* $]-\infty, \infty]$*. Then, the following hold:*

(i) *If f is lower semicontinuous, then* $\operatorname{Min} f$ *is closed;*
(ii) *if f is convex, then* $\operatorname{Min} f$ *is convex.*

Proof. If $\inf_{y\in X} f(y) = -\infty$ or $\inf_{y\in X} f(y) = \infty$, then $\operatorname{Min} f$ is obviously closed and convex. Suppose that $\inf_{y\in X} f(y) \in \mathbb{R}$. Since

$$\operatorname{Lev}_{\inf f} f = \left\{ x \in X \;\middle|\; f(x) \leq \inf_{y\in X} f(y) \right\} = \operatorname{Min} f,$$

we obtain the desired results from the previous theorem. □

Lemma 5.1.3 (Bačák [5], Kimura–Kohsaka [19]). *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into* $]-\infty, \infty]$*. Then,*

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n)$$

whenever a κ-bounded sequence $\{x_n\}$ of X is Δ-convergent to $x_0 \in X$.

Proof. We first assume that a κ-bounded sequence $\{x_n\}$ of X Δ-converges to $x_0 \in \operatorname{dom} f$. Suppose that

$$\liminf_{n\to\infty} f(x_n) < f(x_0).$$

Then, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, $i_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that $f(x_{n_i}) \leq f(x_0) - \varepsilon$ for any $i \in \mathbb{N}$ with $i \geq i_0$. Set $\varepsilon_0 = f(x_0) - \varepsilon$. Then, $\{x_{n_i}\}_{i\geq i_0}$ is included in $\operatorname{Lev}_{\varepsilon_0} f$. Since the level set $\operatorname{Lev}_{\varepsilon_0} f$ is closed and convex, we get $x_0 \in \operatorname{Lev}_{\varepsilon_0} f$ from Lemma 3.2.1 and therefore

$$f(x_0) \leq \varepsilon_0 = f(x_0) - \varepsilon.$$

This is a contradiction and we conclude

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

We next assume that a κ-bounded sequence $\{x_n\}$ of X Δ-converges to $x_0 \in X \setminus \operatorname{dom} f$. Suppose that

$$\liminf_{n\to\infty} f(x_n) < f(x_0) = \infty.$$

Let $m = \liminf_{n\to\infty} f(x_n)$. Then, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} f(x_{n_i}) = m < \infty.$$

Since $f(x_{n_i}) \to m$, there exists $i_0 \in \mathbb{N}$ such that $f(x_{n_i}) \leq m+1$ for all $i \in \mathbb{N}$ with $i \geq i_0$. Thus, $\{x_{n_i}\}_{i\geq i_0}$ is included in $\operatorname{Lev}_{m+1} f$. Since the level set $\operatorname{Lev}_{m+1} f$ is closed and convex, we get $x_0 \in \operatorname{Lev}_{m+1} f$ from Lemma 3.2.1 and therefore

$$\infty = f(x_0) \leq m+1 < \infty.$$

This is a contradiction and we conclude

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

It completes the proof. □

Theorem 5.1.4. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Suppose that for a sequence $\{x_n\}$ of X such that $d(x_n, p) \to D_\kappa/2$ for some $p \in X$, a sequence $\{f(x_n)\}$ diverges to ∞. Then,* $\operatorname{Min} f$ *is nonempty.*

Proof. Let $m = \inf_{y\in X} f(y) < \infty$. Then, we can take a sequence $\{x_n\}$ of X such that $\{f(x_n)\}$ converges to m. Then, $\{x_n\}$ is κ-bounded. Indeed, if it is not κ-bounded, or equivalently,

$$\inf_{x\in X} \limsup_{n\to\infty} d(x, x_n) = \frac{D_\kappa}{2},$$

then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $d(x_{n_i}, p) \to D_\kappa/2$ for some $p \in X$ and hence $f(x_{n_i}) \to \infty$ as $n \to \infty$. This is a contradiction. Now, we can take a Δ-convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Let $x_0 \in X$ be its Δ-limit. Then, it holds from Lemma 5.1.3 that

$$f(x_0) \leq \liminf_{i\to\infty} f(x_{n_i}) = m = \inf_{y\in X} f(y)$$

and hence $x_0 \in \operatorname{Min} f$. It means that $\operatorname{Min} f$ is nonempty. □

Further, we get the following:

Lemma 5.1.5. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper function from X into $]-\infty, \infty]$ such that* $\operatorname{Min} f$ *is nonempty. Suppose that*

$$f\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) < \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

for each $x, y \in \operatorname{dom} f$ with $x \neq y$. Then, $\operatorname{Min} f$ *is a singleton.*

Proof. Let $z_1, z_2 \in \operatorname{Min} f$ with $z_1 \neq z_2$. Then,

$$f\left(\frac{1}{2}z_1 \oplus \frac{1}{2}z_2\right) < \frac{1}{2}f(z_1) + \frac{1}{2}f(z_2) = \inf_{y\in X} f(y).$$

This is a contradiction. Therefore, $\operatorname{Min} f$ is a singleton. □

5.2 Resolvent operators for convex functions

In what follows, we consider a resolvent operator for a convex function. We first prove the following lemma:

Lemma 5.2.1. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that $\operatorname{Min} f$ is nonempty. For a point $x \in X$, define a function f_x by*

$$f_x(z) = f(z) + \phi_\kappa(z, x)$$

for each $z \in X$. Then, $\operatorname{Min} f_x$ is a singleton.

Proof. Fix $x \in X$ arbitrarily. Then, f_x is also proper, lower semicontinuous and convex. Moreover,

$$f_x\left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) < \frac{1}{2}f_x(y_1) + \frac{1}{2}f_x(y_2)$$

for each $y_1, y_2 \in X$ with $y_1 \neq y_2$ from Theorem 1.2.6. For $p \in \operatorname{Min} f$ and $z \in X$, it follows that

$$-\infty < f(p) \leq f(z) \leq f_x(z)$$

and hence f_x is bounded below. Let $m = \inf_{z \in X} f_x(z) \in \mathbb{R}$. Then, we can take a sequence $\{z_n\}$ of X such that $\{f_x(z_n)\}$ converges to m. Then, we obtain

$$f(p) + \phi_\kappa(z_n, x) \leq f(z_n) + \phi_\kappa(z_n, x) = f_x(z_n)$$

and thus

$$\limsup_{n \to \infty} \phi_\kappa(z_n, x) \leq m - f(p) \leq f(p) + \phi_\kappa(p, x) - f(p) = \phi_\kappa(p, x).$$

It implies that $\{z_n\}$ is κ-bounded and it has a Δ-convergent subsequence $\{z_{n_i}\}$ from Lemma 3.2.3. Let $z_0 \in X$ be a Δ-limit of $\{z_{n_i}\}$. From Lemma 5.1.3,

$$f_x(z_0) \leq \liminf_{i \to \infty} f_x(z_{n_i}) = m = \inf_{z \in X} f_x(z),$$

which implies that $z_0 \in \operatorname{Min} f_x$ and therefore $\operatorname{Min} f_x$ is nonempty. Consequently, $\operatorname{Min} f_x$ is a singleton from Lemma 5.1.5. □

Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that $\operatorname{Min} f$ is nonempty. We define a single-valued mapping R_f from X into $\operatorname{dom} f$ as

$$R_f x = \operatorname*{Argmin}_{z \in X} \left(f(z) + \phi_\kappa(z, x)\right)$$

for each $x \in X$. We call R_f a resolvent operator for f.

Remark 14. Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that $\operatorname{Min} f$ is nonempty. Then,

$$R_f x = \begin{cases} \operatorname*{Argmin}_{z \in X} \left(f(z) - \dfrac{\cos\left(\sqrt{\kappa} d(z,x)\right)}{\kappa} \right) & (\kappa > 0); \\ \operatorname*{Argmin}_{z \in X} \left(f(z) + \dfrac{1}{2} d(z,x)^2 \right) & (\kappa = 0); \\ \operatorname*{Argmin}_{z \in X} \left(f(z) + \dfrac{\cosh\left(\sqrt{-\kappa} d(z,x)\right)}{-\kappa} \right) & (\kappa < 0) \end{cases}$$

for $x \in X$.

Remark 15. Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ and C a nonempty closed convex subset of X. We define an indicator function ι_C from X into $]-\infty, \infty]$ as follows: For $x \in X$,

$$\iota_C(x) = \begin{cases} 0 & (x \in C); \\ \infty & (x \in X \setminus C). \end{cases}$$

Then, ι_C is a proper lower semicontinuous convex function. From the definition of ι_C, we get $\operatorname{Min} \iota_C = C$. Further,

$$R_{\iota_C} x = \operatorname*{Argmin}_{z \in X} \left(\iota_C(z) + \phi_\kappa(z,x)\right) = \operatorname*{Argmin}_{z \in C} \phi_\kappa(z,x) = \operatorname*{Argmin}_{z \in C} d(z,x) = P_C x$$

for all $x \in X$, where P_C is a metric projection onto C.

We can prove the following results:

Lemma 5.2.2. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that $\operatorname{Min} f$ is nonempty. Then,*

$$f(R_f x) \le f(w) + \frac{l}{c'_\kappa(l)} \left(\phi_\kappa(w,x) - \phi_\kappa(w, R_f x) - c''_\kappa(l) \phi_\kappa(R_f x, x)\right)$$

for any $w, x \in X$ with $R_f x \neq w$, where $l = d(R_f x, w)$.

Proof. Let $w, x \in X$ with $R_f x \neq w$ and set $l = d(R_f x, w)$. Let $w_\tau = \tau w \oplus (1-\tau) R_f x$ for $\tau \in]0,1[$. Then,

$$f(R_f x) + \phi_\kappa(R_f x, x) \le f(w_\tau) + \phi_\kappa(w_\tau, x) \le \tau f(w) + (1-\tau) f(R_f x) + \phi_\kappa(w_\tau, x)$$

and hence

$$\tau f(R_f x) \le \tau f(w) + \phi_\kappa(w_\tau, x) - \phi_\kappa(R_f x, x).$$

From the parallelogram law of X, we have

$$\tau f(R_f x) \le \tau f(w) + (\tau)_l^\kappa \phi_\kappa(w,x) - (\tau)_l^\kappa \phi_\kappa(w, w_\tau) - \left(1 - (1-\tau)_l^\kappa\right) \phi_\kappa(R_f x, x).$$

Dividing both sides by τ and letting $\tau \searrow 0$, from Lemma 1.1.2, we get

$$f(R_f x) \le f(w) + \frac{l}{c'_\kappa(l)} \left(\phi_\kappa(w,x) - \phi_\kappa(w, R_f x) - c''_\kappa(l) \phi_\kappa(R_f x, x)\right).$$

Thus, we obtain the desired result. □

Lemma 5.2.3. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that* $\operatorname{Min} f$ *is nonempty. Then,* $\operatorname{Min} f = \operatorname{Fix} R_f$.

Proof. Let $p \in \operatorname{Min} f$. Then, from the definition of R_f, we have

$$f(p) + \phi_\kappa(R_f p, p) \le f(R_f p) + \phi_\kappa(R_f p, p) \le f(p) + \phi_\kappa(p, p) = f(p),$$

which implies that $d(R_f p, p) = 0$ and thus $p \in \operatorname{Fix} R_f$. Conversely, let $p \in \operatorname{Fix} R_f$. Fix $w \in X$ with $p \ne w$ arbitrarily. From Lemma 5.2.2,

$$f(p) \le f(w) + \frac{d(p, w)}{c'_\kappa(d(p, w))} \left(\phi_\kappa(w, p) - \phi_\kappa(w, p) - c''_\kappa(d(p, w))\phi_\kappa(p, p)\right) = f(w),$$

which implies that $p \in \operatorname{Min} f$. □

In what follows, we consider nonspreadindness of resolvent operators for convex functions.

Theorem 5.2.4. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that* $\operatorname{Min} f$ *is nonempty. Then,*

$$\begin{aligned} &\lambda\phi_\kappa(R_{\lambda f} x, R_{\mu f} y) + \mu\phi_\kappa(R_{\mu f} y, R_{\lambda f} x) \\ &\le \lambda\phi_\kappa(R_{\lambda f} x, y) + \mu\phi_\kappa(R_{\mu f} y, x) - \mu c''_\kappa(l)\phi_\kappa(R_{\lambda f} x, x) - \lambda c''_\kappa(l)\phi_\kappa(R_{\mu f} y, y) \end{aligned}$$

for each $x, y \in X$ and $\lambda, \mu > 0$, where $l = d(R_{\lambda f} x, R_{\mu f} y)$.

Proof. Let $x, y \in X$, $\lambda, \mu > 0$ and $l = d(R_{\lambda f} x, R_{\mu f} y)$. If $l = 0$, then we get the desired inequality. We assume that $l \ne 0$. From Lemma 5.2.2,

$$f(R_{\lambda f} x) \le f(R_{\mu f} y) + \frac{l}{\lambda c'_\kappa(l)} \left(\phi_\kappa(R_{\mu f} y, x) - \phi_\kappa(R_{\mu f} y, R_{\lambda f} x) - c''_\kappa(l)\phi_\kappa(R_{\lambda f} x, x)\right)$$

and

$$f(R_{\mu f} y) \le f(R_{\lambda f} x) + \frac{l}{\mu c'_\kappa(l)} \left(\phi_\kappa(R_{\lambda f} x, y) - \phi_\kappa(R_{\lambda f} x, R_{\mu f} y) - c''_\kappa(l)\phi_\kappa(R_{\mu f} y, y)\right).$$

Adding these inequalities and multiplying both sides by $(\lambda\mu c'_\kappa(l))/l$, we get

$$\begin{aligned} 0 \le\ & \mu\left(\phi_\kappa(R_{\mu f} y, x) - \phi_\kappa(R_{\mu f} y, R_{\lambda f} x) - c''_\kappa(l)\phi_\kappa(R_{\lambda f} x, x)\right) \\ &+ \lambda\left(\phi_\kappa(R_{\lambda f} x, y) - \phi_\kappa(R_{\lambda f} x, R_{\mu f} y) - c''_\kappa(l)\phi_\kappa(R_{\mu f} y, y)\right). \end{aligned}$$

This is the desired result. □

Remark 16. Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$. Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that $\operatorname{Min} f$ is nonempty. Then, R_f is firmly geodesically nonspreading.

As a direct consequence of Theorem 5.2.4, we have the following corollaries:

Corollary 5.2.5. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that* $\operatorname{Min} f$ *is nonempty. Then,*

$$\lambda\phi_\kappa(R_{\lambda f} x, R_{\mu f} y) + \mu\phi_\kappa(R_{\mu f} y, R_{\lambda f} x) \le \lambda\phi_\kappa(R_{\lambda f} x, y) + \mu\phi_\kappa(R_{\mu f} y, x)$$

for each $x, y \in X$ and $\lambda, \mu > 0$.

Corollary 5.2.6. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that $\operatorname{Min} f$ is nonempty. Then,*

$$\phi_\kappa(p, R_f x) \leq \phi_\kappa(p, x) - c''_\kappa(d(R_f x, p))\phi_\kappa(R_f x, x)$$

for each $x \in X$ and $p \in \operatorname{Min} f$. Therefore, R_f is quasinonexpansive.

5.3 Equilibrium problems

In this section, we will discuss an equilibrium problem on geodesic spaces.

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X. Let F be a function from K^2 into $\mathbb{R}$. We denote the set of all solutions to an equilibrium problem for F by $\operatorname{Equil} F$, namely,

$$\operatorname{Equil} F = \left\{ z \in K \;\middle|\; \inf_{y \in K} F(z, y) \geq 0 \right\}.$$

In this thesis, we suppose the four conditions to F as follows:

(E1) For all $y \in K$, $F(y, y) = 0$;
(E2) for all $y, z \in K$, $F(y, z) + F(z, y) \leq 0$;
(E3) for all $z \in K$, $F(z, \cdot) \colon K \to \mathbb{R}$ is lower semicontinuous and convex;
(E4) for all $y, z \in K$, $\limsup_{t \searrow 0} F(ty \oplus (1-t)z, y) \leq F(z, y)$.

Remark 17. Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that $\operatorname{dom} f$ is closed. We define a function F from $(\operatorname{dom} f)^2$ into $\mathbb{R}$ as

$$F(z, y) = f(y) - f(z)$$

for each $y, z \in \operatorname{dom} f$. Then, F satisfies the conditions (E1)–(E4) and $\operatorname{Equil} F = \operatorname{Min} f$.

Remark 18. Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions (E1)–(E4). Then, for $\lambda > 0$, a function λF from K^2 into $\mathbb{R}$ also satisfies the four conditions. Furthermore, $\operatorname{Equil} F = \operatorname{Equil} \lambda F$.

We get the following facts:

Lemma 5.3.1. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4)*. Then,*

$$\operatorname{Equil} F = \left\{ z \in K \;\middle|\; \sup_{y \in K} F(y, z) \leq 0 \right\}.$$

Proof. Let $z \in K$ with $\sup_{w \in K} F(w, z) \leq 0$. Fix $y \in K$ arbitrarily. Then, from the conditions (E1) and (E3), for $\tau \in]0, 1[$,

$$\begin{aligned} 0 &= F(\tau y \oplus (1-\tau)z, \tau y \oplus (1-\tau)z) \\ &\leq \tau F(\tau y \oplus (1-\tau)z, y) + (1-\tau)F(\tau y \oplus (1-\tau)z, z) \leq \tau F(\tau y \oplus (1-\tau)z, y). \end{aligned}$$

Dividing both sides by τ and letting $\tau \searrow 0$, from the condition (E4), we have

$$0 \leq \limsup_{\tau \searrow 0} F(\tau y \oplus (1-\tau)z, y) \leq F(z, y).$$

Since $y \in K$ is arbitrary, we have $z \in \operatorname{Equil} F$. Inversely, take $z \in \operatorname{Equil} F$. Then, from the condition (E2), for fixed $y \in K$, we get

$$0 \leq F(z, y) \leq -F(y, z)$$

and hence $F(y, z) \leq 0$. It implies that $\sup_{y \in K} F(y, z) \leq 0$. □

Corollary 5.3.2. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4). *Then,* $\operatorname{Equil} F$ *is closed and convex.*

Proof. Let $F_2(\cdot) = \sup_{y \in K} F(y, \cdot)$. Then, F_2 is a lower semicontinuous convex function from K into $]-\infty, \infty]$. We know

$$\operatorname{Equil} F = \left\{ z \in K \;\middle|\; \sup_{y \in K} F(y, z) \leq 0 \right\} = \{z \in K \mid F_2(z) \leq 0\} = \operatorname{Lev}_0 F_2.$$

From Lemma 5.1.1, we get the desired result. □

We next prove the following propositions:

Theorem 5.3.3. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4). *Let C be a nonempty closed convex subset of K such that* $\operatorname{rad} C < D_\kappa/2$. *Then,* $\operatorname{Equil} F|_{C^2}$ *is nonempty, where $F|_{C^2} : C^2 \to \mathbb{R}$ is the restriction of F to C^2. Here,* $\operatorname{rad} C$ *is defined by* $\inf_{y \in C} \sup_{x \in C} d(y, x)$.

Proof. For $y \in C$, define a subset $G(y)$ of C by

$$G(y) = \{v \in C \mid F(y, v) \leq 0\}.$$

Then, $G(y)$ is nonempty, closed and convex. Let $E = \{y_1, y_2, \dots, y_N\}$ be an arbitrary finite subset of C, and we show $\operatorname{co} E \subset \bigcup_{i=1}^N G(y_i)$. Set $N_E = \{1, 2, \dots, N\} \subset \mathbb{N}$. Note that the convex hull is defined by $\operatorname{co} E = \bigcup_{j=1}^\infty W_j$, where $W_1 = E$ and

$$W_{j+1} = \{t u_j \oplus (1-t) v_j \mid u_j, v_j \in W_j, t \in [0,1]\}$$

for $j \in \mathbb{N}$. We prove the following statement by induction: For every $j \in \mathbb{N}$, if $q \in W_j$, then there exists $\{\mu_i \mid i \in N_E\} \subset [0,1]$ such that $\sum_{i=1}^N \mu_i = 1$ and

$$F(y_k, q) \leq \sum_{i=1}^N \mu_i F(y_k, y_i)$$

for any $k \in N_E$. We first suppose $j = 1$. If $q \in W_1 = E$, then $q = y_{i_0}$ for some $i_0 \in N_E$. Thus, letting $\mu_i = 1$ if $i = i_0$; $\mu_i = 0$ if $i \neq i_0$, we have

$$F(y_k, q) = F(y_k, y_{i_0}) = \sum_{i=1}^N \mu_i F(y_k, y_i)$$

for each $k \in N_E$. We next suppose that the statement above holds for fixed $j \in \mathbb{N}$. Fix $q \in W_{j+1}$ arbitrarily. Then, $q = tu_j \oplus (1-t)v_j$ for some $u_j, v_j \in W_j$ and $t \in [0,1]$. From the assumption of the induction, there exist $\{\sigma_i \mid i \in N_E\}, \{\tau_i \mid i \in N_E\} \subset [0,1]$ with $\sum_{i=1}^N \sigma_i = \sum_{i=1}^N \tau_i = 1$ such that

$$F(y_k, u_j) \le \sum_{i=1}^N \sigma_i F(y_k, y_i) \text{ and } F(y_k, v_j) \le \sum_{i=1}^N \tau_i F(y_k, y_i)$$

for any $k \in N_E$. Let $\mu_i = t\sigma_i + (1-t)\tau_i$ for each $i \in N_E$. Note that $\sum_{i=1}^N \mu_i = 1$. Since $F(y_k, \cdot)$ is convex, we obtain

$$\begin{aligned} F(y_k, q) &= F(y_k, tu_j \oplus (1-t)v_j) \le tF(y_k, u_j) + (1-t)F(y_k, v_j) \\ &\le \sum_{i=1}^N t\sigma_i F(y_k, y_i) + \sum_{i=1}^N (1-t)\tau_i F(y_k, y_i) = \sum_{i=1}^N \mu_i F(y_k, y_i) \end{aligned}$$

for any $k \in N_E$ and therefore we get the conclusion. Now, we can prove that $\operatorname{co} E$ is included in $\bigcup_{i=1}^N G(y_i)$ by contradiction. Suppose that there exists $q \in \operatorname{co} E$ such that $F(y_k, q) > 0$ for every $k \in N_E$. Then, there exists $\{\mu_i \mid i \in N_E\} \subset [0,1]$ such that $\sum_{i=1}^N \mu_i = 1$ and that

$$F(y_k, q) \le \sum_{i=1}^N \mu_i F(y_k, y_i)$$

for any $k \in N_E$. Furthermore,

$$\sum_{k=1}^N \sum_{i=1}^N \mu_k \mu_i F(y_k, y_i) = \frac{1}{2} \sum_{k=1}^N \sum_{i=1}^N \mu_k \mu_i (F(y_k, y_i) + F(y_i, y_k)) \le 0.$$

Hence,

$$0 < \sum_{k=1}^N \mu_k F(y_k, q) \le \sum_{k=1}^N \sum_{i=1}^N \mu_k \mu_i F(y_k, y_i) \le 0$$

and therefore this is a contradiction. Hence, there exists $k \in N_E$ such that $q \in G(y_k)$, which implies that $q \in \bigcup_{i=1}^N G(y_i)$. Consequently, we obtain $\operatorname{co} E \subset \bigcup_{i=1}^N G(y_i)$. From Lemma 3.3.4, $\bigcap_{y \in C} G(y)$ is nonempty. Therefore, we can take a point $z_0 \in \bigcap_{y \in C} G(y) \subset C$, which satisfies $F(y, z_0) \le 0$ for any $y \in C$. It implies that $z_0 \in \operatorname{Equil} F|_{C^2}$. □

Corollary 5.3.4. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4). *Assume that* $\operatorname{rad} K < D_\kappa/2$. *Then,* $\operatorname{Equil} F$ *is nonempty.*

Lemma 5.3.5. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1) *and* (E2). *Suppose that* $\operatorname{Equil} F$ *is nonempty and*

$$F\left(z, \frac{1}{2}x \oplus \frac{1}{2}y\right) < \frac{1}{2}F(z, x) + \frac{1}{2}F(z, y)$$

for each $x, y, z \in K$ with $x \neq y$. Then, $\operatorname{Equil} F$ *is a singleton.*

Proof. Let $z_1, z_2 \in \operatorname{Equil} F$ with $z_1 \neq z_2$. Then,

$$\begin{aligned} 0 &\leq F\left(z_1, \frac{1}{2}z_1 \oplus \frac{1}{2}z_2\right) + F\left(z_2, \frac{1}{2}z_1 \oplus \frac{1}{2}z_2\right) \\ &< \frac{1}{2}F(z_1, z_1) + \frac{1}{2}F(z_1, z_2) + \frac{1}{2}F(z_2, z_1) + \frac{1}{2}F(z_2, z_2) = \frac{1}{2}(F(z_1, z_2) + F(z_2, z_1)) \leq 0. \end{aligned}$$

This is a contradiction. Therefore, $\operatorname{Equil} F$ is a singleton. □

5.4 Resolvent operators for equilibrium problems

In what follows, we consider a resolvent operator for an equilibrium problem. We first prove the following lemma:

Lemma 5.4.1. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4). *Suppose that* $\operatorname{Equil} F$ *is nonempty. For $x \in X$, define a function F_x from K^2 into $\mathbb{R}$ by*

$$F_x(z, y) = F(z, y) + \phi_\kappa(y, x) - \phi_\kappa(z, x)$$

for each $y, z \in K$. Then, $\operatorname{Equil} F_x$ *is a singleton.*

Proof. Fix $x \in X$ arbitrarily and let $p \in \operatorname{Equil} F$. Then, we can take a real number

$$L \in \left] d(p, x), \frac{D_\kappa}{2} \right[$$

since X is admissible. We notice that F_x satisfies the conditions (E1)–(E4), and

$$F_x\left(z, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) < \frac{1}{2}F_x(z, y_1) + \frac{1}{2}F_x(z, y_2)$$

for $y_1, y_2, z \in K$ with $y_1 \neq y_2$ from Theorem 1.2.6. For any $z \in K$, since $F(z, p) \leq 0$,

$$F_x(z, p) = F(z, p) + \phi_\kappa(p, x) - \phi_\kappa(z, x) \leq c_\kappa(L) - \phi_\kappa(z, x).$$

If $z \in K$ satisfies that $d(z, x) = L$, then $F_x(z, p) \leq 0$. Let

$$C = \{z \in K \mid d(z, x) \leq L\} = \{z \in K \mid \phi_\kappa(z, x) \leq c_\kappa(L)\}.$$

Then, C is nonempty, closed and convex. Note that C has the convex hull finite property. Moreover,

$$\operatorname{rad} C = \inf_{w \in C} \sup_{z \in C} d(z, w) \leq \sup_{z \in C} d(z, P_C x) \leq \sup_{z \in C} d(z, x) \leq L,$$

where P_C is a metric projection onto C. It means that C is an admissible complete CAT(κ) space such that $\operatorname{rad} C < D_\kappa/2$. From Theorem 5.3.3, there is $z_0 \in C$ such that $\inf_{y \in C} F_x(z_0, y) \geq 0$. We next show $z_0 \in \operatorname{Equil} F_x$. Note that $d(z_0, x) \leq L$. Let

$$u_0 = \begin{cases} p & (d(z_0, x) = L); \\ z_0 & (d(z_0, x) < L). \end{cases}$$

Then, we have $F_x(z_0, u_0) \leq 0$. By the definition of u_0, we have $d(u_0, x) < L$. Fix $y \in K$ arbitrarily. Since

$$\phi_\kappa(\beta y \oplus (1-\beta)u_0, x) \leq \beta\phi_\kappa(y, x) + (1-\beta)\phi_\kappa(u_0, x)$$

for any $\beta \in]0, 1[$ and $\phi_\kappa(u_0, x) < c_\kappa(L)$, there exists $\beta_0 \in]0, 1[$ such that

$$\phi_\kappa(\beta_0 y \oplus (1-\beta_0)u_0, x) \leq \beta_0\phi_\kappa(y, x) + (1-\beta_0)\phi_\kappa(u_0, x) \leq c_\kappa(L),$$

which implies that $\beta_0 y \oplus (1-\beta_0)u_0 \in C$. Since $F_x(z_0, \cdot)$ is convex, we obtain

$$0 \leq \frac{1}{\beta_0} F_x(z_0, \beta_0 y \oplus (1-\beta_0)u_0) \leq F_x(z_0, y) + \frac{1-\beta_0}{\beta_0} F_x(z_0, u_0) \leq F_x(z_0, y)$$

and therefore $z_0 \in \operatorname{Equil} F_x$. It means that $\operatorname{Equil} F_x$ is nonempty. Thus, $\operatorname{Equil} F_x$ is a singleton from Lemma 5.3.5. □

Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions (E1)–(E4). Suppose that $\operatorname{Equil} F$ is nonempty. We define a single-valued mapping R_F from X into K as

$$R_F x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (F(z, y) + \phi_\kappa(y, x)) - \phi_\kappa(z, x) \geq 0 \right\}$$

for each $x \in X$. We call R_F a resolvent operator for F.

Remark 19. Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions (E1)–(E4). Suppose that $\operatorname{Equil} F$ is nonempty. If $\kappa = 0$, then

$$R_F x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(F(z, y) + \frac{1}{2} d(y, x)^2 \right) - \frac{1}{2} d(z, x)^2 \geq 0 \right\}.$$

If $\kappa = 1$, then

$$R_F x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (F(z, y) - \cos d(y, x)) + \cos d(z, x) \geq 0 \right\}.$$

If $\kappa = -1$, then

$$R_F x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (F(z, y) + \cosh d(y, x)) - \cosh d(z, x) \geq 0 \right\}.$$

We get the following results:

Lemma 5.4.2. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4). *Suppose that* $\operatorname{Equil} F$ *is nonempty. Then,*

$$0 \leq F(R_F x, w) + \frac{l}{c'_\kappa(l)} (\phi_\kappa(w, x) - \phi_\kappa(w, R_F x) - c''_\kappa(l)\phi_\kappa(R_F x, x))$$

for any $x \in X$ and $w \in K$ with $R_F x \neq w$, where $l = d(R_F x, w)$.

Proof. Let $x \in X$ and $w \in K$ with $R_F x \neq w$. For $\tau \in]0,1[$, let $w_\tau = \tau w \oplus (1-\tau) R_F x \in K$. Then, from the condition (E3),

$$\begin{aligned} 0 &\leq F(R_F x, w_\tau) + \phi_\kappa(w_\tau, x) - \phi_\kappa(R_F x, x) \\ &\leq \tau F(R_F x, w) + \phi_\kappa(w_\tau, x) - \phi_\kappa(R_F x, x). \end{aligned}$$

From the parallelogram law of X, we obtain

$$0 \leq \tau F(R_F x, w) + (\tau)_l^\kappa \phi_\kappa(w, x) - (\tau)_l^\kappa \phi_\kappa(w, w_\tau) - \left(1 - (1-\tau)_l^\kappa\right) \phi_\kappa(R_F x, x),$$

where $l = d(R_F x, w) > 0$. Dividing both sides by τ and letting $\tau \searrow 0$, from Lemma 1.1.2, we have

$$0 \leq F(R_F x, w) + \frac{l}{c'_\kappa(l)} \left(\phi_\kappa(w, x) - \phi_\kappa(w, R_F x) - c''_\kappa(l) \phi_\kappa(R_F x, x)\right).$$

Thus, we obtain the desired result. □

Lemma 5.4.3. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4). *Suppose that* $\operatorname{Equil} F$ *is nonempty. Then,* $\operatorname{Fix} R_F = \operatorname{Equil} F$.

Proof. Let $p \in \operatorname{Fix} R_F$. Then, from Lemma 5.4.2, we have

$$0 \leq F(p, w) + \frac{d(p, w)}{c'_\kappa(d(p, w))} \left(\phi_\kappa(w, p) - \phi_\kappa(w, p) - c''_\kappa(d(p, w)) \phi_\kappa(p, p)\right) = F(p, w)$$

for every $w \in K$ with $p \neq w$ and hence $p \in \operatorname{Equil} F$. On the other hand, for $p \in \operatorname{Equil} F$,

$$\inf_{y \in K} (F(p, y) + \phi_\kappa(y, p)) - \phi_\kappa(p, p) = \inf_{y \in K} (F(p, y) + \phi_\kappa(y, p)) \geq \inf_{y \in K} F(p, y) \geq 0$$

and therefore $p = R_F p$ and thus $p \in \operatorname{Fix} R_F$. □

Theorem 5.4.4. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4). *Suppose that* $\operatorname{Equil} F$ *is nonempty. Then,*

$$\begin{aligned} &\lambda \phi_\kappa(R_{\lambda F} x, R_{\mu F} y) + \mu \phi_\kappa(R_{\mu F} y, R_{\lambda F} x) \\ &\leq \lambda \phi_\kappa(R_{\lambda F} x, y) + \mu \phi_\kappa(R_{\mu F} y, x) - \mu c''_\kappa(l) \phi_\kappa(R_{\lambda F} x, x) - \lambda c''_\kappa(l) \phi_\kappa(R_{\mu F} y, y) \end{aligned}$$

for each $x, y \in X$ and $\lambda, \mu > 0$, where $l = d(R_{\lambda F} x, R_{\mu F} y)$.

Proof. Let $x, y \in X$, $\lambda, \mu > 0$ and $l = d(R_{\lambda F} x, R_{\mu F} y)$. If $l = 0$, then we get the desired inequality. We assume $l \neq 0$. From Lemma 5.4.2, we have

$$0 \leq F(R_{\lambda F} x, R_{\mu F} y) + \frac{l}{\lambda c'_\kappa(l)} \left(\phi_\kappa(R_{\mu F} y, x) - \phi_\kappa(R_{\mu F} y, R_{\lambda F} x) - c''_\kappa(l) \phi_\kappa(R_{\lambda F} x, x)\right)$$

and

$$0 \leq F(R_{\mu F} y, R_{\lambda F} x) + \frac{l}{\mu c'_\kappa(l)} \left(\phi_\kappa(R_{\lambda F} x, y) - \phi_\kappa(R_{\lambda F} x, R_{\mu F} y) - c''_\kappa(l) \phi_\kappa(R_{\mu F} y, y)\right).$$

Adding these inequalities and multiplying both sides by $(\lambda\mu c'_\kappa(l))/l$, we get

$$\begin{aligned} 0 \le \mu\left(\phi_\kappa(R_{\mu F}y, x) - \phi_\kappa(R_{\mu F}y, R_{\lambda F}x) - c''_\kappa(l)\phi_\kappa(R_{\lambda F}x, x)\right) \\ + \lambda\left(\phi_\kappa(R_{\lambda F}x, y) - \phi_\kappa(R_{\lambda F}x, R_{\mu F}y) - c''_\kappa(l)\phi_\kappa(R_{\mu F}y, y)\right). \end{aligned}$$

This is the desired result. □

Remark 20. Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions (E1)–(E4). Suppose that $\operatorname{Equil} F$ is nonempty. Then, R_F is firmly geodesically nonspreading.

As a direct consequence of Theorem 5.4.4, we have the following corollaries:

Corollary 5.4.5. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4)*. Suppose that* $\operatorname{Equil} F$ *is nonempty. Then,*

$$\lambda\phi_\kappa(R_{\lambda F}x, R_{\mu F}y) + \mu\phi_\kappa(R_{\mu F}y, R_{\lambda F}x) \le \lambda\phi_\kappa(R_{\lambda F}x, y) + \mu\phi_\kappa(R_{\mu F}y, x)$$

for each $x, y \in X$ and $\lambda, \mu > 0$.

Corollary 5.4.6. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4)*. Suppose that* $\operatorname{Equil} F$ *is nonempty. Then,*

$$\phi_\kappa(p, R_F x) \le \phi_\kappa(p, x) - c''_\kappa(d(R_F x, p))\phi_\kappa(R_F x, x)$$

for each $x \in X$ and $p \in \operatorname{Equil} F$. Therefore, R_F is quasinonexpansive.

5.5 Proximal point algorithm

In this section, we get approximation theorems with the proximal point algorithm. We first obtain the following lemma:

Lemma 5.5.1. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that* $\operatorname{Min} f$ *is nonempty. Then,*

$$f(R_f x) - \inf_{y \in X} f(y) \le 2\left(\phi_\kappa(p, x) - \phi_\kappa(p, R_f x)\right)$$

for any $x \in X$ and $p \in \operatorname{Min} f$.

Proof. Let $x \in X$ and $p \in \operatorname{Min} f$. Suppose $R_f x \ne p$. If $\kappa \le 0$, then

$$\frac{d(R_f x, p)}{c'_\kappa(d(R_f x, p))} \le 1 < 2.$$

If $\kappa > 0$, then

$$\frac{d(R_f x, p)}{c'_\kappa(d(R_f x, p))} = \frac{\sqrt{\kappa}d(R_f x, p)}{\sin(\sqrt{\kappa}d(R_f x, p))} \le \frac{\pi}{2} < 2.$$

From Lemma 5.2.2,

$$\begin{aligned} f(R_f x) - f(p) &\le \frac{d(R_f x, p)}{c'_\kappa(d(R_f x, p))} \left(\phi_\kappa(p, x) - \phi_\kappa(p, R_f x) - c''_\kappa(d(R_f x, p))\phi_\kappa(R_f x, x)\right) \\ &\le \frac{d(R_f x, p)}{c'_\kappa(d(R_f x, p))} \left(\phi_\kappa(p, x) - \phi_\kappa(p, R_f x)\right). \end{aligned}$$

Note that $\phi_\kappa(p, x) - \phi_\kappa(p, R_f x) \ge 0$ since R_f is quasinonexpansive. Therefore,

$$f(R_f x) - \inf_{y \in X} f(y) \le 2\left(\phi_\kappa(p, x) - \phi_\kappa(p, R_f x)\right)$$

whenever $R_f x \ne p$. This inequality holds when $R_f x = p$. □

Now, we can prove a Δ-convergence theorem for convex function with the proximal point algorithm:

Theorem 5.5.2. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that $\operatorname{Min} f$ is nonempty. Let $\{\lambda_n\}$ be a positive real sequence. For a given initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = \operatorname*{Argmin}_{z \in X} \left(\lambda_n f(z) + \phi_\kappa(z, x_n)\right) \subset \operatorname{dom} f$$

for each $n \in \mathbb{N}$. Then, the following hold:

(i) *For $n \in \mathbb{N}$,*

$$\left| f(x_{n+1}) - \inf_{y \in X} f(y) \right| \le \frac{2c_\kappa(d(x_1, \operatorname{Min} f))}{\sum_{k=1}^{n} \lambda_k};$$

(ii) *if $\sum_{k=1}^{\infty} \lambda_k = \infty$, then $\{x_n\}$ Δ-converges to a minimiser $x_0 \in \operatorname{Min} f$ which is a limit of $\{P_{\operatorname{Min} f} x_n\}$, where $P_{\operatorname{Min} f}$ is a metric projection onto $\operatorname{Min} f$.*

Proof. Note that $x_{k+1} = R_{\lambda_k f} x_k$ for each $k \in \mathbb{N}$, where $R_{\lambda_k f}$ is the resolvent operator for $\lambda_k f$. From Lemma 4.2.1, a sequence $\{P_{\operatorname{Min} f} x_n\}$ converges to a point $x_0 \in \operatorname{Min} f$. We prove $\{x_n\}$ Δ-converges to x_0. Since $R_{\lambda_k f}$ is quasinonexpansive, we have

$$\phi_\kappa(x_0, x_{k+1}) = \phi_\kappa(x_0, R_{\lambda_k f} x_k) \le \phi_\kappa(x_0, x_k)$$

for each $k \in \mathbb{N}$. Therefore, $\{x_n\}$ is κ-bounded. From the definition of $R_{\lambda_k f}$,

$$\inf_{y \in X} f(y) \le f(x_{k+1}) \le f(R_{\lambda_k f} x_k) + \frac{1}{\lambda_k} \phi_\kappa(R_{\lambda_k f} x_k, x_k) \le f(x_k)$$

for each $k \in \mathbb{N}$, which means that $\{f(x_n)\}$ is decreasing and bounded below. Fix $n \in \mathbb{N}$ arbitrarily. From Lemma 5.5.1, for $k \in \{1, \dots, n\}$ and $p = P_{\operatorname{Min} f} x_1$,

$$\lambda_k \left(f(x_{k+1}) - \inf_{y \in X} f(y) \right) \le 2\left(\phi_\kappa(p, x_k) - \phi_\kappa(p, x_{k+1})\right).$$

Summing up these inequalities with respect to $k \in \{1, \dots, n\}$, we have

$$\sum_{k=1}^{n} \lambda_k \left(f(x_{k+1}) - \inf_{y \in X} f(y) \right) \le 2\left(\phi_\kappa(p, x_1) - \phi_\kappa(p, x_{n+1})\right) \le 2\phi_\kappa(p, x_1).$$

Since $\{f(x_n)\}$ is decreasing, we have

$$\left(f(x_{n+1}) - \inf_{y\in X} f(y)\right)\sum_{k=1}^{n}\lambda_k \leq 2\phi_\kappa(p, x_1).$$

Note that $d(x_1, p) = d(x_1, \operatorname{Min} f)$. Consequently,

$$0 \leq f(x_{n+1}) - \inf_{y\in X} f(y) \leq \frac{2c_\kappa(d(x_1, \operatorname{Min} f))}{\sum_{k=1}^{n}\lambda_k}.$$

Therefore, we obtain (i). We next show (ii). Suppose that $\sum_{k=1}^{\infty}\lambda_k = \infty$. Then,

$$\left|f(x_{n+1}) - \inf_{y\in X} f(y)\right| \leq \frac{2c_\kappa(d(x_1, \operatorname{Min} f))}{\sum_{k=1}^{n}\lambda_k} \to 0$$

as $n \to \infty$, which means that $\{f(x_n)\}$ converges to $\inf_{y\in X} f(y)$. Take a Δ-convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily and let $w \in X$ be its Δ-limit. From Lemma 5.1.3,

$$f(w) \leq \liminf_{i\to\infty} f(x_{n_i}) = \inf_{y\in X} f(y)$$

and thus $w \in \operatorname{Min} f$. Then, we have

$$\begin{aligned}\limsup_{i\to\infty} d(x_0, x_{n_i}) &\leq \limsup_{i\to\infty} \left(d(x_0, P_{\operatorname{Min} f}x_{n_i}) + d(P_{\operatorname{Min} f}x_{n_i}, x_{n_i})\right)\\ &= \limsup_{i\to\infty} d(P_{\operatorname{Min} f}x_{n_i}, x_{n_i}) \leq \limsup_{i\to\infty} d(w, x_{n_i}),\end{aligned}$$

which implies that $x_0 = w$ since w is a unique asymptotic centre of $\{x_{n_i}\}$. From Lemma 3.2.6, $\{x_n\}$ Δ-converges to $x_0 \in \operatorname{Min} f$, which completes the proof. □

In what follows, we consider a Δ-convergence theorem for an equilibrium problem. Before that, we prove the following lemma:

Lemma 5.5.3. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4)*. Suppose that* $\operatorname{Equil} F$ *is nonempty. Then,*

$$0 \leq F(R_F x, w) + 2\left|\phi_\kappa(w, x) - \phi_\kappa(w, R_F x)\right|$$

for any $x \in X$ and $w \in K$.

Proof. Let $x \in X$, $w \in K$ and $l = d(R_F x, w)$. If $l = 0$, then we easily obtain the desired inequality. We assume that $l \neq 0$. Then, from Lemma 5.4.2,

$$0 \leq F(R_F x, w) + \frac{l}{c_\kappa'(l)}\left(\phi_\kappa(w, x) - \phi_\kappa(w, R_F x) - c_\kappa''(l)\phi_\kappa(R_F x, x)\right).$$

Using the same fashions in Lemma 5.5.1, we get $l/c_\kappa'(l) \leq 2$. Therefore,

$$0 \leq F(R_F x, w) + \frac{l}{c_\kappa'(l)}\left(\phi_\kappa(w, x) - \phi_\kappa(w, R_F x) - c_\kappa''(l)\phi_\kappa(R_F x, x)\right)$$

$$\leq F(R_F x, w) + \frac{l}{c'_\kappa(l)} \left|\phi_\kappa(w,x) - \phi_\kappa(w, R_F x)\right|$$
$$\leq F(R_F x, w) + 2\left|\phi_\kappa(w,x) - \phi_\kappa(w, R_F x)\right|.$$

This is the desired result. □

Now, we can prove the following Δ-convergence theorem for an equilibrium problem with the proximal point algorithm:

Theorem 5.5.4. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and K a nonempty closed convex subset of X having the convex hull finite property. Let F be a function from K^2 into $\mathbb{R}$ satisfying the conditions* (E1)–(E4). *Suppose that* Equil F *is nonempty. Let $\{\lambda_n\}$ be a positive real sequence such that $\inf_{n\in\mathbb{N}} \lambda_n > 0$. For a given initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of K as follows:*

$$x_{n+1} = \left\{ z \in K \;\middle|\; \inf_{y\in K} (\lambda_n F(z,y) + \phi_\kappa(y, x_n)) - \phi_\kappa(z, x_n) \geq 0 \right\}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ Δ-converges to a solution $x_0 \in \operatorname{Equil} F$ which is a limit of $\{P_{\operatorname{Equil} F} x_n\}$, where $P_{\operatorname{Equil} F}$ is a metric projection onto Equil F.

Proof. Note that $x_{n+1} = R_{\lambda_n F} x_n$ for each $n \in \mathbb{N}$, where $R_{\lambda_n F}$ is the resolvent operator for $\lambda_n F$. We remark that a sequence $\{P_{\operatorname{Equil} F} x_n\}$ converges to a point $x_0 \in \operatorname{Equil} F$ from Lemma 4.2.1. We prove $\{x_n\}$ Δ-converges to x_0. From Corollary 5.4.6, we have

$$\begin{aligned}\phi_\kappa(x_0, x_{n+1}) &= \phi_\kappa(x_0, R_{\lambda_n F} x_n)\\ &\leq \phi_\kappa(x_0, x_n) - c''_\kappa(d(x_{n+1}, x_0))\phi_\kappa(R_{\lambda_n F} x_n, x_n) \leq \phi_\kappa(x_0, x_n)\end{aligned}$$

for each $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is κ-bounded, and there exists a limit of $\{\phi_\kappa(x_0, x_n)\}$. Note that there exists $L_0 > 0$ such that $L_0 \leq \inf_{n\in\mathbb{N}} c''_\kappa(d(x_n, x_0))$. Then, we get

$$\phi_\kappa(R_{\lambda_n f} x_n, x_n) \leq \frac{\phi_\kappa(x_0, x_n) - \phi_\kappa(x_0, x_{n+1})}{L_0} \to 0$$

and hence $\lim_{n\to\infty} d(R_{\lambda_n F} x_n, x_n) = 0$. Take a Δ-convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily and let $w \in X$ be its Δ-limit. Note that $w \in K$ from Lemma 3.2.1. For each $i \in \mathbb{N}$, set $w_i = R_{\lambda_{n_i} F} x_{n_i}$. We remark that $\lim_{i\to\infty} d(w_i, x_{n_i}) = 0$. Then, $\{w_i\}$ also Δ-converges to w. Indeed, for an arbitrary subsequence $\{w_{i_j}\}$ of $\{w_i\}$ and $\{v\} = \operatorname{AC}(\{w_{i_j}\})$, we obtain

$$\begin{aligned}\limsup_{j\to\infty} d(w, w_{i_j}) &\leq \limsup_{j\to\infty} \left(d(w, x_{n_{i_j}}) + d(x_{n_{i_j}}, w_{i_j}) \right)\\ &= \limsup_{j\to\infty} d(w, x_{n_{i_j}}) \leq \limsup_{j\to\infty} d(v, x_{n_{i_j}})\\ &\leq \limsup_{j\to\infty} \left(d(v, w_{i_j}) + d(x_{n_{i_j}}, w_{i_j}) \right) = \limsup_{j\to\infty} d(v, w_{i_j})\end{aligned}$$

and therefore $w = v$ since v is a unique asymptotic centre of $\{w_{i_j}\}$. Therefore, $\{w_i\}$ also Δ-converges to w. Fix $y \in K$ arbitrarily. From Lemma 5.5.3, we get

$$0 \leq F(w_i, y) + \frac{2\left|\phi_\kappa(y, x_{n_i}) - \phi_\kappa(y, w_i)\right|}{\lambda_{n_i}} \leq F(w_i, y) + \frac{2\left|c_\kappa(d(y, x_{n_i})) - c_\kappa(d(y, w_i))\right|}{\inf_{k\in\mathbb{N}} \lambda_k}$$

and thus

$$F(y,w_i) \leq -F(w_i,y) \leq \frac{2\,|c_\kappa(d(y,x_{n_i})) - c_\kappa(d(y,w_i))|}{\inf_{k\in\mathbb{N}} \lambda_k}$$

for any $i \in \mathbb{N}$. We remark that c_κ is uniformly continuous on a bounded set, and

$$\lim_{i\to\infty} |d(y,x_{n_i}) - d(y,w_i)| \leq \lim_{i\to\infty} d(w_i,x_{n_i}) = 0.$$

Hence, letting $i \to \infty$, since $\{w_i\}$ Δ-converges to w, we have

$$F(y,w) \leq \liminf_{i\to\infty} F(y,w_i) \leq \lim_{i\to\infty} \frac{2\,|c_\kappa(d(y,x_{n_i})) - c_\kappa(d(y,w_i))|}{\inf_{k\in\mathbb{N}} \lambda_k} = 0.$$

Since $y \in K$ is arbitrary, we have $\sup_{y\in K} F(y,w) \leq 0$ and hence $w \in \operatorname{Equil} F$. Then,

$$\begin{aligned}\limsup_{i\to\infty} d(x_0,x_{n_i}) &\leq \limsup_{i\to\infty} \left(d(x_0,P_{\operatorname{Equil} F}x_{n_i}) + d(P_{\operatorname{Equil} F}x_{n_i},x_{n_i})\right) \\ &= \limsup_{i\to\infty} d(P_{\operatorname{Equil} F}x_{n_i},x_{n_i}) \leq \limsup_{i\to\infty} d(w,x_{n_i}),\end{aligned}$$

which implies that $x_0 = w$ since w is a unique asymptotic centre of $\{x_{n_i}\}$. From Lemma 3.2.6, $\{x_n\}$ Δ-converges to $x_0 \in \operatorname{Equil} F$, which completes the proof. □

Chapter 6

Another convex combination

In this chapter, we consider another convex combination defined with the usual convex combination on the considered geodesic space.

6.1 Convex combination

Kimura and Sasaki [24, 25] introduced another convex combination than the usual one on geodesic spaces. In this section, we prove some properties of the convex combination.

Let $\kappa \in \mathbb{R}$. We define a function t_κ from $\{a \in \mathbb{R} \mid c''_\kappa(a) \neq 0\}$ into $\mathbb{R}$ as

$$t_\kappa(a) = \frac{c'_\kappa(a)}{c''_\kappa(a)} = \begin{cases} \dfrac{\tan(\sqrt{\kappa}a)}{\sqrt{\kappa}} & (\kappa > 0); \\ a & (\kappa = 0); \\ \dfrac{\tanh(\sqrt{-\kappa}a)}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases}$$

for each $a \in \mathbb{R}$ with $c''_\kappa(a) \neq 0$. Set $D(t_\kappa) = \{a \in \mathbb{R} \mid c''_\kappa(a) \neq 0\}$. Let

$$T_\kappa = \lim_{\delta \nearrow D_\kappa/2} t_\kappa(\delta) = \begin{cases} \infty & (\kappa \geq 0); \\ \dfrac{1}{\sqrt{-\kappa}} & (\kappa < 0). \end{cases}$$

Then, we know the following:

- $t_\kappa(-a) = -t_\kappa(a)$ for any $a \in D(t_\kappa)$;
- $t_\kappa \colon \,]-D_\kappa/2, D_\kappa/2[\,\to\,]-T_\kappa, T_\kappa[$ is continuous, increasing, bijective and $t_\kappa(0) = 0$;
- $t_\kappa^{-1} \colon \,]-T_\kappa, T_\kappa[\,\to\,]-D_\kappa/2, D_\kappa/2[$ is continuous and increasing.

Fix $a \in D(t_\kappa)$. Since $c''_\kappa(a)^2 + \kappa c'_\kappa(a)^2 = 1$, we obtain

$$c'_\kappa(a) = \sqrt{\frac{t_\kappa(a)^2}{1 + \kappa t_\kappa(a)^2}}; \quad c''_\kappa(a) = \sqrt{\frac{1}{1 + \kappa t_\kappa(a)^2}}.$$

Further,

$$1 + \kappa t_\kappa(a)^2 = \frac{1}{c''_\kappa(a)^2}.$$

Fix $a, b \in D(t_\kappa)$ with $a + b \in D(t_\kappa)$. Then,

$$t_\kappa(a+b) = \frac{t_\kappa(a) + t_\kappa(b)}{1 - \kappa t_\kappa(a) t_\kappa(b)};$$
$$t_\kappa(a-b) = \frac{t_\kappa(a) - t_\kappa(b)}{1 + \kappa t_\kappa(a) t_\kappa(b)}.$$

Moreover, if $2a \in D(t_\kappa)$, then

$$t_\kappa(2a) = \frac{2t_\kappa(a)}{1 - \kappa t_\kappa(a)^2}.$$

If $a/2 \in D(t_\kappa)$, then

$$\kappa t_\kappa \left(\frac{a}{2}\right)^2 = \frac{1 - c''_\kappa(a)}{1 + c''_\kappa(a)}.$$

Furthermore, for $a \in D(t_\kappa)$,

$$t'_\kappa(a) = \frac{1}{c''_\kappa(a)^2}$$

and

$$\kappa \int_0^a t_\kappa(r) \,\mathrm{d}r = -\log |c''_\kappa(a)| .$$

In what follows, to define another convex combination, we prove a lemma. Note that

$$\frac{tc'_\kappa(l)}{1 - t + tc''_\kappa(l)} \in [0, t_\kappa(l)] \subset [0, T_\kappa[$$

for any $l \in [0, D_\kappa/2[$ and $t \in [0,1]$. Indeed,

$$0 \leq \frac{tc'_\kappa(l)}{1 - t + tc''_\kappa(l)} \leq \frac{tc'_\kappa(l)}{tc''_\kappa(l)} = t_\kappa(l)$$

if $t \neq 0$; it obviously holds if $t = 0$. Therefore,

$$\frac{1}{l} t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{1 - t + tc''_\kappa(l)} \right) \in [0,1].$$

Remark 21. Let $\kappa \in \mathbb{R}$ and $l \in]0, D_\kappa/2[$. Then, $t = 0$ if and only if

$$\frac{1}{l} t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{1 - t + tc''_\kappa(l)} \right) = 0.$$

Moreover, $t = 1$ if and only if

$$\frac{1}{l} t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{1 - t + tc''_\kappa(l)} \right) = 1.$$

Indeed, if $t = 1$, then

$$\frac{1}{l}t_\kappa^{-1}\left(\frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)}\right) = \frac{1}{l}t_\kappa^{-1}\left(\frac{c'_\kappa(l)}{c''_\kappa(l)}\right) = \frac{1}{l}t_\kappa^{-1}(t_\kappa(l)) = 1.$$

Inversely, if

$$\frac{1}{l}t_\kappa^{-1}\left(\frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)}\right) = 1,$$

then

$$\frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)} = t_\kappa(l)$$

and thus

$$tc'_\kappa(l) = (1-t)t_\kappa(l) + tc''_\kappa(l)t_\kappa(l) = (1-t)t_\kappa(l) + tc'_\kappa(l).$$

Therefore, $(1-t)t_\kappa(l) = 0$ and hence $t = 1$.

Now, we can prove the following lemma:

Lemma 6.1.1. *Let $\kappa \in \mathbb{R}$. Then,*

$$\frac{1}{l}t_\kappa^{-1}\left(\frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)}\right) + \frac{1}{l}t_\kappa^{-1}\left(\frac{(1-t)c'_\kappa(l)}{t+(1-t)c''_\kappa(l)}\right) = 1.$$

for any $l \in]0, D_\kappa/2[$ and $t \in [0,1]$.

Proof. Let

$$D = \frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)}.$$

Since $t_\kappa^{-1}(D) \in [0,l]$, we get

$$l - t_\kappa^{-1}(D) \in [0,l] \subset \left[0, \frac{D_\kappa}{2}\right[.$$

We show that

$$t_\kappa(l - t_\kappa^{-1}(D)) = \frac{(1-t)c'_\kappa(l)}{t+(1-t)c''_\kappa(l)}.$$

From a property of t_κ, we have

$$t_\kappa\left(l - t_\kappa^{-1}(D)\right) = \frac{t_\kappa(l) - D}{1+\kappa D t_\kappa(l)} = \frac{c'_\kappa(l) - Dc''_\kappa(l)}{c''_\kappa(l) + \kappa D c'_\kappa(l)}.$$

Since $D = (tc'_\kappa(l))/(1-t+tc''_\kappa(l))$, we obtain

$$t_\kappa\left(l - t_\kappa^{-1}(D)\right) = \frac{(1-t+tc''_\kappa(l))c'_\kappa(l) - tc'_\kappa(l)c''_\kappa(l)}{(1-t+tc''_\kappa(l))c''_\kappa(l) + \kappa t c'_\kappa(l)^2}$$

$$= \frac{(1-t)c'_\kappa(l)}{t(c''_\kappa(l)^2 + \kappa c'_\kappa(l)^2) + (1-t)c''_\kappa(l)} = \frac{(1-t)c'_\kappa(l)}{t + (1-t)c''_\kappa(l)}.$$

It implies that

$$\frac{1}{l} t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{1 - t + tc''_\kappa(l)} \right) + \frac{1}{l} t_\kappa^{-1} \left(\frac{(1-t)c'_\kappa(l)}{t + (1-t)c''_\kappa(l)} \right) = 1.$$

This is the desired result. □

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$. Let $x, y \in X$ with $l = d(x, y) < D_\kappa/2$ and $t \in [0, 1]$. We define κ-convex combination as follows:

$$tx \overset{\kappa}{\oplus} (1-t)y = \frac{1}{l} t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{1 - t + tc''_\kappa(l)} \right) x \oplus \frac{1}{l} t_\kappa^{-1} \left(\frac{(1-t)c'_\kappa(l)}{t + (1-t)c''_\kappa(l)} \right) y$$

if $l \neq 0$; $tx \overset{\kappa}{\oplus} (1-t)y = x = y$ if $l = 0$.

Remark 22. Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$. Let $x, y \in X$ with $d(x, y) < D_\kappa/2$. Then, for $\sigma \in [0, 1]$,

$$d(x, \sigma x \oplus (1-\sigma)y) = (1-\sigma)d(x, y) \text{ and } d(y, \sigma x \oplus (1-\sigma)y) = \sigma d(x, y).$$

Therefore, for $t \in [0, 1]$,

$$t_\kappa(d(x, tx \overset{\kappa}{\oplus} (1-t)y)) = \frac{(1-t)c'_\kappa(d(x, y))}{t + (1-t)c''_\kappa(d(x, y))}$$

and

$$t_\kappa(d(y, tx \overset{\kappa}{\oplus} (1-t)y)) = \frac{tc'_\kappa(d(x, y))}{1 - t + tc''_\kappa(d(x, y))}.$$

Remark 23. Let X be a CAT(κ) space and $x, y \in X$ with $l = d(x, y) \in \,]0, D_\kappa/2[$. We define a mapping γ_{xy}^κ from $[0, l]$ into X by

$$\gamma_{xy}^\kappa(t) = \frac{l-t}{l} x \overset{\kappa}{\oplus} \frac{t}{l} y$$

for each $t \in [0, l]$. Then, $\gamma_{xy}^\kappa(0) = x$, $\gamma_{xy}^\kappa(l) = y$ and $\gamma_{xy}^\kappa((1-t)l) = tx \overset{\kappa}{\oplus} (1-t)y$ for $t \in [0, 1]$. Let γ_{xy} be a geodesic from x to y. Then, for $t \in [0, l]$,

$$\begin{aligned} \gamma_{xy}^\kappa(t) &= \frac{l-t}{l} x \overset{\kappa}{\oplus} \frac{t}{l} y = \frac{1}{l} t_\kappa^{-1} \left(\frac{(l-t)c'_\kappa(l)}{t + (l-t)c''_\kappa(l)} \right) x \oplus \frac{1}{l} t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{l - t + tc''_\kappa(l)} \right) y \\ &= \gamma_{xy} \left(t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{l - t + tc''_\kappa(l)} \right) \right). \end{aligned}$$

Fix $s, t \in [0, l]$ arbitrarily. Then,

$$d(\gamma_{xy}^\kappa(s), \gamma_{xy}^\kappa(t)) = d\left(\gamma_{xy} \left(t_\kappa^{-1} \left(\frac{sc'_\kappa(l)}{l - s + sc''_\kappa(l)} \right) \right), \gamma_{xy} \left(t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{l - t + tc''_\kappa(l)} \right) \right) \right)$$

$$= \left| t_\kappa^{-1}\left(\frac{sc'_\kappa(l)}{l-s+sc''_\kappa(l)}\right) - t_\kappa^{-1}\left(\frac{tc'_\kappa(l)}{l-t+tc''_\kappa(l)}\right)\right|.$$

We remark that for $a, b \in [0, T_\kappa[$,

$$t_\kappa^{-1}(a) - t_\kappa^{-1}(b) \in \left]-D_\kappa/2, D_\kappa/2\right[$$

and thus

$$t_\kappa(t_\kappa^{-1}(a) - t_\kappa^{-1}(b)) = \frac{a-b}{1+\kappa ab}.$$

Therefore,

$$\begin{aligned} &t_\kappa\left(t_\kappa^{-1}\left(\frac{sc'_\kappa(l)}{l-s+sc''_\kappa(l)}\right) - t_\kappa^{-1}\left(\frac{tc'_\kappa(l)}{l-t+tc''_\kappa(l)}\right)\right) \\ &= \frac{sc'_\kappa(l)\,(l-t+tc''_\kappa(l)) - tc'_\kappa(l)\,(l-s+sc''_\kappa(l))}{(l-s+sc''_\kappa(l))\,(l-t+tc''_\kappa(l)) + \kappa stc'_\kappa(l)^2}. \end{aligned}$$

Further,

$$\begin{aligned} &sc'_\kappa(l)\,(l-t+tc''_\kappa(l)) - tc'_\kappa(l)\,(l-s+sc''_\kappa(l)) \\ &= s(l-t)c'_\kappa(l) - t(l-s)c'_\kappa(l) = (s(l-t) - t(l-s))c'_\kappa(l) \\ &= (sl - st - tl + st)c'_\kappa(l) = (s-t)lc'_\kappa(l) \end{aligned}$$

and

$$\begin{aligned} &(l-s+sc''_\kappa(l))\,(l-t+tc''_\kappa(l)) + \kappa stc'_\kappa(l)^2 \\ &= (l-s)(l-t) + t(l-s)c''_\kappa(l) + s(l-t)c''_\kappa(l) + stc''_\kappa(l)^2 + \kappa stc'_\kappa(l)^2 \\ &= (l-s)(l-t) + (t(l-s) + s(l-t))c''_\kappa(l) + st \\ &= l^2 + 2st - (s+t)l - (2st - (s+t)l)c''_\kappa(l) \\ &= l^2 + (2st - (s+t)l)(1 - c''_\kappa(l)) = l^2 + \kappa(2st - (s+t)l)c_\kappa(l). \end{aligned}$$

Therefore,

$$t_\kappa\left(t_\kappa^{-1}\left(\frac{sc'_\kappa(l)}{l-s+sc''_\kappa(l)}\right) - t_\kappa^{-1}\left(\frac{tc'_\kappa(l)}{l-t+tc''_\kappa(l)}\right)\right) = \frac{(s-t)lc'_\kappa(l)}{l^2 + \kappa(2st-(s+t)l)c_\kappa(l)}$$

and hence

$$\begin{aligned} d(\gamma_{xy}^\kappa(s), \gamma_{xy}^\kappa(t)) &= \left| t_\kappa^{-1}\left(\frac{sc'_\kappa(l)}{l-s+sc''_\kappa(l)}\right) - t_\kappa^{-1}\left(\frac{tc'_\kappa(l)}{l-t+tc''_\kappa(l)}\right)\right| \\ &= \left| t_\kappa^{-1}\left(\frac{(s-t)lc'_\kappa(l)}{l^2 + \kappa(2st-(s+t)l)c_\kappa(l)}\right)\right| \\ &= t_\kappa^{-1}\left(\frac{lc'_\kappa(l)\,|s-t|}{|l^2 + \kappa(2st-(s+t)l)c_\kappa(l)|}\right). \end{aligned}$$

Consequently,

$$t_\kappa(d(\gamma_{xy}^\kappa(s), \gamma_{xy}^\kappa(t))) = \frac{lc'_\kappa(l)\,|s-t|}{|l^2 + \kappa(2st-(s+t)l)c_\kappa(l)|}.$$

We obtain the following proposition:

Theorem 6.1.2. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$tx \oplus (1-t)y = \left(\frac{(t)_l^\kappa}{(t)_l^\kappa + (1-t)_l^\kappa}\right) x \overset{\kappa}{\oplus} \left(\frac{(1-t)_l^\kappa}{(t)_l^\kappa + (1-t)_l^\kappa}\right) y$$

for every $x, y \in X$ with $l = d(x,y) < D_\kappa/2$ and $t \in [0,1]$.

Proof. Let $x, y \in X$ with $l = d(x,y) < D_\kappa/2$ and $t \in [0,1]$. It obviously holds if $l = 0$. We suppose that $l \neq 0$. Let

$$\sigma = \frac{(t)_l^\kappa}{(t)_l^\kappa + (1-t)_l^\kappa} = \frac{c'_\kappa(tl)}{c'_\kappa(tl) + c'_\kappa((1-t)l)} \in [0,1].$$

We notice that

$$1 - \sigma = \frac{c'_\kappa((1-t)l)}{c'_\kappa(tl) + c'_\kappa((1-t)l)} \in [0,1].$$

From the definition of κ-convex combination,

$$\sigma x \overset{\kappa}{\oplus} (1-\sigma)y = \frac{1}{l} t_\kappa^{-1}\left(\frac{\sigma c'_\kappa(l)}{1-\sigma+\sigma c''_\kappa(l)}\right) x \oplus \frac{1}{l} t_\kappa^{-1}\left(\frac{(1-\sigma)c'_\kappa(l)}{\sigma + (1-\sigma)c''_\kappa(l)}\right) y.$$

Since

$$\sigma c'_\kappa(l) = \frac{c'_\kappa(tl)c'_\kappa(l)}{c'_\kappa(tl) + c'_\kappa((1-t)l)}$$

and

$$1 - \sigma + \sigma c''_\kappa(l) = \frac{c'_\kappa((1-t)l) + c'_\kappa(tl)c''_\kappa(l)}{c'_\kappa(tl) + c'_\kappa((1-t)l)},$$

we have

$$\begin{aligned}
\frac{\sigma c'_\kappa(l)}{1-\sigma+\sigma c''_\kappa(l)} &= \frac{c'_\kappa(tl)c'_\kappa(l)}{c'_\kappa((1-t)l) + c'_\kappa(tl)c''_\kappa(l)} \\
&= \frac{c'_\kappa(tl)c'_\kappa(l)}{c'_\kappa(l)c''_\kappa(tl) - c'_\kappa(tl)c''_\kappa(l) + c'_\kappa(tl)c''_\kappa(l)} \\
&= \frac{c'_\kappa(tl)}{c''_\kappa(tl)} = t_\kappa(tl).
\end{aligned}$$

Similarly, we have

$$\frac{(1-\sigma)c'_\kappa(l)}{\sigma + (1-\sigma)c''_\kappa(l)} = t_\kappa((1-t)l).$$

Therefore, we obtain

$$\sigma x \overset{\kappa}{\oplus} (1-\sigma)y = \frac{1}{l} t_\kappa^{-1}\left(t_\kappa(tl)\right) x \oplus \frac{1}{l} t_\kappa^{-1}\left(t_\kappa((1-t)l)\right) y = tx \oplus (1-t)y.$$

It completes the proof. □

Consequently, we obtain the following:

Theorem 6.1.3. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and C a subset of X. Then, C is convex if and only if $tx \overset{\kappa}{\oplus} (1-t)y \in C$ for any $x, y \in C$ and $t \in [0,1]$.*

Proof. With out loss of generality, we assume that C is nonempty. We first suppose that C is convex. From the definition of κ-convex combination, for any $x, y \in C$ and $t \in [0,1]$, there exists $\sigma \in [0,1]$ such that $tx \overset{\kappa}{\oplus} (1-t)y = \sigma x \oplus (1-\sigma)y$. Since C is convex, $tx \overset{\kappa}{\oplus} (1-t)y = \sigma x \oplus (1-\sigma)y \in C$. We next suppose that $\tau x \overset{\kappa}{\oplus} (1-\tau)y \in C$ for any $x, y \in C$ and $\tau \in [0,1]$. Then, from Theorem 6.1.2,

$$tx \oplus (1-t)y = \left(\frac{(t)_l^\kappa}{(t)_l^\kappa + (1-t)_l^\kappa}\right) x \overset{\kappa}{\oplus} \left(\frac{(1-t)_l^\kappa}{(t)_l^\kappa + (1-t)_l^\kappa}\right) y \in C$$

for any $x, y \in C$ and $t \in [0,1]$, where $l = d(x,y)$. It means that C is convex. Consequently, we obtain the desired conclusion. □

6.2 Parallelogram laws

In this section, we prove other type parallelogram laws in CAT(κ) spaces. Before that, we obtain the following result:

Lemma 6.2.1. *Let $\kappa \in \mathbb{R}$, $l \in]0, D_\kappa/2[$ and $t \in [0,1]$. Let*

$$\sigma = \frac{1}{l} t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)}\right) \in [0,1].$$

Then,

$$(\sigma)_l^\kappa = \frac{t}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}}.$$

Proof. Since

$$c'_\kappa(D) = \sqrt{\frac{t_\kappa(D)^2}{1+\kappa t_\kappa(D)^2}}$$

for any $D \in]0, D_\kappa/2[$, we get

$$(\sigma)_l^\kappa = \frac{c'_\kappa(\sigma l)}{c'_\kappa(l)} = \frac{1}{c'_\kappa(l)} \sqrt{\frac{t_\kappa(\sigma l)^2}{1+\kappa t_\kappa(\sigma l)^2}}.$$

Since

$$\sigma = \frac{1}{l} t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)}\right),$$

we have

$$t_\kappa(\sigma l)^2 = \left(\frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)}\right)^2,$$

which implies that

$$\frac{t_\kappa(\sigma l)^2}{1+\kappa t_\kappa(\sigma l)^2} = \frac{t^2 c_\kappa'(l)^2}{\left(1-t+tc_\kappa''(l)\right)^2 + t^2\kappa c_\kappa'(l)^2}.$$

Hence, we get

$$(\sigma)_l^\kappa = \frac{t}{\sqrt{\left(1-t+tc_\kappa''(l)\right)^2 + t^2\kappa c_\kappa'(l)^2}}.$$

Moreover, we obtain

$$\begin{aligned}(1-t+tc_\kappa''(l))^2 + t^2\kappa c_\kappa'(l)^2 &= (1-t)^2 + 2t(1-t)c_\kappa''(l) + t^2\left(c_\kappa''(l)^2 + \kappa c_\kappa'(l)^2\right)\\ &= t^2 + (1-t)^2 + 2t(1-t)c_\kappa''(l)\end{aligned}$$

and thus

$$(\sigma)_l^\kappa = \frac{t}{\sqrt{t^2+(1-t)^2+2t(1-t)c_\kappa''(l)}}.$$

This is the desired result. □

Remark 24. Let $(M_\kappa^\infty, d_{M_\kappa^\infty})$ be the infinite dimensional model space for $\kappa \neq 0$. For $x, y \in M_\kappa^\infty$ with $l = d_{M_\kappa^\infty}(x,y) \in \,]0, D_\kappa/2[$ and $t \in [0,1]$, set

$$\sigma = \frac{1}{l} t_\kappa^{-1}\left(\frac{tc_\kappa'(l)}{1-t+tc_\kappa''(l)}\right) \in [0,1].$$

Note that

$$(\sigma)_l^\kappa = \frac{t}{\sqrt{t^2+(1-t)^2+2t(1-t)c_\kappa''(l)}}.$$

Therefore,

$$\begin{aligned}tx \overset{\kappa}{\oplus} (1-t)y = \sigma x \oplus (1-\sigma)y &= (\sigma)_l^\kappa x + (1-\sigma)_l^\kappa y\\ &= \frac{tx+(1-t)y}{\sqrt{t^2+(1-t)^2+2t(1-t)c_\kappa''(l)}}.\end{aligned}$$

Suppose that $\kappa = 1$. Then,

$$\begin{aligned}tx \overset{1}{\oplus} (1-t)y &= \frac{tx+(1-t)y}{\sqrt{t^2+(1-t)^2+2t(1-t)\cos l}}\\ &= \frac{tx+(1-t)y}{\sqrt{t^2\|x\|^2+(1-t)^2\|y\|^2+2t(1-t)\langle x,y\rangle}}\\ &= \frac{tx+(1-t)y}{\sqrt{\langle tx+(1-t)y, tx+(1-t)y\rangle}} = \frac{tx+(1-t)y}{\|tx+(1-t)y\|}.\end{aligned}$$

Suppose that $\kappa = -1$. Then,

$$\begin{aligned} tx \overset{-1}{\oplus} (1-t)y &= \frac{tx + (1-t)y}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)\cosh l}} \\ &= \frac{tx + (1-t)y}{\sqrt{-t^2 \langle x \mid x \rangle - (1-t)^2 \langle y \mid y \rangle - 2t(1-t) \langle x \mid y \rangle}} \\ &= \frac{tx + (1-t)y}{\sqrt{-\langle tx + (1-t)y \mid tx + (1-t)y \rangle}}. \end{aligned}$$

Now, we can prove the following theorem:

Theorem 6.2.2. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then, for every $x, y, z \in X$ with $d(x,y) < D_\kappa/2$ and $d(y,z) + d(z,x) + d(x,y) < 2D_\kappa$, and $t \in [0,1]$,*

$$\begin{aligned} \phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \le &\frac{t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(d(x,y))}} \\ &- \frac{t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(d(x,y))}} \end{aligned}$$

and

$$\begin{aligned} \phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \le{} & t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z) \\ & - c''_\kappa(\delta) \left(t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y) \right), \end{aligned}$$

where $\delta = d(tx \overset{\kappa}{\oplus} (1-t)y, z)$.

Proof. Let $x, y, z \in X$ with $l = d(x,y) < D_\kappa/2$ and $d(y,z) + d(z,x) + l < 2D_\kappa$. The desired inequalities obviously hold if $l = 0$. Suppose that $l \neq 0$. Let $t \in [0,1]$ and

$$\sigma = \frac{1}{l} t_\kappa^{-1} \left(\frac{tc'_\kappa(l)}{1 - t + tc''_\kappa(l)} \right) \in [0,1].$$

Then,

$$(\sigma)_l^\kappa = \frac{t}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}}$$

and

$$(1-\sigma)_l^\kappa = \frac{1-t}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}}.$$

Since $tx \overset{\kappa}{\oplus} (1-t)y = \sigma x \oplus (1-\sigma)y$, it holds from the parallelogram law of X that

$$\begin{aligned} \phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \le &\frac{t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}} \\ &- \frac{t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}}. \end{aligned}$$

It implies that

$$\begin{aligned}&\sqrt{t^2+(1-t)^2+2t(1-t)c''_\kappa(l)}\phi_\kappa(tx\overset{\kappa}{\oplus}(1-t)y,z)\\&\le t\phi_\kappa(x,z)+(1-t)\phi_\kappa(y,z)-t\phi_\kappa(x,tx\overset{\kappa}{\oplus}(1-t)y)-(1-t)\phi_\kappa(y,tx\overset{\kappa}{\oplus}(1-t)y).\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\phi_\kappa(tx\overset{\kappa}{\oplus}(1-t)y,z)\\&\le t\phi_\kappa(x,z)+(1-t)\phi_\kappa(y,z)\\&\quad -tc''_\kappa(\delta)\phi_\kappa(x,tx\overset{\kappa}{\oplus}(1-t)y)-(1-t)c''_\kappa(\delta)\phi_\kappa(y,tx\overset{\kappa}{\oplus}(1-t)y)+\mathit{\Gamma},\end{aligned}$$

where $\delta=d(tx\overset{\kappa}{\oplus}(1-t)y,z)$ and

$$\begin{aligned}\mathit{\Gamma}=&\left(1-\sqrt{t^2+(1-t)^2+2t(1-t)c''_\kappa(l)}\right)c_\kappa(\delta)\\&-(1-c''_\kappa(\delta))\left(t\phi_\kappa(x,tx\overset{\kappa}{\oplus}(1-t)y)+(1-t)\phi_\kappa(y,tx\overset{\kappa}{\oplus}(1-t)y)\right).\end{aligned}$$

We show that $\mathit{\Gamma}=0$. Since $1-c''_\kappa(a)=\kappa c_\kappa(a)$ for each $a\in\mathbb{R}$,

$$\begin{aligned}&(1-c''_\kappa(\delta))\left(t\phi_\kappa(x,tx\overset{\kappa}{\oplus}(1-t)y)+(1-t)\phi_\kappa(y,tx\overset{\kappa}{\oplus}(1-t)y)\right)\\&=\kappa c_\kappa(\delta)\left(tc_\kappa(d(x,tx\overset{\kappa}{\oplus}(1-t)y))+(1-t)c_\kappa(d(y,tx\overset{\kappa}{\oplus}(1-t)y))\right)\\&=c_\kappa(\delta)\left(1-tc''_\kappa(d(x,tx\overset{\kappa}{\oplus}(1-t)y))-(1-t)c''_\kappa(d(y,tx\overset{\kappa}{\oplus}(1-t)y))\right)\\&=(1-tc''_\kappa((1-\sigma)l)-(1-t)c''_\kappa(\sigma l))\,c_\kappa(\delta).\end{aligned}$$

Therefore,

$$\mathit{\Gamma}=c_\kappa(\delta)\left(tc''_\kappa((1-\sigma)l)+(1-t)c''_\kappa(\sigma l)-\sqrt{t^2+(1-t)^2+2t(1-t)c''_\kappa(l)}\right).$$

Moreover, we obtain

$$\begin{aligned}c''_\kappa(\sigma l)&=c''_\kappa\left(t_\kappa^{-1}\left(\frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)}\right)\right)=\sqrt{\frac{(1-t+tc''_\kappa(l))^2}{(1-t+tc''_\kappa(l))^2+\kappa t^2c'_\kappa(l)^2}}\\&=\frac{1-t+tc''_\kappa(l)}{\sqrt{(1-t+tc''_\kappa(l))^2+t^2\kappa c'_\kappa(l)^2}}\\&=\frac{1-t+tc''_\kappa(l)}{\sqrt{(1-t)^2+2t(1-t)c''_\kappa(l)+t^2c''_\kappa(l)^2+t^2\kappa c'_\kappa(l)^2}}\\&=\frac{1-t+tc''_\kappa(l)}{\sqrt{t^2+(1-t)^2+2t(1-t)c''_\kappa(l)}}.\end{aligned}$$

Similarly, we get

$$c''_\kappa((1-\sigma)l)=\frac{t+(1-t)c''_\kappa(l)}{\sqrt{t^2+(1-t)^2+2t(1-t)c''_\kappa(l)}}.$$

Therefore, we have

$$\begin{aligned} tc''_\kappa((1-\sigma)l) + (1-t)c''_\kappa(\sigma l) &= \frac{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}} \\ &= \sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}. \end{aligned}$$

Consequently,

$$\Gamma = c_\kappa(\delta)\left(tc''_\kappa((1-\sigma)l) + (1-t)c''_\kappa(\sigma l) - \sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}\right) = 0$$

and hence

$$\begin{aligned} \phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) &\leq t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z) \\ &\quad - tc''_\kappa(\delta)\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) - (1-t)c''_\kappa(\delta)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y). \end{aligned}$$

This is the desired result. □

Corollary 6.2.3. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \leq t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z)$$

for each $x, y, z \in X$ and $t \in [0,1]$.

Using the same methods of Theorem 6.2.2, we get the following:

Theorem 6.2.4. *Let $(M^\infty_\kappa, d_{M^\infty_\kappa})$ be the infinite dimensional model space for $\kappa \in \mathbb{R}$ and $x, y, z \in M^\infty_\kappa$ with $d_{M^\infty_\kappa}(x,y) < D_\kappa/2$ and $d_{M^\infty_\kappa}(y,z) + d_{M^\infty_\kappa}(z,x) + d_{M^\infty_\kappa}(x,y) < 2D_\kappa$. Let $t \in [0,1]$ and set $\delta = d_{M^\infty_\kappa}(tx \overset{\kappa}{\oplus} (1-t)y, z)$. Then,*

$$\begin{aligned} \phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) &= \frac{t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(d_{M^\infty_\kappa}(x,y))}} \\ &\quad - \frac{t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(d_{M^\infty_\kappa}(x,y))}} \end{aligned}$$

and

$$\begin{aligned} \phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) &= t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z) \\ &\quad - c''_\kappa(\delta)\left(t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y)\right). \end{aligned}$$

At the end of this section, we obtain the following theorem:

Theorem 6.2.5. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then, for $x, y, z \in X$ with $l = d(x,y) < D_\kappa/2$ and $d(y,z) + d(z,x) + l < 2D_\kappa$, and $t \in [0,1]$,*

$$\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \leq t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z) - \frac{2t(1-t)\phi_\kappa(x,y) \cdot c''_\kappa(\delta)}{1 + \sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}},$$

where $\delta = d(tx \overset{\kappa}{\oplus} (1-t)y, z)$.

Proof. If $\kappa = 0$, then we easily obtain the desired result. We assume that $\kappa \neq 0$. From Theorem 6.2.2,

$$\begin{aligned}\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) &\leq t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z) \\ &\quad - c''_\kappa(\delta)\left(t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y)\right),\end{aligned}$$

where $\delta = d(tx \overset{\kappa}{\oplus} (1-t)y, z)$. By the same fashion as Theorem 6.2.2, we have

$$c''_\kappa(d(x, tx \overset{\kappa}{\oplus} (1-t)y)) = \frac{t + (1-t)c''_\kappa(l)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}}$$

and

$$c''_\kappa(d(y, tx \overset{\kappa}{\oplus} (1-t)y)) = \frac{1 - t + tc''_\kappa(l)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}}.$$

Set

$$M = \sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}.$$

Then,

$$\begin{aligned}&t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y) \\ &= \frac{1}{\kappa}\left(1 - tc''_\kappa(d(x, tx \overset{\kappa}{\oplus} (1-t)y)) - (1-t)c''_\kappa(d(y, tx \overset{\kappa}{\oplus} (1-t)y))\right) \\ &= \frac{1}{\kappa}\left(1 - \frac{t^2 + t(1-t)c''_\kappa(l)}{M} - \frac{(1-t)^2 + t(1-t)c''_\kappa(l)}{M}\right) \\ &= \frac{1-M}{\kappa} = \frac{1-M^2}{\kappa(1+M)} = \frac{1 - t^2 - (1-t)^2 - 2t(1-t)c''_\kappa(l)}{\kappa(1+M)} \\ &= \frac{2t(1-t)(1-c''_\kappa(l))}{\kappa(1+M)} = \frac{2t(1-t)\phi_\kappa(x,y)}{1+M}.\end{aligned}$$

This is the desired result. □

Using the same methods of Theorem 6.2.5, we get the following:

Theorem 6.2.6. *Let $(M^\infty_\kappa, d_{M^\infty_\kappa})$ be the infinite dimensional model space for $\kappa \in \mathbb{R}$ and $x, y, z \in M^\infty_\kappa$ with $l = d_{M^\infty_\kappa}(x,y) < D_\kappa/2$ and $d_{M^\infty_\kappa}(y,z) + d_{M^\infty_\kappa}(z,x) + l < 2D_\kappa$. Then, for $t \in [0,1]$,*

$$\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) = t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z) - \frac{2t(1-t)\phi_\kappa(x,y) \cdot c''_\kappa(\delta)}{1 + \sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}},$$

where $\delta = d(tx \overset{\kappa}{\oplus} (1-t)y, z)$.

6.3 Convex functions

In this section, we investigate properties of another convex function. We first prove the following lemmas:

Lemma 6.3.1. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $x, y \in X$. Then,*

$$\frac{1}{2}x \overset{\kappa}{\oplus} \frac{1}{2}y = \frac{1}{2}x \oplus \frac{1}{2}y.$$

Proof. Let $x, y \in X$. From Theorem 6.1.2, we obtain

$$\frac{1}{2}x \oplus \frac{1}{2}y = \left(\frac{(1/2)^{\kappa}_{d(x,y)}}{2(1/2)^{\kappa}_{d(x,y)}} \right) x \overset{\kappa}{\oplus} \left(\frac{(1/2)^{\kappa}_{d(x,y)}}{2(1/2)^{\kappa}_{d(x,y)}} \right) y = \frac{1}{2}x \overset{\kappa}{\oplus} \frac{1}{2}y$$

and it completes the proof. □

Lemma 6.3.2. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $x, y \in X$. Then, $tx \overset{\kappa}{\oplus} (1-t)y \to y$ as $t \searrow 0$.*

Proof. Let $x, y \in X$ and $t \in \,]0, 1[$. Then,

$$\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, y) \le t\phi_\kappa(x, y).$$

Letting $t \searrow 0$, we have $tx \overset{\kappa}{\oplus} (1-t)y \to y$. □

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$. A function f from X into $]-\infty, \infty]$ is said to be κ-convex if

$$f(tx \overset{\kappa}{\oplus} (1-t)y) \le tf(x) + (1-t)f(y)$$

for each $x, y \in X$ and $t \in \,]0, 1[$. A function $\phi_\kappa(\cdot, z)$ is κ-convex for each $z \in X$.

Remark 25. Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$, $z \in X$ and h an increasing convex function from $\mathbb{R}$ into itself. We define a function f from X into $\mathbb{R}$ as

$$f(x) = h(\phi_\kappa(x, z))$$

for each $x \in X$. Then, f is κ-convex.

Remark 26. Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$. Let f and g be κ-convex functions from X into $]-\infty, \infty]$. Let $\lambda > 0$. Then, λf and $f + g$ are also κ-convex.

In what follows, we consider a resolvent operator for κ-convex functions. We first obtain the following lemmas:

Lemma 6.3.3. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a κ-convex function from X into $]-\infty, \infty]$. Then, $\operatorname{Lev}_a f$ is convex for all $a \in \mathbb{R}$.*

Proof. Fix $a \in \mathbb{R}$ arbitrary. If $\operatorname{Lev}_a f$ is empty, then it is obviously convex. We assume that $\operatorname{Lev}_a f$ is nonempty. Then, for any $u, v \in \operatorname{Lev}_a f$ and $t \in \,]0, 1[$, we obtain

$$f(tu \overset{\kappa}{\oplus} (1-t)v) \le tf(u) + (1-t)f(v) \le a$$

and hence $tu \overset{\kappa}{\oplus} (1-t)v \in \operatorname{Lev}_a f$. Therefore, from Theorem 6.1.3, $\operatorname{Lev}_a f$ is convex. □

Lemma 6.3.4. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous κ-convex function from X into $]-\infty, \infty]$. Then,*

$$f(x_0) \leq \liminf_{n \to \infty} f(x_n)$$

whenever $\{x_n\}$ Δ-converges to $x_0 \in X$.

Proof. See Lemma 5.1.3. □

Lemma 6.3.5. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous κ-convex function from X into $]-\infty, \infty]$ such that $\operatorname{Min} f$ is nonempty. For a point $x \in X$, define a function f_x by*

$$f_x(z) = f(z) + \phi_\kappa(z, x)$$

for each $z \in X$. Then, $\operatorname{Min} f_x$ is a singleton.

Proof. See Lemma 5.2.1. □

Let X be an admissible complete CAT(κ) space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous κ-convex function from X into $]-\infty, \infty]$ such that $\operatorname{Min} f$ is nonempty. We define a single-valued mapping S_f from X into $\operatorname{dom} f$ as

$$S_f x = \operatorname*{Argmin}_{z \in X} \left(f(z) + \phi_\kappa(z, x) \right)$$

for each $x \in X$. We call S_f a resolvent operator for f.

We can prove the following results:

Lemma 6.3.6. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous κ-convex function from X into $]-\infty, \infty]$ such that $\operatorname{Min} f$ is nonempty. Then,*

$$f(S_f x) \leq f(w) + \phi_\kappa(w, x) - \phi_\kappa(w, S_f x) - c_\kappa''(d(w, S_f x))\phi_\kappa(S_f x, x)$$

for any $x, w \in X$.

Proof. Let $x, w \in X$. For $\tau \in]0, 1[$, let $w_\tau = \tau w \overset{\kappa}{\oplus} (1 - \tau) S_f x$. Then,

$$f(S_f x) + \phi_\kappa(S_f x, x) \leq f(w_\tau) + \phi_\kappa(w_\tau, x) \leq \tau f(w) + (1 - \tau) f(S_f x) + \phi_\kappa(w_\tau, x)$$

and hence

$$\tau f(S_f x) \leq \tau f(w) + \phi_\kappa(w_\tau, x) - \phi_\kappa(S_f x, x).$$

From Theorem 6.2.2,

$$\tau f(S_f x) \leq \tau f(w) + \tau \phi_\kappa(w, x) - \tau c_\kappa''(d(w_\tau, x))\phi_\kappa(w, w_\tau) - \tau \phi_\kappa(S_f x, x).$$

Dividing both sides by τ and letting $\tau \searrow 0$, we have

$$f(S_f x) \leq f(w) + \phi_\kappa(w, x) - c_\kappa''(d(S_f x, x))\phi_\kappa(w, S_f x) - \phi_\kappa(S_f x, x).$$

Moreover,

$$\begin{aligned} &c_\kappa''(d(S_f x, x))\phi_\kappa(w, S_f x) + \phi_\kappa(S_f x, x) \\ &= \phi_\kappa(w, S_f x) - (1 - c_\kappa''(d(S_f x, x)))\phi_\kappa(w, S_f x) + \phi_\kappa(S_f x, x) \end{aligned}$$

$$
\begin{aligned}
&= \phi_\kappa(w, S_f x) - (1 - c''_\kappa(d(w, S_f x)))\phi_\kappa(S_f x, x) + \phi_\kappa(S_f x, x) \\
&= \phi_\kappa(w, S_f x) + c''_\kappa(d(w, S_f x))\phi_\kappa(S_f x, x).
\end{aligned}
$$

Thus, we obtain the desired result. □

Lemma 6.3.7. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous κ-convex function from X into $]-\infty, \infty]$ such that* $\operatorname{Min} f$ *is nonempty. Then,* $\operatorname{Min} f = \operatorname{Fix} S_f$.

Proof. See Lemma 5.2.3. □

Corollary 6.3.8. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous κ-convex function from X into $]-\infty, \infty]$ such that* $\operatorname{Min} f$ *is nonempty. Then, S_f is quasinonexpansive.*

Proof. From Lemma 6.3.6, for $x \in X$ and $p \in \operatorname{Min} f = \operatorname{Fix} S_f$,

$$
f(S_f x) \le f(p) + \phi_\kappa(p, x) - \phi_\kappa(p, S_f x) - c''_\kappa(d(p, S_f x))\phi_\kappa(S_f x, x)
$$

and therefore

$$
\begin{aligned}
0 &\le f(S_f x) - f(p) \le \phi_\kappa(p, x) - \phi_\kappa(p, S_f x) - c''_\kappa(d(p, S_f x))\phi_\kappa(S_f x, x) \\
&\le \phi_\kappa(p, x) - \phi_\kappa(p, S_f x).
\end{aligned}
$$

It implies that S_f is quasinonexpansive. □

Using the same method of Theorem 5.5.2, we obtain the following:

Theorem 6.3.9. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and f a proper lower semicontinuous κ-convex function from X into $]-\infty, \infty]$ such that* $\operatorname{Min} f$ *is nonempty. Let $\{\lambda_n\}$ be a positive real sequence such that $\sum_{k=1}^{\infty} \lambda_k = \infty$. For a given initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$
x_{n+1} = \operatorname*{Argmin}_{z \in X} \left(\lambda_n f(z) + \phi_\kappa(z, x_n) \right)
$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ Δ-converges to a minimiser $x_0 \in$ $\operatorname{Min} f$ *which is a limit of $\{P_{\operatorname{Min} f} x_n\}$, where $P_{\operatorname{Min} f}$ is a metric projection onto* $\operatorname{Min} f$.

Chapter 7

Fixed point approximation

In this chapter, we prove some convergence theorems for a geodesically nonspreading mapping. When generate iterative sequences, we will use κ-combination. We first prove Δ-convergence theorems with the Mann type iterative scheme. We next consider a convergence theorem with the Halpern type iterative scheme.

7.1 Mann type iterative scheme

In this section, we prove a Δ-convergence theorem for a geodesically nonspreading mapping. Before that, we get the following:

Theorem 7.1.1 (Kajimura–Kimura [14], Kimura–Kohsaka [20, 21]). *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$, $\{z_n\}$ a sequence of X such that*

$$\inf_{y \in X} \sup_{n \in \mathbb{N}} d(y, z_n) < \frac{D_\kappa}{2}$$

and $\{\beta_n\}$ a positive real sequence such that $\sum_{n=1}^{\infty} \beta_n = \infty$. Then,

$$\operatorname*{Argmin}_{u \in X} \left(\limsup_{n \to \infty} \frac{1}{\sum_{j=1}^{n} \beta_j} \sum_{k=1}^{n} \beta_k \phi_\kappa(z_k, u) \right)$$

consists of exactly one point.

Proof. Let $m = \inf_{y \in X} \sup_{n \in \mathbb{N}} d(y, z_n)$ and

$$\varepsilon \in \left] 0, \frac{D_\kappa}{2} - m \right[.$$

Then, there exists $y_0 \in X$ such that

$$\sup_{n \in \mathbb{N}} d(y_0, z_n) \le m + \varepsilon < \frac{D_\kappa}{2},$$

or equivalently $\sup_{n \in \mathbb{N}} \phi_\kappa(z_n, y_0) < A_\kappa$. Define a function h by

$$h(x) = \limsup_{n \to \infty} \frac{1}{\sum_{j=1}^{n} \beta_j} \sum_{k=1}^{n} \beta_k \phi_\kappa(z_k, x)$$

for each $x \in X$. Note that $\inf_{y \in X} h(y) \leq h(y_0) < A_\kappa$. Then, from Theorem 3.1.4, $\operatorname{Min} h$ is a singleton. □

Lemma 7.1.2. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a quasinonexpansive mapping from X into itself such that $\operatorname{Fix} T$ is nonempty. For a given initial point $x_1 \in X$ and a real sequence $\{\alpha_n\}$ of $[0, 1[$, generate a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T x_n$$

for each $n \in \mathbb{N}$. Then, $\{P_{\operatorname{Fix} T} x_n\}$ converges to a fixed point $x_0 \in \operatorname{Fix} T$, where $P_{\operatorname{Fix} T}$ is a metric projection onto $\operatorname{Fix} T$.

Proof. Fix $n \in \mathbb{N}$ and define a mapping S_n by $S_n x = \alpha_n x \oplus (1 - \alpha_n) T x$ for each $x \in X$. Then, $\operatorname{Fix} S_n = \operatorname{Fix} T$. Indeed, for arbitrary $x \in X$,

$$d(x, S_n x) = d(x, \alpha_n x \oplus (1 - \alpha_n) T x) = (1 - \alpha_n) d(x, T x).$$

Since $\alpha_n < 1$, we have $\operatorname{Fix} S_n = \operatorname{Fix} T$ and hence $\bigcap_{n=1}^{\infty} \operatorname{Fix} S_n = \operatorname{Fix} T$. Further, we easily obtain $\{S_n\}$ is a sequence of quasinonexpansive mappings. From Lemma 4.2.1, we get the desired result. □

Now we can prove the following convergence theorem with the Mann type iterative scheme:

Theorem 7.1.3. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a geodesically nonspreading mapping on X. Let $\{\alpha_n\}$ be a real sequence of $[0, 1[$ such that $\sum_{k=1}^{\infty} (1 - \alpha_k) = \infty$. For a given initial point $x_1 \in X$ and a subset I of $\mathbb{N}$, generate a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = \begin{cases} \alpha_n x_n \overset{\kappa}{\oplus} (1 - \alpha_n) T x_n & (n \in I); \\ \alpha_n x_n \oplus (1 - \alpha_n) T x_n & (n \in \mathbb{N} \setminus I) \end{cases}$$

for each $n \in \mathbb{N}$. Then, the following hold:

- (i) $\operatorname{Fix} T$ *is nonempty if and only if* $\inf_{y \in X} \sup_{n \in \mathbb{N}} d(y, T x_n) < D_\kappa / 2$;
- (ii) *if* $\operatorname{Fix} T$ *is nonempty,* $\sup_{n \in \mathbb{N}} \alpha_n < 1$ *and T is firmly geodesically nonspreading, then $\{x_n\}$ Δ-converges to a fixed point $x_0 \in \operatorname{Fix} T$ which is a limit of $\{P_{\operatorname{Fix} T} x_n\}$, where $P_{\operatorname{Fix} T}$ is a metric projection onto* $\operatorname{Fix} T$.

Proof. From the definition of $\{x_n\}$,

$$\phi_\kappa(w, x_{n+1}) \leq \alpha_n \phi_\kappa(w, x_n) + (1 - \alpha_n) \phi_\kappa(w, T x_n)$$

for any $n \in \mathbb{N}$ and $w \in X$. Suppose that $\operatorname{Fix} T$ is nonempty. Then, since T is quasinonexpansive, for $p \in \operatorname{Fix} T$,

$$\phi_\kappa(p, x_{n+1}) \leq \alpha_n \phi_\kappa(p, x_n) + (1 - \alpha_n) \phi_\kappa(p, T x_n) \leq \phi_\kappa(p, x_n)$$

and thus $d(p, x_n) \leq d(p, x_1)$ for each $n \in \mathbb{N}$. Since $d(p, T x_n) \leq d(p, x_n)$ for all $n \in \mathbb{N}$, we have

$$\inf_{y \in X} \sup_{n \in \mathbb{N}} d(y, T x_n) \leq \sup_{n \in \mathbb{N}} d(p, T x_n) \leq d(p, x_1) < \frac{D_\kappa}{2}.$$

We next prove the "if" part of (i). Assume that $\inf_{y \in X} \sup_{n \in \mathbb{N}} d(y, Tx_n) < D_\kappa/2$. Then, we can take a unique point

$$p \in \operatorname*{Argmin}_{u \in X} \left(\limsup_{n \to \infty} \frac{\sum_{k=1}^n (1-\alpha_k)\phi_\kappa(Tx_k, u)}{\sum_{j=1}^n (1-\alpha_j)} \right).$$

Fix $k \in \mathbb{N}$ arbitrarily. Then,

$$\phi_\kappa(Tp, x_{k+1}) \leq \alpha_k \phi_\kappa(Tp, x_k) + (1-\alpha_k)\phi_\kappa(Tp, Tx_k).$$

Since T is geodesically nonspreading,

$$\phi_\kappa(Tp, Tx_k) + \phi_\kappa(Tx_k, Tp) \leq \phi_\kappa(Tp, x_k) + \phi_\kappa(Tx_k, p)$$

and hence

$$\phi_\kappa(Tp, Tx_k) \leq \phi_\kappa(Tp, x_k) + \phi_\kappa(Tx_k, p) - \phi_\kappa(Tx_k, Tp).$$

Therefore, we obtain

$$\begin{aligned}\phi_\kappa(Tp, x_{k+1}) &\leq \alpha_k \phi_\kappa(Tp, x_k) + (1-\alpha_k)\phi_\kappa(Tp, Tx_k) \\ &\leq \phi_\kappa(Tp, x_k) + (1-\alpha_k)\phi_\kappa(Tx_k, p) - (1-\alpha_k)\phi_\kappa(Tx_k, Tp)\end{aligned}$$

and thus

$$(1-\alpha_k)\phi_\kappa(Tx_k, Tp) \leq (1-\alpha_k)\phi_\kappa(Tx_k, p) + \phi_\kappa(Tp, x_k) - \phi_\kappa(Tp, x_{k+1}).$$

For fixed $n \in \mathbb{N}$, summing up these inequalities with respect to $k \in \{1, \ldots, n\}$, we have

$$\begin{aligned}\sum_{k=1}^n (1-\alpha_k)\phi_\kappa(Tx_k, Tp) &\leq \sum_{k=1}^n (1-\alpha_k)\phi_\kappa(Tx_k, p) + \phi_\kappa(Tp, x_1) - \phi_\kappa(Tp, x_{n+1}) \\ &\leq \sum_{k=1}^n (1-\alpha_k)\phi_\kappa(Tx_k, p) + \phi_\kappa(Tp, x_1).\end{aligned}$$

Dividing both sides by $\sum_{j=1}^n (1-\alpha_j)$ and letting $n \to \infty$, we obtain

$$\limsup_{n \to \infty} \frac{\sum_{k=1}^n (1-\alpha_k)\phi_\kappa(Tx_k, Tp)}{\sum_{j=1}^n (1-\alpha_j)} \leq \limsup_{n \to \infty} \frac{\sum_{k=1}^n (1-\alpha_k)\phi_\kappa(Tx_k, p)}{\sum_{j=1}^n (1-\alpha_j)}$$

and thus $p = Tp$. It means that $\operatorname{Fix} T$ is nonempty.

We finally show (ii). Suppose that $\operatorname{Fix} T$ is nonempty, $\sup_{n \in \mathbb{N}} \alpha_n < 1$ and T is firmly geodesically nonspreading. We first prove that a sequence $\{P_{\operatorname{Fix} T} x_n\}$ converges to a point in $\operatorname{Fix} T$. From the definition of $\{x_n\}$, there exists a real sequence $\{\sigma_n\}$ of $[0, 1[$ such that

$$x_{n+1} = \sigma_n x_n \oplus (1-\sigma_n)Tx_n$$

for all $n \in \mathbb{N}$. Therefore, from Lemma 7.1.2, $\{P_{\operatorname{Fix} T} x_n\}$ converges to some $x_0 \in \operatorname{Fix} T$. We next prove that $\{x_n\}$ Δ-converges to x_0. Since T is quasinonexpansive, we have

$$\phi_\kappa(x_0, x_{n+1}) \leq \alpha_n \phi_\kappa(x_0, x_n) + (1-\alpha_n)\phi_\kappa(x_0, Tx_n) \leq \phi_\kappa(x_0, x_n)$$

for each $n \in \mathbb{N}$. Therefore, there exists a limit of $\{\phi_\kappa(x_0, x_n)\}$. Note that there exists $L_0 > 0$ such that $L_0 \le \inf_{n\in\mathbb{N}} c''_\kappa(d(x_0, Tx_n))$ since $d(x_0, Tx_n) \le d(x_0, x_n) \le d(x_0, x_1)$ for all $n \in \mathbb{N}$. Since T is firmly geodesically nonspreading, for fixed $n \in \mathbb{N}$,

$$2\phi_\kappa(Tx_n, x_0) \le \phi_\kappa(Tx_n, x_0) + \phi_\kappa(x_0, x_n) - c''_\kappa(d(Tx_n, x_0))\phi_\kappa(Tx_n, x_n)$$

and hence

$$\phi_\kappa(Tx_n, x_0) \le \phi_\kappa(x_0, x_n) - c''_\kappa(d(Tx_n, x_0))\phi_\kappa(Tx_n, x_n).$$

Therefore,

$$\begin{aligned}\phi_\kappa(x_0, x_{n+1}) &\le \alpha_n\phi_\kappa(x_0, x_n) + (1-\alpha_n)\phi_\kappa(x_0, Tx_n)\\ &\le \alpha_n\phi_\kappa(x_0, x_n) + (1-\alpha_n)\phi_\kappa(x_0, x_n) - c''_\kappa(d(Tx_n, x_0))(1-\alpha_n)\phi_\kappa(Tx_n, x_n)\\ &\le \phi_\kappa(x_0, x_n) - L_0\left(1 - \sup_{n\in\mathbb{N}}\alpha_n\right)\phi_\kappa(Tx_n, x_n)\end{aligned}$$

and thus

$$\phi_\kappa(Tx_n, x_n) \le \frac{\phi_\kappa(x_0, x_n) - \phi_\kappa(x_0, x_{n+1})}{L_0\left(1 - \sup_{n\in\mathbb{N}}\alpha_n\right)}.$$

Letting $n \to \infty$, we obtain $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ as $n \to \infty$. Take a Δ-converging subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily and let $z \in X$ be its Δ-limit. From Lemma 4.1.3, we obtain $z \in \operatorname{Fix} T$. Then,

$$\begin{aligned}\limsup_{i\to\infty} d(x_0, x_{n_i}) &\le \limsup_{i\to\infty}\left(d(x_0, P_{\operatorname{Fix} T}x_{n_i}) + d(P_{\operatorname{Fix} T}x_{n_i}, x_{n_i})\right)\\ &= \limsup_{i\to\infty} d(P_{\operatorname{Fix} T}x_{n_i}, x_{n_i}) \le \limsup_{i\to\infty} d(z, x_{n_i}),\end{aligned}$$

which implies that $x_0 = z$ since z is a unique asymptotic centre of $\{x_{n_i}\}$. From Lemma 3.2.6, $\{x_n\}$ Δ-converges to x_0, which completes the proof. □

To prove the convergence theorem above, we use firm nonspreadingness. However, if we devise how to take the coefficients, we get the following:

Theorem 7.1.4. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a geodesically nonspreading mapping from X into itself which has a fixed point. Let $\{\alpha_n\}$ be a real sequence of $[a, 1-a]$ with $a \in\,]0, 1/2]$. For a given initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n)Tx_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ Δ-converges to a fixed point $x_0 \in \operatorname{Fix} T$ which is a limit of $\{P_{\operatorname{Fix} T}x_n\}$, where $P_{\operatorname{Fix} T}$ is a metric projection onto Fix T.

Proof. We know $\{P_{\operatorname{Fix} T}x_n\}$ converges to some $x_0 \in \operatorname{Fix} T$. If we obtain $d(x_n, Tx_n) \to 0$ as $n \to \infty$, then we get the desired result in the same way as Theorem 7.1.3. Since T is quasinonexpansive, we have

$$\begin{aligned}\phi_\kappa(x_0, x_{n+1}) &= \phi_\kappa(x_0, \alpha_n x_n \overset{\kappa}{\oplus} (1-\alpha_n)Tx_n)\\ &\le \alpha_n\phi_\kappa(x_0, x_n) + (1-\alpha_n)\phi_\kappa(x_0, Tx_n) \le \phi_\kappa(x_0, x_n)\end{aligned}$$

for each $n \in \mathbb{N}$. Therefore, there exists a limit of $\{\phi_\kappa(x_0, x_n)\}$. Note that there exists $L_0 > 0$ such that $L_0 \leq \inf_{n\in\mathbb{N}} c''_\kappa(d(x_0, x_n))$. Further, for any $n \in \mathbb{N}$,

$$\begin{aligned}\phi_\kappa(x_0, x_{n+1}) &\leq \alpha_n\phi_\kappa(x_0, x_n) + (1-\alpha_n)\phi_\kappa(x_0, Tx_n) \\ &\quad - c''_\kappa(d(x_0, x_{n+1}))\left(\alpha_n\phi_\kappa(x_n, x_{n+1}) + (1-\alpha_n)\phi_\kappa(Tx_n, x_{n+1})\right) \\ &\leq \phi_\kappa(x_0, x_n) - aL_0\left(\phi_\kappa(x_n, x_{n+1}) + \phi_\kappa(Tx_n, x_{n+1})\right)\end{aligned}$$

and hence

$$0 \leq \phi_\kappa(x_n, x_{n+1}) + \phi_\kappa(Tx_n, x_{n+1}) \leq \frac{\phi_\kappa(x_0, x_n) - \phi_\kappa(x_0, x_{n+1})}{aL_0} \to 0$$

as $n \to \infty$. It implies that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(Tx_n, x_{n+1}) = 0$, and therefore

$$0 \leq \lim_{n\to\infty} d(x_n, Tx_n) \leq \lim_{n\to\infty} d(x_n, x_{n+1}) + \lim_{n\to\infty} d(Tx_n, x_{n+1}) = 0.$$

Consequently, we obtain the desired result using the same way as Theorem 7.1.3. □

7.2 Halpern type iterative scheme

In this section, we obtain convergence theorem with the Halpern type iterative scheme. We first prove the following parallelogram law:

Theorem 7.2.1. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \leq \frac{t\phi_\kappa(x, z) + (1-t)\phi_\kappa(y, z)}{M} - \frac{2t(1-t)\phi_\kappa(x, y)}{M(1+M)}$$

for every $x, y, z \in X$ and $t \in [0, 1]$, where $M = \sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(d(x, y))}$.

Proof. From Theorem 6.2.2,

$$\begin{aligned}\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \leq{}& \frac{t\phi_\kappa(x, z) + (1-t)\phi_\kappa(y, z)}{M} \\ &- \frac{t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y)}{M}.\end{aligned}$$

Further, using the same fashion as Theorem 6.2.5,

$$t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y) = \frac{2t(1-t)\phi_\kappa(x, y)}{1+M}.$$

Hence, we obtain the desired result. □

Using this result, we get the following:

Lemma 7.2.2. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Let $x, y, z \in X$, and $\alpha \in]0, 1[$. Set*

$$M = \sqrt{\alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha)c''_\kappa(d(x, y))} \text{ and } \beta = 1 - \frac{1-\alpha}{M} \neq 0.$$

Then,

$$\begin{aligned}
&\phi_\kappa(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z) \\
&\le (1-\beta)\phi_\kappa(y,z) + \beta\left(\frac{(1+M)(M+1-\alpha)\phi_\kappa(x,z) - 2(1-\alpha)(M+1-\alpha)\phi_\kappa(x,y)}{(1+M)(\alpha+2(1-\alpha)c''_\kappa(d(x,y)))}\right).
\end{aligned}$$

Proof. From the previous theorem, we get

$$\phi_\kappa(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z) \le (1-\beta)\phi_\kappa(y,z) + \frac{\alpha\phi_\kappa(x,z)}{M} - \frac{2\alpha(1-\alpha)\phi_\kappa(x,y)}{M(1+M)}.$$

Then,

$$\begin{aligned}
&\frac{\alpha\phi_\kappa(x,z)}{M} - \frac{2\alpha(1-\alpha)\phi_\kappa(x,y)}{M(1+M)} = \beta\left(\frac{1}{\beta}\right)\left(\frac{\alpha\phi_\kappa(x,z)}{M} - \frac{2\alpha(1-\alpha)\phi_\kappa(x,y)}{M(1+M)}\right) \\
&= \beta\left(\frac{M}{M-(1-\alpha)}\right)\left(\frac{\alpha\phi_\kappa(x,z)}{M} - \frac{2\alpha(1-\alpha)\phi_\kappa(x,y)}{M(1+M)}\right) \\
&= \beta\left(\frac{\alpha\phi_\kappa(x,z)}{M-(1-\alpha)} - \frac{2\alpha(1-\alpha)\phi_\kappa(x,y)}{(1+M)(M-(1-\alpha))}\right) \\
&= \beta\left(\frac{\alpha(M+1-\alpha)\phi_\kappa(x,z)}{M^2-(1-\alpha)^2} - \frac{2\alpha(1-\alpha)(M+1-\alpha)\phi_\kappa(x,y)}{(1+M)(M^2-(1-\alpha)^2)}\right) \\
&= \beta\left(\frac{\alpha(M+1-\alpha)\phi_\kappa(x,z)}{\alpha^2+2\alpha(1-\alpha)c''_\kappa(d(x,y))} - \frac{2\alpha(1-\alpha)(M+1-\alpha)\phi_\kappa(x,y)}{(1+M)(\alpha^2+2\alpha(1-\alpha)c''_\kappa(d(x,y)))}\right) \\
&= \beta\left(\frac{(M+1-\alpha)\phi_\kappa(x,z)}{\alpha+2(1-\alpha)c''_\kappa(d(x,y))} - \frac{2(1-\alpha)(M+1-\alpha)\phi_\kappa(x,y)}{(1+M)(\alpha+2(1-\alpha)c''_\kappa(d(x,y)))}\right) \\
&= \beta\left(\frac{(1+M)(M+1-\alpha)\phi_\kappa(x,z) - 2(1-\alpha)(M+1-\alpha)\phi_\kappa(x,y)}{(1+M)(\alpha+2(1-\alpha)c''_\kappa(d(x,y)))}\right).
\end{aligned}$$

It completes the proof. □

Moreover, we get the following lemmas:

Lemma 7.2.3. *Let $\kappa \in \mathbb{R}$ and $\{l_n\}$ a bounded real sequence of $[0, D_\kappa/2[$. Let $\{\alpha_n\}$ be a real sequence of $]0,1[$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Define a real sequence $\{\beta_n\}$ of $]0,1[$ by*

$$\beta_n = 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)c''_\kappa(l_n)}}$$

for each $n \in \mathbb{N}$. Further, assume one of the following:

(a) $\sup_{n\in\mathbb{N}} l_n < D_\kappa/2$;
(b) $\sum_{n=1}^{\infty}\alpha_n^2 = \infty$.

Then, $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\beta_n = \infty$.

Proof. Since $\{l_n\}$ is bounded, there exist $L > 0$ such that $c''_\kappa(l_n) \le L$ for any $n \in \mathbb{N}$. We first show $\lim_{n\to\infty}\beta_n = 0$. From the definition of $\{\beta_n\}$,

$$0 \le \beta_n \le 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)L}}.$$

Since $\alpha_n \to 0$, we have $\beta_n \to 0$ as $n \to \infty$. We next show $\sum_{n=1}^{\infty} \beta_n = \infty$. Since $\lim_{n\to\infty} \alpha_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $1 - \alpha_n \geq 1/2$ for any $n \geq n_0$. Note that

$$\alpha_n^2 + (1-\alpha_n)^2 \leq 2 \text{ and } \alpha_n(1-\alpha_n) \leq 1$$

for any $n \in \mathbb{N}$. Let

$$M_n = \sqrt{\alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)c''_\kappa(l_n)}$$

for each $n \in \mathbb{N}$. We remark that $M_n \leq \sqrt{2(1+L)}$ for all $n \in \mathbb{N}$. Then, for $n \geq n_0$, we obtain

$$\begin{aligned}
\beta_n &= 1 - \frac{1-\alpha_n}{M_n} = \frac{M_n - (1-\alpha_n)}{M_n} = \frac{M_n^2 - (1-\alpha_n)^2}{M_n(M_n + 1 - \alpha_n)} \\
&= \frac{\alpha_n^2 + 2\alpha_n(1-\alpha_n)c''_\kappa(l_n)}{M_n(M_n+1-\alpha_n)} = \frac{1}{M_n(M_n+1-\alpha_n)}\alpha_n^2 + \frac{2(1-\alpha_n)c''_\kappa(l_n)}{M_n(M_n+1-\alpha_n)}\alpha_n \\
&\geq \frac{1}{\sqrt{2(1+L)}(\sqrt{2(1+L)}+1)}\alpha_n^2 + \frac{c''_\kappa(l_n)}{\sqrt{2(1+L)}(\sqrt{2(1+L)}+1)}\alpha_n.
\end{aligned}$$

From (a) or (b), we get $\sum_{n=1}^{\infty} \beta_n = \infty$. □

Lemma 7.2.4. *Let $\kappa \in \mathbb{R}$, $\{l_n\}$ a bounded real sequence of $[0, D_\kappa/2[$ and $l \in [0, D_\kappa/2[$. Let $\{\alpha_n\}$ be a real sequence of $]0,1[$ which converges to 0. Let*

$$M_n = \sqrt{\alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)c''_\kappa(l_n)}$$

and

$$t_n = \frac{(1+M_n)(M_n+1-\alpha_n)c_\kappa(l) - 2(1-\alpha_n)(M_n+1-\alpha_n)c_\kappa(l_n)}{(1+M_n)(\alpha_n + 2(1-\alpha_n)c''_\kappa(l_n))}$$

for each $n \in \mathbb{N}$. Then, $\limsup_{n\to\infty} t_n \leq 0$ whenever $\liminf_{n\to\infty} l_n \geq l$.

Proof. Note that $M_n \to 1$ as $n \to \infty$ since $\alpha_n \to 0$ and $\{l_n\}$ is bounded. We can take a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\lim_{i\to\infty} t_{n_i} = \limsup_{n\to\infty} t_n.$$

Moreover, there exists a subsequence $\{l_{n_{i_j}}\}$ of $\{l_{n_i}\}$ which converges to $l_0 = \liminf_{i\to\infty} l_{n_i}$. Henceforth, we denote n_{i_j} by j simply. We remark that

$$l_0 = \lim_{j\to\infty} l_j = \liminf_{i\to\infty} l_{n_i} \geq \liminf_{n\to\infty} l_n \geq l.$$

Since $\alpha_j \to 0$ as $j \to \infty$, we have

$$\begin{aligned}
\limsup_{n\to\infty} t_n &= \lim_{j\to\infty} \frac{(1+M_j)(M_j+1-\alpha_j)c_\kappa(l) - 2(1-\alpha_j)(M_j+1-\alpha_j)c_\kappa(l_j)}{(1+M_j)(\alpha_j + 2(1-\alpha_j)c''_\kappa(l_j))} \\
&= \lim_{j\to\infty} \frac{4c_\kappa(l) - 4c_\kappa(l_j)}{4c''_\kappa(l_j)} = \lim_{j\to\infty} \frac{c_\kappa(l) - c_\kappa(l_j)}{c''_\kappa(l_j)}.
\end{aligned}$$

Whenever $c''_\kappa(l_0) \neq 0$, we obtain the desired result. In what follows, we assume that $\kappa > 0$ and $l_0 = D_\kappa/2$. Then,

$$\begin{aligned}\limsup_{n\to\infty} t_n &= \lim_{j\to\infty} \frac{c_\kappa(l) - c_\kappa(l_j)}{c''_\kappa(l_j)} = \lim_{j\to\infty} \frac{1 - c''_\kappa(l) - 1 + c''_\kappa(l_j)}{\kappa c''_\kappa(l_j)} \\ &= \lim_{j\to\infty} \frac{c''_\kappa(l_j) - c''_\kappa(l)}{\kappa c''_\kappa(l_j)} = \frac{1}{\kappa} - \lim_{j\to\infty} \frac{c''_\kappa(l)}{\kappa c''_\kappa(l_j)} \\ &= -\infty < 0.\end{aligned}$$

It completes the proof. □

In what follows, we prove convergence theorems for a firmly geodesically nonspreading mapping. To obtain convergence theorems, we use the following lemma:

Lemma 7.2.5 (Kimura–Saejung [23], Saejung–Yotkaew [37]). *Let $\{s_n\}$ be a real sequence of $[0, \infty[$ and $\{t_n\}$ a real sequence. Let $\{\beta_n\}$ be a real sequence of $]0, 1]$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that*

$$s_{n+1} \leq (1 - \beta_n)s_n + \beta_n t_n$$

for all $n \in \mathbb{N}$ and that $\limsup_{i\to\infty} t_{n_i} \leq 0$ for every subsequence $\{s_{n_i}\}$ of $\{s_n\}$ satisfying that

$$\limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) \leq 0.$$

Then, $\lim_{n\to\infty} s_n = 0$.

We first obtain the following convergence theorem with the Halpern type iterative scheme:

Theorem 7.2.6. *Let X be an admissible complete $\mathrm{CAT}(\kappa)$ space for $\kappa \in \mathbb{R}$ and T a firmly geodesically nonspreading mapping from X into itself which has a fixed point. Let $\{\alpha_n\}$ be a real sequence of $]0, 1[$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. For an anchor point $u \in X$ and an initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = \alpha_n u \overset{\kappa}{\oplus} (1 - \alpha_n)Tx_n$$

for each $n \in \mathbb{N}$. Further, assume one of the following:

(a) $\sup_{n\in\mathbb{N}} d(u, Tx_n) < D_\kappa/2$;
(b) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then, $\{x_n\}$ converges to a fixed point $P_{\operatorname{Fix} T}u$, where $P_{\operatorname{Fix} T}$ is a metric projection onto $\operatorname{Fix} T$.

Proof. Set $p = P_{\operatorname{Fix} T}u$. Since T is quasinonexpansive, for each $n \in \mathbb{N}$,

$$\begin{aligned}\phi_\kappa(p, x_{n+1}) &\leq \alpha_n \phi_\kappa(p, u) + (1 - \alpha_n)\phi_\kappa(p, Tx_n) \\ &\leq \alpha_n \phi_\kappa(p, u) + (1 - \alpha_n)\phi_\kappa(p, x_n) \\ &\leq \max\{\phi_\kappa(p, u), \phi_\kappa(p, x_n)\}.\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$,

$$d(p, Tx_n) \leq d(p, x_n) \leq \max\{d(p, u), d(p, x_1)\} < \frac{D_\kappa}{2},$$

which implies that $\{x_n\}$ is κ-bounded. Note that there exists $L > 0$ which satisfies

$$L \leq c''_\kappa(d(p, Tx_n))$$

for any $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Let $l = d(u,p)$, $l_n = d(u, Tx_n)$,

$$M_n = \sqrt{\alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)c''_\kappa(l_n)} \text{ and } \beta_n = 1 - \frac{1-\alpha_n}{M_n}.$$

Further, set

$$\begin{aligned} t_n &= \frac{(1+M_n)(M_n+1-\alpha_n)\phi_\kappa(p,u) - 2(1-\alpha_n)(M_n+1-\alpha_n)\phi_\kappa(u,Tx_n)}{(1+M_n)(\alpha_n + 2(1-\alpha_n)c''_\kappa(d(u,Tx_n)))} \\ &= \frac{(1+M_n)(M_n+1-\alpha_n)c_\kappa(l) - 2(1-\alpha_n)(M_n+1-\alpha_n)c_\kappa(l_n)}{(1+M_n)(\alpha_n + 2(1-\alpha_n)c''_\kappa(l_n))} \end{aligned}$$

and $s_n = \phi_\kappa(p, x_n)$. From Lemma 7.2.2,

$$\begin{aligned} s_{n+1} = \phi_\kappa(p, x_{n+1}) &= \phi_\kappa(p, \alpha_n u \overset{\kappa}{\oplus} (1-\alpha_n)Tx_n) \\ &\leq (1-\beta_n)\phi_\kappa(p, Tx_n) + \beta_n t_n \\ &\leq (1-\beta_n)\phi_\kappa(p, x_n) + \beta_n t_n = (1-\beta_n)s_n + \beta_n t_n. \end{aligned}$$

Since one of (a) and (b) holds, from Lemma 7.2.3, we have

$$\lim_{n\to\infty} \beta_n = 0 \text{ and } \sum_{n=1}^{\infty} \beta_n = \infty.$$

We show the following statement: For any subsequence $\{s_{n_i}\}$ of $\{s_n\}$ satisfying that

$$\limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) \leq 0,$$

we have $\limsup_{i\to\infty} t_{n_i} \leq 0$. Let $\{s_{n_i}\}$ be a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that

$$\limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) \leq 0.$$

Then, we get

$$\begin{aligned} 0 &\geq \limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) = \limsup_{i\to\infty} (\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, x_{n_i+1})) \\ &= \limsup_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, \alpha_{n_i} u \overset{\kappa}{\oplus} (1-\alpha_{n_i})Tx_{n_i})\right) \\ &\geq \limsup_{i\to\infty} (\phi_\kappa(p, x_{n_i}) - \alpha_{n_i}\phi_\kappa(p,u) - (1-\alpha_{n_i})\phi_\kappa(p, Tx_{n_i})) \\ &= \limsup_{i\to\infty} (\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, Tx_{n_i})) \geq \liminf_{i\to\infty} (\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, Tx_{n_i})) \geq 0 \end{aligned}$$

and thus $\lim_{n\to\infty} (\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, Tx_{n_i})) = 0$. Since T is firmly geodesically nonspreading,

$$\begin{aligned} \phi_\kappa(p, Tx_{n_i}) &\leq \phi_\kappa(p, x_{n_i}) - c''_\kappa(d(Tx_{n_i}, p))\phi_\kappa(Tx_{n_i}, x_{n_i}) \\ &\leq \phi_\kappa(p, x_{n_i}) - L\phi_\kappa(Tx_{n_i}, x_{n_i}). \end{aligned}$$

It implies that

$$\phi_\kappa(Tx_{n_i}, x_{n_i}) \leq \frac{\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, Tx_{n_i})}{L}$$

and therefore $d(Tx_{n_i}, x_{n_i}) \to 0$ as $i \to \infty$. Take a subsequence $\{w_j\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} d(u, w_j) = \liminf_{i\to\infty} d(u, x_{n_i})$$

and that it Δ-converges to some $w \in X$. From Lemma 4.1.3, we get $w \in \operatorname{Fix} T$. Further, since

$$d(u, x_{n_i}) \leq d(u, Tx_{n_i}) + d(Tx_{n_i}, x_{n_i}) \leq d(u, x_{n_i}) + 2d(Tx_{n_i}, x_{n_i}),$$

we have

$$\liminf_{i\to\infty} l_{n_i} = \liminf_{i\to\infty} d(u, Tx_{n_i}) = \liminf_{i\to\infty} d(u, x_{n_i}).$$

Hence, from Lemma 3.2.4,

$$\liminf_{i\to\infty} l_{n_i} = \liminf_{i\to\infty} d(u, x_{n_i}) = \lim_{j\to\infty} d(u, w_j) \geq d(u, w) \geq d(u, p) = l.$$

From Lemma 7.2.4, we have

$$\limsup_{i\to\infty} t_{n_i} \leq 0.$$

Consequently, from Lemma 7.2.5, we obtain $\lim_{n\to\infty} s_n = 0$. It means that $\{x_n\}$ converges to $P_{\operatorname{Fix} T} u$. □

In Theorem 7.2.6, to get a convergence theorem, we need to use κ-convex combination. To obtain a convergence theorem with the usual convex combination as a direct consequence of Theorem 7.2.6, we need to prove the following lemma:

Lemma 7.2.7. *Let $\kappa \in \mathbb{R}$. Let $\{\alpha_n\}$ be a real sequence of $]0, 1[$ which converges to 0 and let $\{l_n\}$ be a bounded real sequence of $[0, D_\kappa/2[$. Let*

$$\sigma_n = \frac{(\alpha_n)^\kappa_{l_n}}{(\alpha_n)^\kappa_{l_n} + (1-\alpha_n)^\kappa_{l_n}}$$

for each $n \in \mathbb{N}$. Then, there exist positive real numbers r_1 and r_2, and $n_0 \in \mathbb{N}$ such that

$$r_1\alpha_n \leq \sigma_n \leq r_2\alpha_n$$

for all $n \in \mathbb{N}$ with $n \geq n_0$.

Proof. Let $l_0 = \sup_{n\in\mathbb{N}} l_n$. We first show that there exists a real number r_2 such that $\sigma_n \leq r_2\alpha_n$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ arbitrarily. If $l_n = 0$, then

$$\sigma_n = \frac{(\alpha_n)^\kappa_0}{(\alpha_n)^\kappa_0 + (1-\alpha_n)^\kappa_0} = \alpha_n.$$

Suppose $l_n \neq 0$. Then,

$$\sigma_n = \frac{(\alpha_n)^\kappa_{l_n}}{(\alpha_n)^\kappa_{l_n} + (1-\alpha_n)^\kappa_{l_n}} = \frac{c'_\kappa(\alpha_n l_n)}{c'_\kappa(\alpha_n l_n) + c'_\kappa((1-\alpha_n)l_n)}.$$

We notice that

$$\frac{\tau l_n}{c'_\kappa(\tau l_n)} \leq 2$$

for any $\kappa \in \mathbb{R}$ and $\tau \in]0,1[$. Therefore,

$$c'_\kappa(\alpha_n l_n) + c'_\kappa((1-\alpha_n)l_n) \geq \frac{\alpha_n l_n}{2} + \frac{(1-\alpha_n)l_n}{2} = \frac{l_n}{2}$$

and hence

$$\sigma_n \leq \frac{2c'_\kappa(\alpha_n l_n)}{l_n} = 2\alpha_n \cdot \frac{c'_\kappa(\alpha_n l_n)}{\alpha_n l_n}.$$

Since $\{l_n\}$ is bounded and $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\lim_{n\to\infty} \frac{c'_\kappa(\alpha_n l_n)}{\alpha_n l_n} = 1.$$

Therefore, setting $r_2 = 4$, we obtain the desired evaluation. We next show that there exists a real number r_1 such that $r_1\alpha_n \leq \sigma_n$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ arbitrarily. Suppose $l_n \neq 0$. If $\kappa > 0$, then

$$\sigma_n = \frac{(\alpha_n)^\kappa_{l_n}}{(\alpha_n)^\kappa_{l_n} + (1-\alpha_n)^\kappa_{l_n}} \geq \frac{(\alpha_n)^\kappa_{l_n}}{2} = \frac{c'_\kappa(\alpha_n l_n)}{2c'_\kappa(l_n)} \geq \frac{\alpha_n c'_\kappa(l_n)}{2c'_\kappa(l_n)} = \frac{\alpha_n}{2}.$$

On the other hand, if $\kappa \leq 0$, then

$$\sigma_n = \frac{(\alpha_n)^\kappa_{l_n}}{(\alpha_n)^\kappa_{l_n} + (1-\alpha_n)^\kappa_{l_n}} \geq (\alpha_n)^\kappa_{l_n} = \frac{c'_\kappa(\alpha_n l_n)}{c'_\kappa(l_n)} \geq \frac{\alpha_n l_n}{c'_\kappa(l_n)}.$$

Since

$$c'_\kappa(a) \leq \frac{c'_\kappa(l_0)}{l_0} a$$

for all $a \in [0, l_0]$, we have

$$\sigma_n \geq \frac{\alpha_n l_n}{c'_\kappa(l_n)} \geq \frac{\alpha_n l_0}{c'_\kappa(l_0)}.$$

Set $r_1 = \min\{1/2, l_0/c'_\kappa(l_0)\}$. Then, for any $\kappa \in \mathbb{R}$, we have $\sigma_n \geq r_1\alpha_n$ whenever $l_n \neq 0$. However, this inequality holds if $l_n = 0$. Consequently, we obtain the desired result. □

Now, we obtain the following:

Theorem 7.2.8. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a firmly geodesically nonspreading mapping from X into itself has a fixed point. Let $\{\alpha_n\}$ be a real sequence of $]0,1[$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. For an anchor point $u \in X$ and an initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)Tx_n$$

for each $n \in \mathbb{N}$. Further, assume one of the following:

(a) $\sup_{n\in\mathbb{N}} d(u, Tx_n) < D_\kappa/2$;
(b) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then, $\{x_n\}$ *converges to a fixed point* $P_{\operatorname{Fix} T} u$*, where* $P_{\operatorname{Fix} T}$ *is a metric projection onto* $\operatorname{Fix} T$.

Proof. Set

$$\sigma_n = \frac{(\alpha_n)_{l_n}^{\kappa}}{(\alpha_n)_{l_n}^{\kappa} + (1-\alpha_n)_{l_n}^{\kappa}}$$

for each $n \in \mathbb{N}$, where $l_n = d(u, Tx_n)$. Then, from Lemma 7.2.7,

$$\lim_{n\to\infty} \sigma_n = 0 \text{ and } \sum_{n=1}^{\infty} \sigma_n = \infty.$$

We remark that

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n) T x_n = \sigma_n u \overset{\kappa}{\oplus} (1-\sigma_n) T x_n$$

for each $n \in \mathbb{N}$. Consequently, from Theorem 7.2.6, $\{x_n\}$ converges to $P_{\operatorname{Fix} T} u$. □

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