A modification of approximating sequences for a Δ-convergence theorem in CAT(1) spaces

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Master Thesis
March 2022

Contents

Chapter 1

Introduction

The notion of resolvents plays an important role in convex minimization problems. Mayer [17] showed that the resolvent J_f of a proper lower semicontinuous convex function in a complete CAT(0) space X can be defined by

$$J_f x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2} d(y,x)^2 \right\}$$

for all $x \in X$. Recently, Kimura and Kohsaka [7] discovered a new resolvent in CAT(1) space. That is, let X be a complete admissible CAT(1) space and f a proper lower semicontinuous convex function. The resolvent R_f of f is defined by

$$R_f x = \operatorname*{argmin}_{y \in X} \{ f(y) + \tan d(y,x) \sin d(y,x) \}$$

for all $x \in X$. In this thesis, we will adopt this resolvent. Furthermore, Kajimura and Kimura [5] discovered another resolvent. That is, let X be a complete CAT(1) space and f a proper lower semicontinuous convex function. The resolvent Q_f of f is defined by

$$Q_f x = \operatorname*{argmin}_{y \in X} \{ f(y) - \log(\cos d(y,x)) \}$$

for all $x \in X$.

Next, we introduce some types of iterative sequences converging to a fixed point of a mapping. The Mann type iteration [15] is as follows:

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T x_n$$

for all $n \in \mathbb{N}$, where T is a nonexpansive mapping. The first result of the Mann type iteration on a complete CAT(0) space and CAT(1) space was showed by He, Fang, López, and Li [4]. The Halpern type iteration [3] is as follows:

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T x_n$$

for all $n \in \mathbb{N}$, where T is a nonexpansive mapping and $u \in X$. The first result of the Halpern type iteration on a complete CAT(0) space was showed by Saejung [19] for nonexpansive mappings. Recently, Kimura and Satô [13] showed a remarkable result on a complete CAT(1) space for quasinonexpansive mappings.

The proximal point algorithm, introduced by Martinet [16], is an iterative method for approximating zero points of maximal monotone operators in Hilbert spaces. In 1976, Rockafellar [18] proved its weak convergence in the setting of Banach spaces. In 2013, Bačák [1] proved a Δ-convergence theorem in Hadamard spaces.

Theorem 1.1 ([1]). *Let X be a Hadamard space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$, $J_{\eta f}$ the resolvent of ηf for all $\eta > 0$ and $\{x_n\}$ a sequence defined by $x_1 \in X$ and $x_{n+1} = J_{\lambda_n f} x_n$, where $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. If $\operatorname{argmin}_X f$ is nonempty, then $\{x_n\}$ is Δ-convergent to an element of $\operatorname{argmin}_X f$.*

In 2019, Kimura and Kohsaka (Theorem 3.1) proved a Δ-convergence theorem of Mann type iterative sequence in CAT(1) spaces. They also proved a convergence theorem of Halpern type iterative sequence in CAT(1) spaces [10].

The following theorem was showed by Kimura.

Theorem 1.2 ([6]). *Let E be a uniformly convex Banach space satisfying either the Opial property or the Fréchet differentiability of the norm. Let $\{A_n\}$ be a sequence of m-accretive operators on E such that $\bigcap_{n=1}^{\infty} A_n^{-1}0$ is nonempty and that for any sequences $\{u_n\}$ and $\{v_n\}$ in E satisfying that $u_n \in A_n v_n$ for every $n \in \mathbb{N}$ and that $\{u_n\}$ converges strongly to 0, every subsequential weak limit point of $\{v_n\}$ belongs to $\bigcap_{n=1}^{\infty} A_n^{-1}0$.*

Let $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$ and that both $\{\beta_n\}$ and $\{\gamma_n\}$ converge to 0. $\{e_n\}$ is a sequence in E such that $\sum_{n=1}^{\infty} ||e_n|| < \infty$. For an initial point $x_1 \in H$, generate an iterative sequence $\{x_n\}$ as follows:

$$\begin{aligned}
y_n &= (I + A_n)^{-1} x_n, \\
\alpha_n &\in [\min\{\beta_n, ||x_n - y_n|| - \gamma_n\}, 1] \cap [0, 1], \\
x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n y_n + e_n.
\end{aligned}$$

Then, $x_n \rightharpoonup x_0 \in \bigcap_{n=1}^{\infty} A_n^{-1}0$.

In this thesis, we apply this theorem to convex minimization problems in CAT(1) spaces. Furthermore, we try to modify the coefficient condition.

Chapter 2

Preliminaries

In this chapter, we introduce various notations that will be need in the main result. Let κ be a nonnegative real number and D_κ the extended real number defined by $D_\kappa = \infty$ if $\kappa = 0$ and $\pi/\sqrt{\kappa}$ if $\kappa > 0$. A metric space X is said to be D_κ-geodesic if for each $x, y \in X$ with $d(x, y) < D_\kappa$, there exists a mapping $c : [0, l] \to X$ such that $c(0) = x, c(l) = y$, and

$$d(c(t_1), c(t_2)) = |t_1 - t_2|$$

for all $t_1, t_2 \in [0, l]$, where $l = d(x, y)$. The mapping c is called a geodesic from x to y. If a geodesic c from x to y is unique, the geodesic segment $[x, y]$ is defined by

$$[x, y] = \{c(t) : 0 \leq t \leq l\}$$

and the point $\alpha x \oplus (1 - \alpha)y$ is defined by

$$\alpha x \oplus (1 - \alpha)y = c((1 - \alpha)l)$$

for all $\alpha \in [0, 1]$. A subset C of a D_κ-geodesic space X such that $d(v, v') < D_\kappa$ for all $v, v' \in C$ is said to be convex if

$$\alpha x \oplus (1 - \alpha)y \in C$$

whenever $x, y \in C$ and $\alpha \in [0, 1]$.

Let (M_κ, d_κ) be the uniquely D_κ-geodesic space given by

$$(M_\kappa, d_\kappa) = \begin{cases} (\mathbb{R}^2, \rho_{\mathbb{R}^2}), & (\kappa = 0); \\ (\mathbb{S}^2, \frac{1}{\sqrt{\kappa}}\rho_{\mathbb{S}^2}) & (\kappa > 0). \end{cases}$$

Suppose κ is a nonnegative real number, X is a D_κ-geodesic space, and x_1, x_2, x_3 are points of X satisfying

$$d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_\kappa. \tag{2.1}$$

Then there exist $\bar{x_1}, \bar{x_2}, \bar{x_3} \in M_\kappa$ such that

$$d(x_i, x_j) = d_\kappa(\bar{x_i}, \bar{x_j})$$

for all $i, j \in \{1, 2, 3\}$. The sets Δ and $\bar{\Delta}$ given by

$$\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$$

and

$$\bar{\Delta} = [\bar{x_1}, \bar{x_2}] \cup [\bar{x_2}, \bar{x_3}] \cup [\bar{x_3}, \bar{x_1}]$$

are called a geodesic triangle with vertices x_1, x_2, x_3 in X and a comparison triangle for Δ, respectively. A point $\bar{p} \in \bar{\Delta}$ is called a comparison point for $p \in \Delta$ if

$$p \in [x_i, x_j], \ \bar{p} \in [\bar{x_i}, \bar{x_j}],$$

and

$$d(x_i, p) = d_\kappa(\bar{x_i}, \bar{p})$$

for some distinct $i, j \in \{1, 2, 3\}$. A metric space X is said to be a CAT(κ) space if it is D_κ-geodesic and the CAT(κ) inequality

$$d(p, q) \leq d_\kappa(\bar{p}, \bar{q})$$

holds whenever Δ is a geodesic triangle with vertices $x_1, x_2, x_3 \in X$ satisfying (2.1), $\bar{\Delta}$ is a comparison triangle for Δ, and $\bar{p}, \bar{q} \in \bar{\Delta}$ are comparison points for $p, q \in \Delta$, respectively. Every a CAT(κ) space is a CAT(κ') space for all $\kappa' \in \,]\kappa, \infty[$. A complete CAT(0) space is particularly called a Hadamard space. In CAT(1) spaces, we can consider the following convex combination. Let X be a uniquely geodesic space. For $u, v \in X$ with $d(u, v) < \pi/2$ and $\alpha \in [0, 1]$, we define a 1-convex combination [12] of u and v by

$$\alpha u \overset{1}{\oplus} (1 - \alpha) v \overset{\text{def}}{=} \operatorname*{argmax}_{x \in X} (\alpha \cos d(u, x) + (1 - \alpha) \cos d(v, x)).$$

We say that a CAT(1) space X is admissible if

$$d(w, w') < \frac{\pi}{2}$$

for all $w, w' \in X$. A sequence $\{x_n\}$ in X is said to be spherically bounded if

$$\inf_{y \in X} \limsup_{n \to \infty} d(x_n, y) < \frac{\pi}{2}.$$

Let X be a CAT(1) space, $x_1, x_2, x_3 \in X$, and $\alpha \in [0, 1]$. Then

$$\begin{aligned} &\cos d(\alpha x_1 \oplus (1 - \alpha) x_2, x_3) \sin d(x_1, x_2) \geq \\ &\qquad \cos d(x_1, x_3) \sin(\alpha d(x_1, x_2)) + \cos d(x_2, x_3) \sin((1 - \alpha) d(x_1, x_2)). \end{aligned} \tag{2.2}$$

This inequality is called the parallelogram law in CAT(1) spaces. We also know the following inequality.

$$\cos d(\alpha x_1 \oplus (1 - \alpha) x_2, x_3) \geq \alpha \cos d(x_1, x_3) + (1 - \alpha) \cos d(x_2, x_3). \tag{2.3}$$

For a bounded sequence $\{x_n\} \subset X$, the asymptotic center $\mathcal{A}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathcal{A}(\{x_n\}) = \left\{ u \in X \;\middle|\; \limsup_{n \to \infty} d(u, x_n) = \inf_{y \in X} \limsup_{n \to \infty} d(y, x_n) \right\}.$$

A sequence $\{x_n\}$ is said to be Δ-convergent to a point $p \in X$ if

$$\mathcal{A}(\{x_{n_i}\}) = \{p\}$$

for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. For a sequence $\{x_n\} \subset X$, we denote by $\omega_\Delta(\{x_n\})$ the set of all $z \in X$ such that there exists a subsequence of $\{x_n\}$ which is Δ-convergent to z. For a mapping T of X into itself, we denote by $F(T)$ the set of all $u \in X$ such that $Tu = u$. For a function f of X into $]-\infty, \infty]$, we denote by $\operatorname{argmin}_X f$ the set of all $u \in X$ such that $f(u) = \inf f(X)$. R_f is defined by

$$R_f x = \operatorname*{argmin}_{y \in X} \{f(y) + \tan d(y,x) \sin d(y,x)\}$$

for all $x \in X$. We know [8] that R_f is a well-defined and single-valued mapping of X into itself, and it satisfies

$$F(R_f) = \operatorname*{argmin}_X f. \tag{2.4}$$

The following is the definition of vicinal mapping.

Definition 2.1 ([14]). Let X be a metric space such that $d(v, v') \le \pi/2$ for all $v, v' \in X$, T a mapping of X into itself, and C_z the real number given by

$$C_z = \cos d(Tz, z)$$

for all $z \in X$.

The mapping T is said to be vicinal if

$$\begin{aligned}(C_x^2(1 + C_y^2) &+ C_y^2(1 + C_x^2)) \cos d(Tx, Ty) \\ &\ge C_x^2(1 + C_y^2) \cos d(Tx, y) + C_y^2(1 + C_x^2) \cos d(Ty, x)\end{aligned}$$

for all $x, y \in X$.

We know the following lemma regarding to vicinal mapping.

Lemma 2.2 ([14]). *Let X be a metric space such that $d(v, v') < \pi/2$ for all $v, v' \in X$, T a vicinal mapping of X into itself, p an element of X, and $\{x_n\}$ a sequence in X such that $\mathcal{A}(\{x_n\}) = \{p\}$ and $d(Tx_n, x_n) \to 0$. Then p is a fixed point of T.*

Let X be an admissible CAT(1) space and f a function of X into $]-\infty, \infty]$. Then f is said to be proper if $f(a) \in \mathbb{R}$ for some $a \in X$. It is said to be convex if

$$f(\alpha x \oplus (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y)$$

whenever $x, y \in X$ and $\alpha \in]0, 1[$. Also, f is said to be lower semicontinuous if for any $a \in \mathbb{R}$, the set

$$\{x \in X \mid f(x) \le a\}$$

is closed.

We know the following lemmas.

Lemma 2.3 ([2]). *Let X be a complete* CAT(1) *space and $\{x_n\}$ a spherically bounded sequence in X. Then $\mathcal{A}(\{x_n\})$ is a singleton and $\{x_n\}$ has a Δ-convergent subsequence.*

Lemma 2.4 ([11]). *Let X be a complete* CAT(1) *space and $\{x_n\}$ a spherically bounded sequence in X. If $\{d(z, x_n)\}$ is convergent for each element z of $\omega_\Delta(\{x_n\})$, then $\{x_n\}$ is Δ-convergent to an element of X.*

Lemma 2.5 ([10]). *Let X be an admissible complete* CAT(1) *space, f a proper convex lower semicontinuous functions of X into $]-\infty, \infty]$. $R_{\eta f}$ the resolvent of ηf for all $\eta > 0$ and $C_{\eta,z}$ the real number given by*

$$C_{\eta,z} = \cos d(R_{\eta f} z, z) \tag{2.5}$$

for all $\eta > 0$ *and* $z \in X$. *Then*

$$(\lambda C_{\lambda,x}^2(1+C_{\mu,y}^2)C_{\mu,y} + \mu C_{\mu,y}^2(1+C_{\lambda,x}^2)C_{\lambda,x}) \cos d(R_{\lambda f}x, R_{\mu f}y)$$
$$\geq \lambda C_{\lambda,x}^2(1+C_{\mu,y}^2) \cos d(R_{\lambda f}x, y) + \mu C_{\mu,y}^2(1+C_{\lambda,x}^2) \cos d(R_{\mu f}y, x) \quad (2.6)$$

holds for all $x, y \in X$ *and* $\lambda, \mu > 0$. *Further,*

$$\frac{\pi}{2}\left(\frac{1}{C_{\lambda,x}^2} + 1\right)(C_{\lambda,x} \cos d(u, R_{\lambda f}x) - \cos d(u, x)) \geq \lambda(f(R_{\lambda f}x) - f(u)) \quad (2.7)$$

and

$$\cos d(R_{\lambda f}x, x) \cos d(u, R_{\lambda f}x) \geq \cos d(u, x) \quad (2.8)$$

hold for all $x \in X, u \in \operatorname{argmin}_X f$ *and* $\lambda > 0$.

Lemma 2.6 ([9])**.** *Let* X *be an admissible complete* CAT(1) *space,* $\{z_n\}$ *a spherically bounded sequence in* X, $\{\beta_n\}$ *a sequence of positive real numbers such that* $\sum_{n=1}^{\infty} \beta_n = \infty$ *and* g *the real function on* X *defined by*

$$g(y) = \liminf_{n\to\infty} \frac{1}{\sum_{l=1}^{n} \beta_l} \sum_{k=1}^{n} \cos d(y, z_k) \quad (2.9)$$

for all $y \in X$. *Then* g *is a 1-Lipschitz continuous and concave function of* X *into* $[0, 1]$ *which has a unique maximizer.*

Chapter 3

Convex minimization problems

In this chapter, we apply Theorem 1.2 to convex minimization problems. Furthermore, we try to modify the coefficient condition. We know the following result.

Theorem 3.1 ([10])**.** *Let X be an admissible complete* CAT(1) *space, f a proper convex lower semicontinuous function, $R_{\eta f}$ the resolvent of ηf for all $\eta > 0$ and $\{x_n\}$ a sequence defined by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) R_{\lambda_n f} x_n (n = 1, 2, \cdots),$$

where $\{\alpha_n\}$ is a sequence in $[0, 1[$ and $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty}(1 - \alpha_n)\lambda_n = \infty$. Then the following hold.

(i) *The set $\operatorname{argmin}_X f$ is nonempty if and only if $\{R_{\lambda_n f} x_n\}$ is spherically bounded and $\sup_n d(R_{\lambda_n f} x_n, x_n) < \pi/2$;*

(ii) *if $\operatorname{argmin}_X f$ is nonempty and $\sup_n \alpha_n < 1$, then both $\{x_n\}$ and $\{R_{\lambda_n f} x_n\}$ are Δ-convergent to an element x_∞ of $\operatorname{argmin}_X f$.*

For the sake of completeness, we will give the proof.

Proof. Let $C_{\eta,z}$ be the real number given by (2.5) for all $\eta > 0$ and $z \in X$ and let $\{z_n\}$ be the sequence in X given by $z_n = R_{\lambda_n f} x_n$ for all $n \in \mathbb{N}$.

We first show the if part of (i). Suppose that $\{z_n\}$ is spherically bounded and $\sup_n d(z_n, x_n) < \pi/2$ and let $\{\beta_n\}$ and $\{\sigma_n\}$ be the real sequences given by

$$\beta_n = \frac{(1 - \alpha_n)\lambda_n C^2_{\lambda_n, x_n}}{1 + C^2_{\lambda_n, x_n}} \quad \text{and} \quad \sigma_n = \sum_{l=1}^{n} \beta_l$$

for all $n \in \mathbb{N}$. Since $\alpha_n < 1$ and $\lambda_n > 0$ for all $n \in \mathbb{N}$, and X is admissible, we know that $\{\beta_n\}$ is a sequence of positive real numbers. Since $\sup_n d(z_n, x_n) < \pi/2$, we also know that

$$0 < \cos\left(\sup_n d(z_n, x_n)\right) = \inf_n \cos d(z_n, x_n) = \inf_n C_{\lambda_n, x_n}.$$

Thus, since

$$\beta_n \geq \frac{(1 - \alpha_n)\lambda_n (\inf_m C_{\lambda_m, x_m})^2}{2}$$

and $\sum_{n=1}^{\infty}(1 - \alpha_n)\lambda_n = \infty$, we obtain $\sum_{n=1}^{\infty} \beta_n = \infty$. According to Theorem 2.6, the real function g on X defined by (2.9) for all $y \in X$ has a unique maximizer $p \in X$. It follows from (2.6) that

$$\frac{\lambda_k C^2_{\lambda_k,x_k}}{1+C^2_{\lambda_k,x_k}}\cos d(R_{\lambda_k f}x_k, R_f p) \geq \frac{\lambda_k C^2_{\lambda_k,x_k}}{1+C^2_{\lambda_k,x_k}}\cos d(R_{\lambda_k f}x_k, p) + \frac{C^2_{1,p}}{1+C^2_{1,p}}(\cos d(R_f p, x_k) - \cos d(R_f p, R_{\lambda_k f}x_k))$$

and hence

$$\beta_k \cos d(z_k, R_f p) \geq \beta_k \cos d(z_k, p) + \frac{(1-\alpha_k)C^2_{1,p}}{1+C^2_{1,p}}(\cos d(R_f p, x_k) - \cos d(R_f p, z_k)) \tag{3.1}$$

for all $k \in \mathbb{N}$. On the other hand, it follows from (2.3) and the definition of $\{x_n\}$ that

$$\cos d(R_f p, x_{k+1}) \geq \alpha_k \cos d(R_f p, x_k) + (1-\alpha_k)\cos d(R_f p, z_k) \tag{3.2}$$

and hence, by (3.1) and (3.2), we have

$$\beta_k \cos d(z_k, R_f p) \geq \beta_k \cos d(z_k, p) + \frac{C^2_{1,p}}{1+C^2_{1,p}}(\cos d(R_f p, x_k) - \cos d(R_f p, x_{k+1})) \tag{3.3}$$

for all $k \in \mathbb{N}$. Summing up (3.3) with respect to $k \in \{1, 2, \cdots, n\}$, we obtain

$$\frac{1}{\sigma_n}\sum_{k=1}^{n}\beta_k \cos d(z_k, R_f p) \geq \frac{1}{\sigma_n}\sum_{k=1}^{n}\beta_k \cos d(z_k, p) + \frac{C^2_{1,p}}{1+C^2_{1,p}}\cdot\frac{1}{\sigma_n}(\cos d(R_f p, x_1) - \cos d(R_f p, x_{n+1})) \tag{3.4}$$

for all $n \in \mathbb{N}$. Since $\lim_n \sigma_n = \infty$, it follows from (3.4) that $g(R_f p) \geq g(p)$. Since p is the unique maximizer of g, we obtain $R_f p = p$. Consequently, it follows from (2.4) that $p \in \operatorname{argmin}_X f$. Therefore, the set $\operatorname{argmin}_X f$ is nonempty.

We next show the only if part of (i). Suppose that $\operatorname{argmin}_X f$ is nonempty and let $u \in \operatorname{argmin}_X f$ be given. Then it follows from (2.8) that

$$\begin{aligned}\min\{\cos d(u, z_n), \cos d(z_n, x_n)\} &\geq \cos d(u, z_n)\cos d(z_n, x_n)\\ &\geq \cos d(u, x_n)\end{aligned} \tag{3.5}$$

and hence

$$\max\{d(u, z_n), d(z_n, x_n)\} \leq d(u, x_n) \tag{3.6}$$

for all $n \in \mathbb{N}$. By (2.3) and (3.5), we have

$$\cos d(u, x_{n+1}) \geq \alpha_n \cos d(u, x_n) + (1-\alpha_n)\cos d(u, z_n) \geq \cos d(u, x_n). \tag{3.7}$$

It follows from (3.7) and the admissibility of X that

$$d(u, x_{n+1}) \leq d(u, x_n) \leq d(u, x_1) < \frac{\pi}{2} \tag{3.8}$$

for all $n \in \mathbb{N}$. Thus it follows from (3.6) and (3.8) that

$$\limsup_{n\to\infty} d(u, z_n) \leq \lim_{n\to\infty} d(u, x_n) \leq d(u, x_1) < \frac{\pi}{2}.$$

This implies the spherical boundedness of $\{x_n\}$ and $\{z_n\}$. It also follows from (3.6) and (3.8) that $\sup_n d(z_n, x_n) < \pi/2$.

We finally show (ii). Suppose that $\operatorname{argmin}_X f$ is nonempty and $\sup_n \alpha_n < 1$. Then we know that (3.5), (3.6), (3.7) and (3.8) hold and that both $\{x_n\}$ and $\{z_n\}$ are spherically bounded. Let $u \in \operatorname{argmin}_X f$ be given. It follows from (3.8) that $\{d(u, x_n)\}$ tends to some $\beta \in [0, \pi/2[$. By (2.3) and (3.5), we have

$$\begin{aligned}\cos d(u, x_{n+1}) &\geq \alpha_n \cos d(u, x_n) + (1 - \alpha_n) \cos d(u, z_n) \\ &\geq \alpha_n \cos d(u, x_n) + (1 - \alpha_n) \cdot \frac{\cos d(u, x_n)}{\cos d(z_n, x_n)} \\ &= \cos d(u, x_n) + (1 - \alpha_n) \cos d(u, x_n) \left(\frac{1}{\cos d(z_n, x_n)} - 1 \right)\end{aligned}$$

and hence

$$0 \leq (1 - \alpha_n) \left(\frac{1}{\cos d(z_n, x_n)} - 1 \right) \leq \frac{\cos d(u, x_{n+1})}{\cos d(u, x_n)} - 1 \to \frac{\cos \beta}{\cos \beta} - 1 = 0 \tag{3.9}$$

as $n \to \infty$. Since $\sup_n \alpha_n < 1$, it follows from (3.9) that

$$\lim_{n \to \infty} d(z_n, x_n) = 0. \tag{3.10}$$

On the other hand, it follows from (2.7) and (3.10) that there exists a positive real number K such that

$$\lambda_n (f(z_n) - f(u)) \leq \frac{K\pi}{2} (\cos d(u, z_n) - \cos d(u, x_n)) \tag{3.11}$$

for all $n \in \mathbb{N}$. It then follows from (3.7) and (3.11) that

$$(1 - \alpha_n) \lambda_n (f(z_n) - f(u)) \leq \frac{K\pi}{2} (\cos d(u, x_{n+1}) - \cos d(u, x_n))$$

and hence

$$\sum_{n=1}^{\infty} (1 - \alpha_n) \lambda_n (f(z_n) - f(u)) \leq \frac{K\pi}{2} (\cos \beta - \cos d(u, x_1)) < \infty. \tag{3.12}$$

Since $\sum_{n=1}^{\infty} (1 - \alpha_n) \lambda_n = \infty$, it follows from (3.12) that

$$\liminf_{n \to \infty} (f(z_n) - f(u)) = 0. \tag{3.13}$$

By the definitions of $\{x_n\}$ and $\{z_n\}$ and the convexity of f, we also have

$$-\infty < \inf f(X) \leq f(z_n) \leq f(z_n) + \frac{1}{\lambda_n} \tan d(z_n, x_n) \sin d(z_n, x_n) \leq f(x_n)$$

and

$$-\infty < \inf f(X) \leq f(x_{n+1}) \leq \alpha_n f(x_n) + (1 - \alpha_n) f(z_n) \leq f(x_n)$$

for all $n \in \mathbb{N}$. Thus $\{f(x_n)\}$ tends to some $\gamma \in \mathbb{R}$ and $\{f(z_n)\}$ is bounded. Let $\{n_i\}$ be any increasing sequence in $\mathbb{N}$. Since $\sup_n \alpha_n < 1$, we have a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that $\{\alpha_{n_{i_j}}\}$ tends to some $\delta \in [0, 1[$. Then, letting $j \to \infty$ in

$$\frac{1}{1 - \alpha_{n_{i_j}}} \left(f(x_{n_{i_j}+1}) - \alpha_{n_{i_j}} f(x_{n_{i_j}}) \right) \leq f(z_{n_{i_j}}) \leq f(x_{n_{i_j}}),$$

we obtain $f(z_{n_{i_j}}) \to \gamma$. Thus $\{f(z_n)\}$ also tends to γ. Consequently, it follows from (3.13) that

$$\lim_{n\to\infty} f(x_n) = \gamma = f(u) = \inf f(X). \tag{3.14}$$

Hence we obtain the desired result. □

We obtain the following theorem. This is the first one of our two main results in this thesis.

Theorem 3.2. *Let X be an admissible complete* CAT(1) *space. Let $f : X \to]-\infty, \infty]$ be a proper convex lower semicontinuous function such that* $\operatorname{argmin} f \neq \emptyset$. *Let $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$ and that both $\{\beta_n\}$ and $\{\gamma_n\}$ converge to 0. For an initial point $x_1 \in X$ such that $f(x_1) < \infty$, generate a sequence $\{x_n\}$ as follows:*

$$\begin{aligned}
&y_n = R_f x_n,\\
&\alpha_n \in [\min\{\beta_n, d(x_n, y_n) - \gamma_n\}, 1] \cap [0,1],\\
&x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n y_n.
\end{aligned}$$

If $\sup \alpha_n < 1$, $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \operatorname{argmin} f$.

Proof. We consider the cases that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\sum_{n=1}^{\infty} \alpha_n = \infty$, it is already shown by Theorem 3.1. So, we consider the case that $\sum_{n=1}^{\infty} \alpha_n < \infty$. We show $x_n \overset{\Delta}{\rightharpoonup} x_0$. Put $M = \sup_{j\in\mathbb{N}} d(y_j, x_j)$,

$$\begin{aligned}
d(x_{n+1}, x_n) &= d((1-\alpha_n)x_n \oplus \alpha_n y_n, x_n)\\
&= \alpha_n d(y_n, x_n)\\
&\leq \alpha_n M.
\end{aligned}$$

For $m, n \in \mathbb{N}$, let $m < n$,

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)\\
&= \sum_{j=m}^{n-1} d(x_{j+1}, x_j) \leq \sum_{j=m}^{n-1} \alpha_j M \leq \sum_{j=m}^{\infty} \alpha_j M.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \alpha_n$ is finite, $\{x_n\}$ is a Cauchy sequence. Therefore, $x_n \to x_0$, and hence $x_n \overset{\Delta}{\rightharpoonup} x_0$.

Next, we show that there exists $\{n_i\}$ such that $d(x_{n_i}, y_{n_i}) \to 0$. We focus on the range of α_n. Put $P = \{n \in \mathbb{N} \mid \alpha_n \in [d(x_n, y_n) - \gamma_n, 1] \cap [0,1]\}, Q = \{n \in \mathbb{N} \mid \alpha_n \in [\beta_n, 1]\}$. Assume that there exists $n_0 \in \mathbb{N}$ such that $n \in Q$ for all $n \geq n_0$,

$$\infty = \sum_{n=n_0}^{\infty} \beta_n \leq \sum_{n=n_0}^{\infty} \alpha_n < \infty.$$

This is a contradiction. Therefore for all $n_0 \in \mathbb{N}$, there exists $n \geq n_0$ such that $n \in P$. So, for all $i \in \mathbb{N}$, there exists $n_i \geq i$ such that $n_i \in P$. Hence $\alpha_{n_i} \in [d(x_{n_i}, y_{n_i}) - \gamma_{n_i}, 1]$, we get $d(x_{n_i}, y_{n_i}) - \gamma_{n_i} \leq \alpha_{n_i} \leq 1$. Then we know that $\sum_{n=1}^{\infty} \alpha_n < \infty$, and this means $\alpha_n \to 0$. Hence we get $\lim_{i\to\infty} d(x_{n_i}, y_{n_i}) \leq 0$. Therefore $d(x_{n_i}, y_{n_i}) \to 0$. We also get that $y_{n_i} \to x_0$ since $x_n \to x_0$ and $d(x_{n_i}, y_{n_i}) \to 0$.

Next we show $x_0 \in \operatorname{argmin} f$. From the property of resolvents defined by

$$R_f x_n = \operatorname*{argmin}_{y\in X}\{f(y) + \tan d(y, x_n) \sin d(y, x_n)\},$$

for all $y \in X$, we have

$$f(y_{n_i}) + \tan d(y_{n_i}, x_{n_i}) \sin d(y_{n_i}, x_{n_i}) \le f(y) + \tan d(y, x_{n_i}) \sin d(y, x_{n_i}).$$

Let $t \in]0,1[\,, w \in X$, and $y = ty_{n_i} \oplus (1-t)w$. Then,

$$\begin{aligned}
& f(y_{n_i}) + \tan d(y_{n_i}, x_{n_i}) \sin d(y_{n_i}, x_{n_i}) \\
& \le f(ty_{n_i} \oplus (1-t)w) + \tan d(ty_{n_i} \oplus (1-t)w, x_{n_i}) \sin d(ty_{n_i} \oplus (1-t)w, x_{n_i}) \\
& \le tf(y_{n_i}) + (1-t)f(w) + \frac{1}{\cos d(ty_{n_i} \oplus (1-t)w, x_{n_i})} \\
& \quad - \cos d(ty_{n_i} \oplus (1-t)w, x_{n_i}) \\
& \le tf(y_{n_i}) + (1-t)f(w) \\
& \quad + \frac{\sin d(y_{n_i}, w)}{\cos d(y_{n_i}, x_{n_i}) \sin(td(y_{n_i}, w)) + \cos d(w, x_{n_i}) \sin((1-t)d(y_{n_i}, w))} \\
& \quad - \frac{\cos d(y_{n_i}, x_{n_i}) \sin(td(y_{n_i}, w)) + \cos d(w, x_{n_i}) \sin((1-t)d(y_{n_i}, w))}{\sin d(y_{n_i}, w)}
\end{aligned}$$

Putting $A_i = d(y_{n_i}, x_{n_i}), B_i = d(y_{n_i}, w)$, and $C_i = d(w, x_{n_i})$, we get

$$\begin{aligned}
f(y_{n_i}) + \frac{1}{1-t}\Bigg(\tan A_i \sin A_i + \frac{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i}{\sin B_i} \\
- \frac{\sin B_i}{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i}\Bigg) \le f(w).
\end{aligned}$$

Letting $t \to 1$, we have

$$\begin{aligned}
& \lim_{t\to1}\Bigg(\frac{1}{1-t}(\tan A_i \sin A_i + \frac{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i}{\sin B_i} \\
& \quad - \frac{\sin B_i}{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i})\Bigg) \\
& = -\lim_{t\to1} \frac{d}{dt}\Bigg(\tan A_i \sin A_i + \frac{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i}{\sin B_i} \\
& \quad - \frac{\sin B_i}{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i}\Bigg) \\
& = -\lim_{t\to1}\Bigg(\frac{\cos A_i \cos tB_i \cdot B_i + \cos C_i \cos(1-t)B_i \cdot (-B_i)}{\sin B_i} \\
& \quad + \frac{\sin B_i(\cos A_i \cos tB_i \cdot B_i + \cos C_i \cos(1-t)B_i \cdot (-B_i))}{(\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i)^2}\Bigg) \\
& = -\Bigg(\frac{\cos A_i \cos B_i \cdot B_i + \cos C_i \cdot (-B_i)}{\sin B_i} + \frac{\sin B_i(\cos A_i \cos B_i \cdot B_i + \cos C_i \cdot (-B_i))}{(\cos A_i \sin B_i)^2}\Bigg) \\
& = -\Bigg(\frac{\cos A_i \cos B_i \cdot B_i + \cos C_i \cdot (-B_i)}{\sin B_i} + \frac{\cos A_i \cos B_i \cdot B_i + \cos C_i \cdot (-B_i)}{\cos^2 A_i \sin B_i}\Bigg) \\
& = -\frac{B_i}{\sin B_i}\Bigg(\cos A_i \cos B_i - \cos C_i + \frac{\cos A_i \cos B_i - \cos C_i}{\cos^2 A_i}\Bigg) \\
& = \frac{B_i}{\sin B_i}\Bigg(\cos C_i - \cos A_i \cos B_i + \frac{\cos C_i - \cos A_i \cos B_i}{\cos^2 A_i}\Bigg)
\end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&= \frac{B_i}{\sin B_i}(\cos C_i - \cos A_i \cos B_i)\left(1 + \frac{1}{\cos^2 A_i}\right) \\
&= \frac{B_i}{\sin B_i}(\cos C_i - \cos B_i + \cos B_i(1 - \cos A_i))\left(1 + \frac{1}{\cos^2 A_i}\right).
\end{aligned}
$$

It is obvious that $\cos C_i - \cos B_i \to 0$. Letting $i \to \infty$, we have

$$
\begin{aligned}
&\frac{B_i}{\sin B_i}(\cos C_i - \cos B_i + \cos B_i(1 - \cos A_i))\left(1 + \frac{1}{\cos^2 A_i}\right) \\
\to\ &\frac{d(x_0, w)}{\sin d(x_0, w)}(0 + 0)\left(1 + \frac{1}{1}\right) \\
=\ &0.
\end{aligned}
$$

Hence we get

$$
f(x_0) \leq \liminf_{i \to \infty} f(y_{n_i}) \leq f(w).
$$

This inequality implies $x_0 \in \operatorname{argmin} f$. □

The following is the second one of our two main results in this thesis. We modify the coefficient condition from Theorem 3.2. We note that this theorem consists of independently Kimura and Kohsaka [10].

Theorem 3.3. *Let X be an admissible complete* CAT(1) *space. Let $f : X \to]-\infty, \infty]$ be a proper convex lower semicontinuous function such that* $\operatorname{argmin} f \neq \emptyset$. *Let $\{\gamma_n\}$ be a real sequence in $[0, 1]$ converging to 0. For an initial point $x_1 \in X$ such that $f(x_1) < \infty$, generate a sequence $\{x_n\}$ as follows:*

$$
\begin{aligned}
&y_n = R_f x_n, \\
&\alpha_n \in \left[\frac{1}{2} d(x_n, y_n) - \gamma_n, 1\right] \cap [0, 1], \\
&x_{n+1} = (1 - \alpha_n) x_n \oplus \alpha_n y_n.
\end{aligned}
$$

Then, $x_n \xrightarrow{\Delta} x_0 \in \operatorname{argmin} f$.

Proof. Take $\{d(x_{n_i}, y_{n_i})\} \subset \{d(x_n, y_n)\}$. There exists $\{\alpha_{n_{i_j}}\} \subset \{\alpha_{n_i}\}$ such that $\alpha_{n_{i_j}} \to \alpha_0 \in [0, 1]$. If $\alpha_0 = 0$, then

$$
\frac{1}{2} d(x_{n_{i_j}}, y_{n_{i_j}}) \leq \alpha_{n_{i_j}} + \gamma_{n_{i_j}} \to 0.
$$

We get $d(x_{n_{i_j}}, y_{n_{i_j}}) \to 0$.

If $\alpha_0 \in]0, 1]$, for $u \in \operatorname{argmin} f$, using (2.8) in Lemma 2.5, we have

$$
\begin{aligned}
\cos d(u, x_{n+1}) &\geq (1 - \alpha_n) \cos d(u, x_n) + \alpha_n \cos d(u, y_n) \\
&\geq (1 - \alpha_n) \cos d(u, x_n) + \alpha_n \frac{\cos d(u, x_n)}{\cos d(y_n, x_n)} \\
&= \cos d(u, x_n) + \alpha_n \cos d(u, x_n) \left(\frac{1}{\cos d(y_n, x_n)} - 1\right).
\end{aligned}
$$

It implies

$$
\frac{\cos d(u, x_{n+1})}{\cos d(u, x_n)} - 1 \geq \alpha_n \left(\frac{1}{\cos d(y_n, x_n)} - 1\right).
$$

We know that $d(u, x_n) \to [0, \pi/2[$. In fact, since

$$\cos d(u, x_{n+1}) \geq (1 - \alpha_n) \cos d(u, x_n) + \alpha_n \cos d(u, y_n) \geq \cos d(u, x_n),$$

we have

$$d(u, x_{n+1}) \leq d(u, x_n).$$

Thus $d(u, x_n) \to [0, \pi/2[$. Hence we get,

$$0 \leq \alpha_{n_{i_j}} \left(\frac{1}{\cos d(y_{n_{i_j}}, x_{n_{i_j}})} - 1 \right) \leq \frac{\cos d(u, x_{n_{i_j}+1})}{\cos d(u, x_{n_{i_j}})} - 1 \to 0.$$

We get $d(x_{n_{i_j}}, y_{n_{i_j}}) \to 0$. It means that $d(x_n, y_n) \to 0$.

Let $\{x_{n_i}\} \subset \{x_n\}$ with $w = \mathcal{A}(\{x_{n_i}\})$. There exists $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} z$. In fact, since

$$d(u, x_{n+1}) \leq d(u, x_n) \leq d(u, x_1) < \frac{\pi}{2},$$

we have $\limsup_{n\to\infty} d(u, x_n) < \pi/2$. Using Lemma 2.2, we have $z \in F(J_f) = \operatorname{argmin} f$. We put $v = \mathcal{A}(\{x_n\})$. Then,

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, z) &= \lim_{i\to\infty} d(x_{n_i}, z) \\
&= \lim_{j\to\infty} d(x_{n_{i_j}}, z) \\
&\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, w) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, w) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, v) \\
&\leq \limsup_{n\to\infty} d(x_n, v).
\end{aligned}$$

We get $v = w = z$. Hence, $x_n \overset{\Delta}{\rightharpoonup} v = z \in \operatorname{argmin} f$. □

Furthermore, we can generalize this result as follows.

Theorem 3.4. *Let X be an admissible complete* CAT(1) *space. Let $f : X \to]-\infty, \infty]$ be a proper convex lower semicontinuous function such that* $\operatorname{argmin} f \neq \emptyset$. *Let $\{\gamma_n\}$ be a real sequence in $[0, 1]$ converging to 0, and $\{\lambda_n\}$ a real sequence in $]0, \infty[$ converging to $\lambda_0 > 0$. For an initial point $x_1 \in X$ such that $f(x_1) < \infty$, generate a sequence $\{x_n\}$ as follows:*

$$\begin{aligned}
y_n &= R_{\lambda_n f} x_n, \\
\alpha_n &\in \left[\frac{1}{2} d(x_n, y_n) - \gamma_n, 1 \right] \cap [0, 1], \\
x_{n+1} &= (1 - \alpha_n) x_n \oplus \alpha_n y_n.
\end{aligned}$$

Then, $x_n \overset{\Delta}{\rightharpoonup} x_0 \in$ $\operatorname{argmin} f$.

Proof. As in the proof of the previous theorem, we may show that $d(y_n, x_n) \to 0$ and $\sup_{n\in\mathbb{N}} d(x_n, u) < \pi/2$ for every $u \in \operatorname{argmin} f$.

We show $d(R_{\lambda_0 f} x_n, x_n) \to 0$. From (2.6), for $\{x_n\} \subset X$ and $\lambda_n, \lambda_0 > 0$,

$$(\lambda_n C^2_{\lambda_n,x_n}(1+C^2_{\lambda_0,x_n})C_{\lambda_0,x_n} + \lambda_0 C^2_{\lambda_0,x_n}(1+C^2_{\lambda_n,x_n})C_{\lambda_n,x_n})\cos d(R_{\lambda_n f}x_n, R_{\lambda_0 f}x_n)$$
$$\geq \lambda_n C^2_{\lambda_n,x_n}(1+C^2_{\lambda_0,x_n})\cos d(R_{\lambda_n f}x_n, x_n) + \lambda_0 C^2_{\lambda_0,x_n}(1+C^2_{\lambda_n,x_n})\cos d(R_{\lambda_0 f}x_n, x_n) \quad (3.16)$$

Put $A_n = C_{\lambda_n,x_n}$ and $B_n = C_{\lambda_0,x_n}$. Then we have

$$\cos d(R_{\lambda_n f}x_n, R_{\lambda_0 f}x_n) \geq \frac{\lambda_n {A_n}^2(1+{B_n}^2)A_n + \lambda_0 {B_n}^2(1+{A_n}^2)B_n}{\lambda_n {A_n}^2(1+{B_n}^2)B_n + \lambda_0 {B_n}^2(1+{A_n}^2)A_n}.$$

We know that

$$\frac{{A_n}^2(1+{B_n}^2)A_n + {B_n}^2(1+{A_n}^2)B_n}{{A_n}^2(1+{B_n}^2)B_n + {B_n}^2(1+{A_n}^2)A_n} \geq 1.$$

In fact, we have

$$\begin{aligned}\frac{{A_n}^2(1+{B_n}^2)A_n + {B_n}^2(1+{A_n}^2)B_n}{{A_n}^2(1+{B_n}^2)B_n + {B_n}^2(1+{A_n}^2)A_n} &= \frac{(A_n+B_n)({A_n}^2{B_n}^2 + {A_n}^2 - A_nB_n + {B_n}^2)}{A_nB_n(A_n+B_n)(A_nB_n+1)} \\ &= 1 + \frac{{A_n}^2 - 2A_nB_n + {B_n}^2}{A_nB_n(A_nB_n+1)} \\ &= 1 + \frac{(A_n-B_n)^2}{A_nB_n(A_nB_n+1)} \geq 1.\end{aligned}$$

Putting $\mu_n = \frac{\lambda_n}{\lambda_0}$, $L_n = {A_n}^2(1+{B_n}^2)A_n$, $M_n = {B_n}^2(1+{A_n}^2)B_n$, $L'_n = {A_n}^2(1+{B_n}^2)B_n$, and $M'_n = {B_n}^2(1+{A_n}^2)A_n$. Since $R_{\lambda_n f}$ and $R_{\lambda_0 f}$ are quasinonexpansive and $\sup_{n\in\mathbb{N}} d(x_n,u) < \pi/2$ for $u \in \operatorname{argmin} f$, there exists $m > 0$ such that $A_n, B_n \geq m > 0$. So, $L_n \geq m^2(1+m^2)m = m_0$ and $M_n, L'_n, M'_n \geq m_0$. Hence we get

$$\begin{aligned}&\left|\frac{\mu_n L_n + M_n}{\mu_n L'_n + M'_n} - \frac{L_n + M_n}{L'_n + M'_n}\right| \\ &= \frac{|(\mu_n L_n + M_n)(L'_n + M'_n) - (\mu_n L'_n + M'_n)(L_n + M_n)|}{(\mu_n L'_n + M'_n)(L'_n + M'_n)} \\ &\leq \frac{|(\mu_n - 1)|\,|(L_n M'_n - L'_n M_n)|}{(\mu_n m_0 + m_0)(m_0 + m_0)} \to 0.\end{aligned}$$

Since

$$\frac{L_n + M_n}{L'_n + M'_n} = \frac{{A_n}^2(1+{B_n}^2)A_n + {B_n}^2(1+{A_n}^2)B_n}{{A_n}^2(1+{B_n}^2)B_n + {B_n}^2(1+{A_n}^2)A_n} \geq 1,$$

we get

$$\limsup_{n\to\infty} \frac{\mu_n L_n + M_n}{\mu_n L'_n + M'_n} \geq 1.$$

This means $\lim_{n\to\infty} \cos d(R_{\lambda_n f}x_n, R_{\lambda_0 f}x_n) \geq 1$. We get $\lim_{n\to\infty} d(R_{\lambda_n f}x_n, R_{\lambda_0 f}x_n) = 0$. Since

$$d(R_{\lambda_0 f}x_n, x_n) \leq d(R_{\lambda_0 f}x_n, R_{\lambda_n f}x_n) + d(R_{\lambda_n f}x_n, x_n),$$

we obtain $d(R_{\lambda_0 f}x_n, x_n) \to 0$. From this fact, we can show that

$$x_n \xrightarrow{\Delta} z \in \operatorname{argmin} f$$

by using the same method as the proof of Theorem 3.3. □

At the end of this thesis, we discuss the following conjecture, whose iterative sequence is constructed by modifying the convex combination from the previous theorem.

Conjecture 1. Let X be an admissible complete CAT(1) space. Let $f : X \to]-\infty, \infty]$ be a proper convex lower semicontinuous function such that $\operatorname{argmin} f \neq \emptyset$. Let $\{\gamma_n\}$ be a real sequence in $[0, 1]$ converging to 0. For an initial point $x_1 \in X$ such that $f(x_1) < \infty$, generate a sequence $\{x_n\}$ as follows:

$$\begin{aligned} y_n &= R_f x_n, \\ \alpha_n &\in \left[\frac{1}{2} d(x_n, y_n) - \gamma_n, 1\right] \cap [0, 1], \\ x_{n+1} &= (1 - \alpha_n) x_n \overset{1}{\oplus} \alpha_n y_n. \end{aligned}$$

Then, does it hold that $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \operatorname{argmin} f$?

Acknowledgements

Professor Yasunori Kimura taught me mathematics so kind. Thanks to his advices, I get results and complete Master Thesis. I am grateful for him.

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