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Convergence theorems of implicit type sequences and quadrilateral inequalities in geodesic spaces

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Chapter 1

Introduction

In recent years, fixed point theory has been investigated by many mathematicians. In particular, approximating a common fixed point of a nonlinear mapping is one of the main topics in this theory. We have been investigating some types of approximating iteration to find a fixed point in several spaces such as Banach spaces, Hilbert spaces and geodesic spaces. For example, Mann type iteration and Helpern type iteration are explicit types which are very major methods in this research field. Moreover, there are some implicit type methods such as Browder type [7] and Xu-Ori type [8] as shown below.

In this thesis, we consider some implicit type methods in CAT(κ) spaces, whose name comes from E. Cartan, A. D. Alexandrov and A. Toponogov. They have attracted the attention of many mathematicians as they have played a very important role in different aspects of geometry. We know that since any CAT(κ) space is a CAT(κ') space for $\kappa < \kappa'$, every results for CAT(0) spaces can be applied to any CAT(κ) spaces with $\kappa \le 0$. See the book written by Bridson and Haefliger [2] for more detail.

Above all, Browder proved the following convergence theorem which is a very popular and important method to find a fixed point.

Theorem 1.1 (Browder [7])**.** *Let H be a Hilbert space and C a nonempty closed convex subset of H. Let $T\colon C \to C$ be a nonexpansive mapping such that $F(T) \neq \varnothing$ and $P\colon H \to F(T)$ the metric projection. Suppose $\{\alpha_n\} \subset\,]0,1[$ and fix $x_0 \in C$.*

(1) *For any $n \in \mathbb{N}$, define a mapping $T_n\colon C \to C$*

$$T_n x = (1-\alpha_n)Tx + \alpha_n x_0,$$

for all $x \in C$. Then, T_n has a unique fixed point $u_n \in C$;

(2) *If $\alpha_n \to 0$, then $\{u_n\}$ converges strongly to $Px_0 \in F(T)$.*

As another method, Xu and Ori proposed a different kind of iterative scheme converging weakly to a common fixed point of a finite family of nonexpansive mappings in the setting of a Hilbert space.

Theorem 1.2 (Xu and Ori [8])**.** *Let C be a closed convex subset of a Hilbert space and $T_k\colon C \to C$ nonexpansive mappings for $k = 1, 2, \ldots, N$ with $\bigcap_{k=1}^{N} F(T_k) \neq \varnothing$. For a positive real sequence $\{\alpha_n\} \subset\,]0,1[$ converging to 0 and for given $x_1 \in C$, generate $\{x_n\}$ by the following implicit iterative formula:*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{(n \bmod N)+1} x_{n+1}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ is well-defined and convergent weakly to some common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

Recently, Kimura [4] proved the following convergence theorem with multiple anchor points in a complete CAT(0) space. The implicit scheme in the following theorem was inspired by the idea of Theorem 1.1.

Theorem 1.3 (Kimura [4]). *Let X be a Hadamard space, $T\colon X \to X$ be a nonexpansive mapping such that $F(T) \neq \varnothing$ and $u_1, u_2, \dots, u_r \in X$. Suppose $\{\alpha_n\} \subset]0,1[$ is a real sequence such that $\alpha_n \to 0$. For $k = 1, 2, \cdots, r$, let $\{\beta_n^k\} \subset [0,1]$ such that $\sum_{k=1}^r \beta_n^k = 1$ and $\beta_n^k \to \beta^k \in [0,1]$. Define $\{x_n\} \subset X$ by*

$$x_n = \operatorname*{argmin}_{y \in X} \left(\alpha_n \sum_{k=1}^{r} \beta_n^k d(y, u_k)^2 + (1 - \alpha_n) d(y, T x_n)^2 \right)$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges to a unique minimizer of a function g on $F(T)$, where $g\colon X \to \mathbb{R}$ is defined by

$$g(y) = \sum_{k=1}^{r} \beta^k d(y, u_k)^2,$$

for $y \in X$.

On the other hand, in the following Δ-convergence theorem with implicit iterative scheme for a finite family of nonexpansive mappings proved by Kimura [5] took in the idea of Theorem 1.2.

Theorem 1.4 (Kimura [5]). *Let X be a Hadamard space and $T_k\colon X \to X$ a nonexpansive mapping for $k = 1, 2, \dots, N$ such that $\bigcap_{k=1}^N F(T_k) \neq \varnothing$. Suppose $\{\alpha_n^k\} \subset]0,1[$ is a real sequence for $k = 0, 1, \dots, N$ such that $\sum_{k=0}^N \alpha_n^k = 1$. For given $x_1 \in X$, generate a sequence $\{x_n\} \subset X$ satisfying that*

$$x_{n+1} = \operatorname*{argmin}_{y \in X} \left(\alpha_n^0 d(x_n, y)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_{n+1}, y)^2 \right)$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ is well-defined and Δ-convergent to some $x_0 \in \bigcap_{k=1}^N F(T_k)$.

The following theorems proved by Bačák are helpful to show Theorem 1.3 and Theorem 1.4.

Theorem 1.5 (Bačák [1]). *Let X be a* CAT(0) *space and $x_1, x_2, x_3, x_4 \in X$. Then,*

$$d(x_1, x_3)^2 + d(x_2, x_4)^2 \leq d(x_1, x_2)^2 + d(x_3, x_4)^2 + 2d(x_1, x_4)d(x_2, x_3).$$

Theorem 1.6 (Bačák [1]). *Let X be a* CAT(0) *space and $x_1, x_2, x_3, x_4 \in X$. Then,*

$$d(x_1, x_3)^2 + d(x_2, x_4)^2 \leq d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(x_3, x_4)^2 + d(x_4, x_1)^2.$$

These inequalities in Theorem 1.5 and Theorem 1.6 are called quadrilateral inequalities in CAT(0) spaces.

In this thesis, we propose some quadrilateral inequalities in CAT(κ) spaces for $\kappa \in \{-1, 0, 1\}$. Moreover, we consider a relation of these inequalities and the equivalence of CAT(κ) spaces for $\kappa \in \{-1, 0, 1\}$ in Chapter 3. In Chapter 4, we consider some convergence results of Browder type and Xu-Ori type sequences with some mappings in CAT(1) spaces. In Chapter 5, we prove convergence theorems of two type of sequence considered in Chapter 4 with (-1)-convex combination in CAT(-1) spaces.

Chapter 2

Preliminaries

Let X be a metric space with a metric d. For $x, y \in X$ and $l \geq 0$, a mapping $c : [0, l] \to X$ is called a geodesic with endpoints $x, y \in X$ if it satisfies $c(0) = x, c(l) = y$, and $d(c(t), c(s)) = |t - s|$ for every $t, s \in [0, l]$. For $D \in]0, \infty]$, the metric space X is called a D-geodesic if for any $x, y \in X$ with $d(u, v) < D$, there exists a geodesic joining x and y. In this thesis, we assume X has a unique geodesic for every x, y in D-geodesic space X and then X is called a uniquely D-geodesic space. We denote the image of the geodesic with endpoints $x, y \in X$ by $[x, y]$, which is well defined.

Let X be a uniquely D-geodesic space. For $x, y \in X$ with $d(x, y) < D$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y)$. We denote it by $tx \oplus (1 - t)y$.

For $\kappa \in \mathbb{R}$, let M_κ^2 be a two dimensional model space with a curvature κ, that is,

$$M_\kappa^2 = \begin{cases} \frac{1}{\sqrt{\kappa}}\mathbb{S}^2 & (\kappa > 0); \\ \mathbb{E}^2 & (\kappa = 0); \\ \frac{1}{\sqrt{-\kappa}}\mathbb{H}^2 & (\kappa < 0). \end{cases}$$

The diameter D_κ of M_κ is defined by

$$D_\kappa = \begin{cases} \frac{\pi}{\sqrt{\kappa}} & (\kappa > 0); \\ \infty & (\kappa \leq 0). \end{cases}$$

For $x, y, z \in X$, a geodesic triangle $\triangle(x, y, z) \subset X$ is defined as the union of three segments $[x, y], [y, z]$, and $[z, x]$. For a geodesic triangle $\triangle(x, y, z)$ with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, its comparison triangle $\bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$ is defined as the triangle in M_κ^2 whose length of each corresponding edge is identical with that of the original triangle;

$$d(x, y) = d_{M_\kappa^2}(\bar{x}, \bar{y}), \quad d(y, z) = d_{M_\kappa^2}(\bar{y}, \bar{z}), \quad d(z, x) = d_{M_\kappa^2}(\bar{z}, \bar{x}).$$

A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = d_{M_\kappa^2}(\bar{x}, \bar{p})$. If for any $p, q \in \triangle(x, y, z)$ and their comparison points $\bar{p}, \bar{q} \in \bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$, the inequality

$$d(p, q) \leq d_{M_\kappa^2}(\bar{p}, \bar{q})$$

holds for all triangles in X, then we call X a CAT(κ) space. The above inequality is called a CAT(κ) inequality. A CAT(κ) is called an admissible if $d(x, y) < \frac{D_\kappa}{2}$ for any $x, y \in X$.

In CAT(κ) space, the following lemmas play an important role in this thesis.

Lemma 2.1. *Let X be an admissible CAT(κ) space. Then the following inequality holds:*

- *If* $\kappa = -1$,

$$\cosh d(z, tx \oplus (1-t)y) \sinh d(x, y) \\ \leq \cosh d(z, x) \sinh td(x, y) + \cosh d(z, y) \sinh(1-t)d(x, y),$$

- *if* $\kappa = 0$,

$$d(z, tx \oplus (1-t)y)^2 \leq td(z, x)^2 + (1-t)d(z, y)^2 - t(1-t)d(x, y)^2,$$

- *if* $\kappa = 1$,

$$\cos d(z, tx \oplus (1-t)y) \sin d(x, y) \\ \geq \cos d(z, x) \sin td(x, y) + \cos d(z, y) \sin(1-t)d(x, y),$$

for any $x, y, z \in X$ *and* $t \in [0, 1]$.

A mapping $T : X \to X$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$

for every $x, y \in X$. $F(T) = \{z \in X \mid z = Tz\}$ stands for the set of all fixed points of T. Furthermore, $U : X \to X$ is a contraction if there exists $\alpha \in [0, 1[$ such that

$$d(Ux, Uy) \leq \alpha d(x, y)$$

for every $x, y \in X$. The famous Banach contraction principle guarantees the existence and uniqueness of a fixed point of U.

For a bounded sequence $\{x_n\}$ in X, let $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$ for $x \in X$, and define the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ by

$$r(\{x_n\}) = \inf_{x\in X} r(x, \{x_n\}).$$

The asymptotic center $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is a set of point $p \in X$ satisfying

$$r(p, \{x_n\}) = r(\{x_n\}).$$

We say $\{x_n\}$ is Δ-convergent to $x_0 \in X$ if x_0 is the unique asymptotic center of any subsequence of $\{x_n\}$. Then, we denotes it by $x_n \overset{\Delta}{\rightharpoonup} x_0$.

Theorem 2.2 (He, Fang, Lopez and Li [3])**.** *Let* X *be a Hadamard space and* $\{x_n\}$ *a bounded sequence on* X *such that* $x_n \overset{\Delta}{\rightharpoonup} x \in X$. *Then, for any* $u \in X$, *the following holds:*

$$d(u, x) \leq \liminf_{n\to\infty} d(u, x_n).$$

Let X be an admissible complete $\mathrm{CAT}(\kappa)$ space and C a nonempty closed convex subset of X. Then there exists a unique point $p \in C$ such that $d(x, p) = \inf_{y\in C} d(x, y)$ for each $x \in X$. We define the metric projection P_C from X onto C by $P_C x = p$ for any $x \in X$.

Theorem 2.3 (Kimura and Sasaki [6])**.** *Let* X *be an admissible complete* $\mathrm{CAT}(1)$ *space and* C *a nonempty closed convex subset of* X. *For* $u_1, u_2, \dots, u_N \in X$ *and* $\beta_1, \beta_2, \dots, \beta_N \in [0, 1]$ *with* $\sum_{k=1}^N \beta_k = 1$, *define a function* $g : X \to [0, \infty[$ *by*

$$g(x) = \sum_{k=1}^N \beta_k \cos d(u_k, x)$$

for all $x \in X$. *Then,* g *has a unique maximizer on* C.

Definition 2.4 (Kimura and Sasaki [6]). Let X be a uniquely π-geodesic space. Then, for any $u, v \in X$, and $\alpha \in [0,1]$, we define a 1-convex combination of u and v by

$$\alpha u \overset{1}{\oplus} (1-\alpha)v = \operatorname*{argmax}_{x \in X}\{\alpha \cos d(u,x) + (1-\alpha)\cos d(v,x)\}.$$

Lemma 2.5 (Kimura and Sasaki [6]). *Let X be a uniquely geodesic space. For $x, y \in X$ with $x \neq y$ and $\alpha \in]0,1[$, the following equation holds:*

$$\alpha x \overset{1}{\oplus} (1-\alpha)y = \sigma x \oplus (1-\sigma)y,$$

where

$$\sigma = \frac{1}{d(x,y)} \tan^{-1} \frac{\alpha \sin d(x,y)}{1-\alpha+\alpha\cos d(x,y)}.$$

Lemma 2.6 (Kimura and Sasaki [6]). *Let X be a* CAT(1) *space and $x, y, z \in X$. Then, for any $\alpha \in]0,1[$,*

$$\cos d(\alpha x \overset{1}{\oplus} (1-\alpha)y, z) \geq \alpha \cos d(x,z) + (1-\alpha)\cos d(y,z).$$

Theorem 2.7 (Kimura and Sasaki [6]). *Let X be a complete* CAT(−1) *space and C a nonempty closed convex subset of X. For $u_1, u_2, \ldots, u_N \in X$ and $\beta_1, \beta_2, \ldots, \beta_N \in [0,1]$ with $\sum_{k=1}^{N} \beta_k = 1$, define a function $g : X \to [1, \infty[$ by*

$$g(x) = \sum_{k=1}^{N} \beta_k \cosh d(u_k, x)$$

for all $x \in X$. Then, g has a unique maximizer on C.

Definition 2.8 (Kimura and Sasaki [6]). Let X be a uniquely geodesic space. Then, for any $u, v \in X$, and $\alpha \in [0,1]$, we define a (-1)-convex combination of u and v by

$$\alpha u \overset{-1}{\oplus} (1-\alpha)v = \operatorname*{argmin}_{x \in X}\{\alpha \cosh d(u,x) + (1-\alpha)\cosh d(v,x)\}.$$

Lemma 2.9 (Kimura and Sasaki [6]). *Let X be a* CAT(−1) *space and $x, y \in X$. Then, for any $\alpha \in]0,1[$,*

$$\alpha x \overset{-1}{\oplus} (1-\alpha)y = \sigma x \oplus (1-\sigma)y,$$

where,

$$\sigma = \frac{1}{d(x,y)} \tanh^{-1} \frac{\alpha \sinh d(x,y)}{1-\alpha+\alpha\cosh d(x,y)}.$$

Lemma 2.10 (Kimura and Sasaki [6]). *Let X be a* CAT(−1) *space and $x, y, z \in X$. Then, for any $\alpha \in]0,1[$,*

$$\cosh d(\alpha x \overset{-1}{\oplus} (1-\alpha)y, z) \leq \alpha \cosh d(x,z) + (1-\alpha)\cosh d(y,z).$$

Lemma 2.11. *Let X be a* CAT(−1) *space and $x, y, z \in X$. Let $\alpha \in]0, \frac{1}{2}]$. Let*

$$\sigma_1 = \frac{1}{d(x,y)} \tanh^{-1} \frac{\alpha \sinh d(x,y)}{1-\alpha+\alpha\cosh d(x,y)}, \quad \sigma_2 = \frac{1}{d(x,z)} \tanh^{-1} \frac{\alpha \sinh d(x,z)}{1-\alpha+\alpha\cosh d(x,z)}.$$

Then,

$$\sigma_1 \geq \sigma_2,$$

if and only if

$$d(x, y) \geq d(x, z).$$

Lemma 2.12 (Kimura [5])**.** *Let $\{x_n\}$ be a Δ-convergent sequence in a Hadamard space X with its Δ-limit $x \in X$. If $\{d(x_n, u)\}$ converges for some $u \in X$, then $\{x_n\}$ converges to x.*

Chapter 3

Quadrilateral inequalities

In this chapter, we consider quadrilateral inequalities in CAT(κ) and conditions of the equivalence of CAT(κ) spaces for $\kappa \in \{-1, 0, 1\}$.

3.1 Quadrilateral inequalities and the equivalence of CAT(0) spaces

Bačák showed conditions of the equivalence of CAT(0) spaces.

Theorem 3.1 (Bačák [1])**.** *Let X be a geodesic space. The following assertions are equivalent:*

(a) *X is a* CAT(0) *space;*
(b) *for every $x, y, z \in X$,*

$$d(z, m)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2$$

where m is a midpoint of x and y;
(c) *for every geodesic $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$ and every $z \in X$,*

$$d(z, \gamma_t)^2 \leq (1 - t)d(z, x)^2 + td(z, y)^2 - t(1 - t)d(x, y)^2$$

for every $t \in [0, 1]$;
(d) *for every $x, y, u, v \in X$,*

$$d(x, u)^2 + d(y, v)^2 \leq d(x, y)^2 + d(u, v)^2 + 2d(x, v)d(y, u);$$

(e) *for every $x, y, u, v \in X$,*

$$d(x, u)^2 + d(y, v)^2 \leq d(x, y)^2 + d(y, u)^2 + d(u, v)^2 + d(v, x)^2.$$

The condition (d) or (e), which is called a quadrilateral inequality, is very important for proving Theorem 1.3 and Theorem 1.4. We consider quadrilateral inequalities in CAT(1) spaces and a condition of the equivalence of CAT(1) spaces in the next section.

3.2 Quadrilateral inequalities and the equivalence of CAT(1) spaces

The following assertions are written in Bridson and Haefliger [2].

Theorem 3.2. *Let X be a geodesic space. The following assertions are equivalent:*

(1) *X is a* CAT(1) *space;*

(2) *for every* $x, y, z \in X$,

$$\cos(z, m) \cos \frac{d(x,y)}{2} \geq \frac{1}{2} \cos d(z,x) + \frac{1}{2} \cos d(z,y)$$

where m *is a midpoint of* x *and* y*;*

(3) *for every geodesic* $\gamma\colon [0,1] \to X$ *such that* $\gamma(0) = x, \gamma(1) = y$ *and every* $z \in X$,

$$\cos d(z, \gamma_t) \sin d(x,y) \geq \cos d(z,x) \sin(1-t) d(x,y) + \cos d(z,y) \sin t d(x,y)$$

for every $t \in [0,1]$.

Now, we consider quadrilateral inequalities in CAT(1) spaces in this section.

Lemma 3.3. *Let* X *be an admissible* CAT(1) *space and* $x_1, x_2, x_3, x_4 \in X$*. Then, the following inequality holds:*

$$\cos d(x_1,x_2) + \cos d(x_2,x_3) + \cos d(x_3,x_4) + \cos d(x_4,x_1) \leq 4 \cos \frac{d(x_1,x_3)}{2} \cos \frac{d(x_2,x_4)}{2}. \tag{4}$$

Proof. Let $m_1 = \frac{1}{2}x_2 \oplus \frac{1}{2}x_4, m_2 = \frac{1}{2}x_1 \oplus \frac{1}{2}x_3$. Then we have

$$\begin{aligned}
&\cos d(m_1,m_2) \cos \frac{d(x_1,x_3)}{2} \cos \frac{d(x_2,x_4)}{2} \\
&\geq \frac{1}{2} \cos d(m_1,x_1) \cos \frac{d(x_2,x_4)}{2} + \frac{1}{2} \cos d(m_1,x_3) \cos \frac{d(x_2,x_4)}{2} \\
&= \frac{1}{2} \cos d\left(\frac{1}{2}x_2 \oplus \frac{1}{2}x_4, x_1\right) \cos \frac{d(x_2,x_4)}{2} + \frac{1}{2} \cos d\left(\frac{1}{2}x_2 \oplus \frac{1}{2}x_4, x_3\right) \cos \frac{d(x_2,x_4)}{2} \\
&\geq \frac{1}{2}\left(\frac{1}{2} \cos d(x_2,x_1) + \frac{1}{2} \cos d(x_4,x_1)\right) + \frac{1}{2}\left(\frac{1}{2} \cos d(x_2,x_3) + \frac{1}{2} \cos d(x_4,x_3)\right) \\
&= \frac{1}{4}\left(\cos d(x_1,x_2) + \cos d(x_2,x_3) + \cos d(x_3,x_4) + \cos d(x_4,x_1)\right).
\end{aligned}$$

Since $1 \geq \cos d(m_1,m_2)$, we get

$$\cos d(x_1,x_2) + \cos d(x_2,x_3) + \cos d(x_3,x_4) + \cos d(x_4,x_1) \leq 4 \cos \frac{d(x_1,x_3)}{2} \cos \frac{d(x_2,x_4)}{2},$$

the desired result. □

Lemma 3.4. *Let* X *be an admissible* CAT(1) *space and* $x_1, x_2, x_3, x_4 \in X$*. Then, the following inequality holds:*

$$\begin{aligned}
&(1 - \cos d(x_1,x_3)) + (1 - \cos d(x_2,x_4)) \\
&\quad \leq (1 - \cos d(x_1,x_2)) + (1 - \cos d(x_2,x_3)) + (1 - \cos d(x_3,x_4)) + (1 - \cos d(x_4,x_1)).
\end{aligned} \tag{5}$$

Proof. From Lemma 3.3, we get

$$\begin{aligned}
&\frac{1}{4}(\cos d(x_1,x_2)+\cos d(x_2,x_3)+\cos d(x_3,x_4)+\cos d(x_4,x_1))\\
&\le \cos\frac{d(x_1,x_3)}{2}\cos\frac{d(x_2,x_4)}{2}=\sqrt{\cos^2\frac{d(x_1,x_3)}{2}\cos^2\frac{d(x_2,x_4)}{2}}\\
&\le \frac{1}{2}\left(\cos^2\frac{d(x_1,x_3)}{2}+\cos^2\frac{d(x_2,x_4)}{2}\right)\\
&=\frac{1}{2}\left(\frac{1+\cos d(x_1,x_3)}{2}+\frac{1+\cos d(x_2,x_4)}{2}\right)\\
&=\frac{1}{4}(\cos d(x_1,x_3)+\cos d(x_2,x_4)+2).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&(1-\cos d(x_1,x_3))+(1-\cos d(x_2,x_4))\\
&\quad\le(1-\cos d(x_1,x_2))+(1-\cos d(x_2,x_3))+(1-\cos d(x_3,x_4))+(1-\cos d(x_4,x_1)).
\end{aligned}$$

This is the desired result. □

We consider (4) and (5) are quadrilateral inequalities in CAT(1) spaces. In particular, (5) is very important to prove a convergence theorem of implicit type sequences in CAT(1) spaces. See Chapter 4 for more detail. Next, we consider the equivalence of CAT(1) spaces.

Remark 3.5. Let X be a geodesic space with $d(u,v)<\frac{\pi}{2}$ for $u,v\in X$. From Lemma 5.4, Lemma 3.3, and Lemma 3.4, we get the following result:

$$(1)\Longleftrightarrow(2)\Longleftrightarrow(3)\Longrightarrow(4)\Longrightarrow(5).$$

If we could show (5) $\Longrightarrow$ (2), we would get the equivalence of CAT(1) spaces, that is,

$$(1)\Longleftrightarrow(2)\Longleftrightarrow(3)\Longleftrightarrow(4)\Longleftrightarrow(5).$$

This is a difficult problem and it remains unsolved. However, if X is a geodesic space and (4) holds, we get X is a uniquely geodesic space.

Lemma 3.6. *Let X be a geodesic space with $d(u,v)<\frac{\pi}{2}$ for $u,v\in X$ and suppose the following inequality holds for every $x_1,x_2,x_3,x_4\in X$:*

$$\cos d(x_1,x_2)+\cos d(x_2,x_3)+\cos d(x_3,x_4)+\cos d(x_4,x_1)\le 4\cos\frac{d(x_1,x_3)}{2}\cos\frac{d(x_2,x_4)}{2}.$$

Then, X is a uniquely geodesic space.

Proof. Let $x,y\in X$ and $\gamma,\gamma'\colon[0,l]\to X$ such that $\gamma(0)=\gamma'(0)=x$, $\gamma(l)=\gamma'(l)=y$ where $l=d(x,y)$. Let $\gamma_{\frac{1}{2}}=\gamma\left(\frac{l}{2}\right)$, $\gamma'_{\frac{1}{2}}=\gamma\left(\frac{l}{2}\right)$. Then, we have

$$\begin{aligned}
&4\cos\frac{d\left(\gamma_{\frac{1}{2}},\gamma'_{\frac{1}{2}}\right)}{2}\cos\frac{d(x,y)}{2}\\
&\ge\cos d\left(\gamma_{\frac{1}{2}},x\right)+\cos d\left(x,\gamma'_{\frac{1}{2}}\right)+\cos d\left(\gamma'_{\frac{1}{2}},y\right)+\cos d\left(y,\gamma_{\frac{1}{2}}\right)\\
&=4\cos\frac{d(x,y)}{2}.
\end{aligned}$$

Thus we get

$$\cos \frac{d\left(\gamma_{\frac{1}{2}}, \gamma'_{\frac{1}{2}}\right)}{2} \geq 1,$$

and hence we have $\gamma_{\frac{1}{2}} = \gamma'_{\frac{1}{2}}$. Repeating this argument inductively, we get

$$\gamma\left(\frac{k}{2^n} l\right) = \gamma'\left(\frac{k}{2^n} l\right)$$

for $n \in \mathbb{N}$ and $k \in \{0, 1, 2, \ldots, 2^n\}$. Since γ, γ' are continuous and the set

$$\left\{ \frac{k}{2^n} l \;\middle|\; n \in \mathbb{N}, k = 0, 1, 2, \ldots, 2^n \right\}$$

is dense in $[0, l]$, we get $\gamma(t) = \gamma'(t)$ for all $t \in [0, l]$. Therefore X is a uniquely geodesic space. □

Remark 3.7. If X is a geodesic space and (4) holds, we only have X is a uniquely geodesic space by Lemma 3.6. However, if we assume the following inequality

$$\begin{aligned} (6)\quad & \cos d(x_1, x_2) + \cos d(x_2, x_3) + \cos d(x_3, x_4) + \cos d(x_4, x_1) \\ & \leq 4 \cos \frac{d(x_1, x_3)}{2} \cos \frac{d(x_2, x_4)}{2} \cos d(m_1, m_2) \end{aligned}$$

on a geodesic space X, we get the next result.

Theorem 3.8. *Let X be a geodesic space with $d(u, v) < \frac{\pi}{2}$ for $u, v \in X$ and the following inequality holds for every $x_1, x_2, x_3, x_4 \in X$:*

$$\begin{aligned} (6)\quad & \cos d(x_1, x_2) + \cos d(x_2, x_3) + \cos d(x_3, x_4) + \cos d(x_4, x_1) \\ & \leq 4 \cos \frac{d(x_1, x_3)}{2} \cos \frac{d(x_2, x_4)}{2} \cos d(m_1, m_2), \end{aligned}$$

where m_1 is a midpoint of x_2 and x_4, and m_2 is a midpoint of x_1 and x_3. Then X is a CAT(1) *space.*

Proof. It is easy to see that (6) implies (4). From Lemma 3.6, we get X is a uniquely geodesic space and hence we denote $m_1 = \frac{1}{2}x_2 \oplus \frac{1}{2}x_4$ and $m_2 = \frac{1}{2}x_1 \oplus \frac{1}{2}x_3$. Letting $x_2 = x_4$, we have

$$4 \cos \frac{d(x_1, x_3)}{2} \cos d(m_2, x_2) \geq \cos d(x_1, x_2) + \cos d(x_2, x_3) + \cos d(x_3, x_2) + \cos d(x_2, x_1).$$

Thus we obtain

$$\cos d\left(\frac{1}{2}x_1 \oplus \frac{1}{2}x_3, x_2\right) \cos \frac{d(x_1, x_3)}{2} \geq \frac{1}{2}(\cos d(x_1, x_2) + \cos d(x_2, x_3)).$$

Therefore we get X is a CAT(1) space. □

We get the following proposition from the above arguments.

Theorem 3.9. *Let X be a geodesic space with $d(u, v) < \frac{\pi}{2}$ for $u, v \in X$. The following assertions are equivalent:*

(1) *X is a* CAT(1) *space;*

(2) *for every* $x, y, z \in X$,

$$\cos(z, m) \cos \frac{d(x,y)}{2} \geq \frac{1}{2} \cos d(z,x) + \frac{1}{2} \cos d(z,y)$$

where m *is a midpoint of* x *and* y;

(3) *for every geodesic* $\gamma\colon [0,1] \to X$ *such that* $\gamma(0) = x, \gamma(1) = y$ *and every* $z \in X$, *then*

$$\cos d(z, \gamma_t) \sin d(x,y) \geq \cos d(z,x) \sin(1-t)d(x,y) + \cos d(z,y) \sin td(x,y)$$

for every $t \in [0,1]$;

(6) *for every* $x_1, x_2, x_3, x_4 \in X$,

$$\cos d(x_1,x_2) + \cos d(x_2,x_3) + \cos d(x_3,x_4) + \cos d(x_4,x_1) \leq 4 \cos \frac{d(x_1,x_3)}{2} \cos \frac{d(x_2,x_4)}{2} \cos d(m_1,m_2),$$

where m_1 *is a midpoint of* x_2 *and* x_4, *and* m_2 *is a midpoint of* x_1 *and* x_3.

3.3 Quadrilateral inequality and the equivalence of CAT(−1) spaces

The following assertions are written in [2].

Theorem 3.10. *Let* X *be a geodesic space. The following assertions are equivalent:*

(*i*) X *is a* $\mathrm{CAT}(-1)$ *space;*

(*ii*) *for every* $x, y, z \in X$,

$$\cosh(z, m) \cosh \frac{d(x,y)}{2} \leq \frac{1}{2} \cosh d(z,x) + \frac{1}{2} \cosh d(z,y)$$

where m *is a midpoint of* x *and* y;

(*iii*) *for every geodesic* $\gamma\colon [0,1] \to X$ *such that* $\gamma(0) = x, \gamma(1) = y$ *and every* $z \in X$,

$$\cosh d(z, \gamma_t) \sinh d(x,y) \leq \cosh d(z,x) \sinh(1-t)d(x,y) + \cosh d(z,y) \sinh td(x,y)$$

for every $t \in [0,1]$.

Now, we consider quadrilateral inequalities in $\mathrm{CAT}(-1)$ spaces in this section.

Lemma 3.11. *Let* X *be a* $\mathrm{CAT}(-1)$ *space and* $x_1, x_2, x_3, x_4 \in X$. *Then, the following inequality holds:*

$$(iv)\quad \cosh d(x_1,x_2) + \cosh d(x_2,x_3) + \cosh d(x_3,x_4) + \cosh d(x_4,x_1) \geq 4 \cosh \frac{d(x_1,x_3)}{2} \cosh \frac{d(x_2,x_4)}{2}.$$

Proof. Let $m_1 = \frac{1}{2}x_2 \oplus \frac{1}{2}x_4, m_2 = \frac{1}{2}x_1 \oplus \frac{1}{2}x_3$. Then we have

$$\begin{aligned}
&\cosh d(m_1, m_2) \cosh \frac{d(x_1, x_3)}{2} \cosh \frac{d(x_2, x_4)}{2} \\
&\leq \frac{1}{2} \cosh d(m_1, x_1) \cosh \frac{d(x_2, x_4)}{2} + \frac{1}{2} \cosh d(m_1, x_3) \cosh \frac{d(x_2, x_4)}{2} \\
&= \frac{1}{2} \cosh d \left(\frac{1}{2}x_2 \oplus \frac{1}{2}x_4, x_1 \right) \cosh \frac{d(x_2, x_4)}{2} + \frac{1}{2} \cosh d \left(\frac{1}{2}x_2 \oplus \frac{1}{2}x_4, x_3 \right) \cosh \frac{d(x_2, x_4)}{2} \\
&\leq \frac{1}{2} \left(\frac{1}{2} \cosh d(x_2, x_1) + \frac{1}{2} \cosh d(x_4, x_1) \right) + \frac{1}{2} \left(\frac{1}{2} \cosh d(x_2, x_3) + \frac{1}{2} \cosh d(x_4, x_3) \right) \\
&= \frac{1}{4} \left(\cosh d(x_1, x_2) + \cosh d(x_2, x_3) + \cosh d(x_3, x_4) + \cosh d(x_4, x_1) \right).
\end{aligned}$$

Since $1 \leq \cosh d(m_1, m_2)$, we get

$$\begin{aligned}
&\cosh d(x_1, x_2) + \cosh d(x_2, x_3) + \cosh d(x_3, x_4) + \cosh d(x_4, x_1) \\
&\qquad \geq 4 \cosh \frac{d(x_1, x_3)}{2} \cosh \frac{d(x_2, x_4)}{2},
\end{aligned}$$

the desired result. □

Remark 3.12. Let X be a geodesic space. From Lemma 3.10 and Lemma 3.11, we get the following result:

$$(i) \iff (ii) \iff (iii) \implies (iv).$$

We have not proved an inequality corresponding to (5) in CAT(1) spaces at present.

Remark 3.13. Let X be a geodesic space and (iv) holds. We have not proved the following inequality.

$$\begin{aligned}
(v)\ &(\cosh d(x_1, x_3) - 1) + (\cosh d(x_2, x_4) - 1) \\
&\leq (\cosh d(x_1, x_2) - 1) + (\cosh d(x_2, x_3) - 1) + (\cosh d(x_3, x_4) - 1) + (\cosh d(x_4, x_1) - 1).
\end{aligned}$$

Next, we consider the equivalence of CAT(-1) spaces. Similarly to the case of CAT(1) spaces, if we show $(iv) \implies (ii)$, we get the equivalence of CAT(-1) spaces. If a geodesic space X satisfies (iv) holds, then we get X is a uniquely geodesic space.

Lemma 3.14. *Let X be a geodesic space and the following inequality holds for every $x_1, x_2, x_3, x_4 \in X$:*

$$\begin{aligned}
&\cosh d(x_1, x_2) + \cosh d(x_2, x_3) + \cosh d(x_3, x_4) + \cosh d(x_4, x_1) \\
&\qquad \geq 4 \cosh \frac{d(x_1, x_3)}{2} \cosh \frac{d(x_2, x_4)}{2}.
\end{aligned}$$

Then, X is a uniquely geodesic space.

Proof. Let $x, y \in X$ and $\gamma, \gamma' \colon [0, l] \to X$ such that $\gamma(0) = \gamma'(0) = x$, $\gamma(l) = \gamma'(l) = y$ where

$l = d(x, y)$. Let $\gamma_{\frac{1}{2}} = \gamma\left(\frac{l}{2}\right)$, $\gamma'_{\frac{1}{2}} = \gamma\left(\frac{l}{2}\right)$. Then, we have

$$\begin{aligned}
&4\cosh\frac{d\left(\gamma_{\frac{1}{2}}, \gamma'_{\frac{1}{2}}\right)}{2}\cosh\frac{d(x,y)}{2} \\
&\leq \cosh d\left(\gamma_{\frac{1}{2}}, x\right) + \cosh d\left(x, \gamma'_{\frac{1}{2}}\right) + \cosh d\left(\gamma'_{\frac{1}{2}}, y\right) + \cosh d\left(y, \gamma_{\frac{1}{2}}\right) \\
&= 4\cosh\frac{d(x,y)}{2}.
\end{aligned}$$

Thus we get

$$\cosh\frac{d\left(\gamma_{\frac{1}{2}}, \gamma'_{\frac{1}{2}}\right)}{2} \leq 1,$$

and hence we have $\gamma_{\frac{1}{2}} = \gamma'_{\frac{1}{2}}$. Repeating this argument inductively, we get

$$\gamma\left(\frac{k}{2^n}l\right) = \gamma'\left(\frac{k}{2^n}l\right)$$

for $n \in \mathbb{N}$ and $k \in \{0, 1, 2, \ldots, 2^n\}$. Since γ, γ' are continuous and the set

$$\left\{\frac{k}{2^n}l \;\middle|\; n \in \mathbb{N}, k = 0, 1, 2, \ldots, 2^n\right\}$$

is dense in $[0, l]$, we get $\gamma(t) = \gamma'(t)$ for all $t \in [0, l]$. Therefore X is a uniquely geodesic space. $\square$

Remark 3.15. If X is a geodesic space and (iv) holds, we only have X is a uniquely geodesic space by Lemma 3.6. However, if we assume the following inequality

$$\begin{aligned}
(vi)\quad &\cosh d(x_1, x_2) + \cosh d(x_2, x_3) + \cosh d(x_3, x_4) + \cosh d(x_4, x_1) \\
&\geq 4\cosh\frac{d(x_1, x_3)}{2}\cosh\frac{d(x_2, x_4)}{2}\cosh d(m_1, m_2)
\end{aligned}$$

on a geodesic space X, we get the next result.

Lemma 3.16. *Let X be a geodesic space and the following inequality holds for every $x_1, x_2, x_3, x_4 \in X$:*

$$\begin{aligned}
(vi)\quad &\cosh d(x_1, x_2) + \cosh d(x_2, x_3) + \cosh d(x_3, x_4) + \cosh d(x_4, x_1) \\
&\geq 4\cosh\frac{d(x_1, x_3)}{2}\cosh\frac{d(x_2, x_4)}{2}\cosh d(m_1, m_2)
\end{aligned}$$

where, m_1 is a midpoint of x_2 and x_4, and m_2 is a midpoint of x_1 and x_3. Then X is a CAT(-1) *space.*

Proof. It is easy to see (vi) implies (iv). From Lemma 3.14, we get X is a uniquely geodesic space and hence we denote $m_1 = \frac{1}{2}x_2 \oplus \frac{1}{2}x_4$ and $m_2 = \frac{1}{2}x_1 \oplus \frac{1}{2}x_3$. Letting $x_2 = x_4$, we have

$$\begin{aligned}
&4\cosh\frac{d(x_1, x_3)}{2}\cosh d(m_2, x_2) \\
&\qquad\leq \cosh d(x_1, x_2) + \cosh d(x_2, x_3) + \cosh d(x_3, x_2) + \cosh d(x_2, x_1).
\end{aligned}$$

Thus we obtain

$$\cosh d\left(\frac{1}{2}x_1 \oplus \frac{1}{2}x_3, x_2\right)\cosh\frac{d(x_1,x_3)}{2} \leq \frac{1}{2}(\cosh d(x_1,x_2) + \cosh d(x_2,x_3)).$$

Therefore we get X is a CAT(-1) space. □

We get the following proposition from the above arguments.

Theorem 3.17. *Let X be a geodesic space. The following assertions are equivalent:*

(i) *X is a* CAT(-1) *space;*
(ii) *for every $x, y, z \in X$,*

$$\cosh(z,m)\cosh\frac{d(x,y)}{2} \leq \frac{1}{2}\cosh d(z,x) + \frac{1}{2}\cosh d(z,y)$$

where m is a midpoint of x and y;
(iii) *for every geodesic $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$ and every $z \in X$,*

$$\cosh d(z,\gamma_t)\sinh d(x,y) \leq \cosh d(z,x)\sinh(1-t)d(x,y) + \cosh d(z,y)\sinh td(x,y)$$

for every $t \in [0,1]$;
(vi) *for every $x_1, x_2, x_3, x_4 \in X$,*

$$\begin{aligned}&\cosh d(x_1,x_2) + \cosh d(x_2,x_3) + \cosh d(x_3,x_4) + \cosh d(x_4,x_1)\\ &\qquad\geq 4\cosh\frac{d(x_1,x_3)}{2}\cosh\frac{d(x_2,x_4)}{2}\cosh d(m_1,m_2),\end{aligned}$$

where m_1 is a midpoint of x_2 and x_4, and m_2 is a midpoint of x_1 and x_3.

Chapter 4

A convergence theorem in CAT(1) spaces

In this chapter, we consider some convergences results of Browder and Xu-Ori type sequences in CAT(1) spaces.

Lemma 4.1. *Let X be a metric space with $d(u,v) < \frac{\pi}{2}$ for any $u, v \in X$ and $U : X \to X$ a mapping. Suppose $\beta \in [0,1[$. If for any $x, y \in X$,*

$$1 - \cos d(Ux, Uy) \leq \beta(1 - \cos d(x,y)),$$

then U is a contraction.

Proof. Since

$$\frac{1 - \cos d(Ux, Uy)}{2} \leq \frac{\beta(1 - \cos d(x,y))}{2},$$

we have

$$\sin^2 \frac{d(Ux, Uy)}{2} \leq \beta \sin^2 \frac{d(x,y)}{2}.$$

Since $\sin(\cdot)$ is a monotone and concave function on $[0, \frac{\pi}{2}]$, we get

$$\sin \frac{d(Ux, Uy)}{2} \leq \sqrt{\beta} \sin \frac{d(x,y)}{2} \leq \sin \frac{\sqrt{\beta} d(x,y)}{2}.$$

Since $\sin(\cdot)$ is increasing on $[0, \frac{\pi}{2}]$, we get

$$\frac{d(Ux, Uy)}{2} \leq \frac{\sqrt{\beta} d(x,y)}{2}.$$

Therefore, we have

$$d(Ux, Uy) \leq \sqrt{\beta} d(x,y).$$

Since $\beta \in [0,1[$, this implies U is a contraction. □

By Lemma 3.4 and Lemma 4.1, we can show the following theorem.

Theorem 4.2. *Let X be a complete* CAT(1) *space with $d(u,v) < \frac{\pi}{3}$ for any $u, v \in X$ and $T_i : X \to X$ nonexpansive mappings for $i = 1, 2, \ldots, k$. Let $u \in X$ and $\alpha \in]\frac{1}{2}, 1[$. For $i = 1, 2, \ldots, k$, let $\beta_i \in]0,1[$ such that $\sum_{i=1}^{k} \beta^i = 1$. Define $U \colon X \to X$ by*

$$Ux = \operatorname*{argmax}_{z \in X} \left\{ \alpha \cos d(z,u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cos d(z, T_i x) \right\}$$

for every $x \in X$. Then U is well-defined and a contraction.

Proof. U is well-defined as a single-valued mapping on X by Theorem 2.3. Let $x, y \in X$. If $d(Ux, Uy) = 0$, then it is obvious that $d(Ux, Uy) \leq \beta d(x, y)$ for any $\beta \in [0, 1[$. Thus, we consider the case of $d(Ux, Uy) \neq 0$. For $t \in]0, 1[$, we have

$$\begin{aligned}
&\left(\alpha \cos d(Ux, u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Ux, T_i x)\right) \sin d(Ux, Uy) \\
&\geq \alpha \cos d(tUx \oplus (1-t)Uy, u) \sin d(Ux, Uy) \\
&\quad + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(tUx \oplus (1-t)Uy, T_i x) \sin d(Ux, Uy) \\
&\geq \alpha(\cos d(Ux, u) \sin td(Ux, Uy) + \cos d(Uy, u) \sin(1-t)d(Ux, Uy)) \\
&\quad + (1-\alpha)\sum_{i=1}^{k} \beta^i (\cos d(Ux, T_i x) \sin td(Ux, Uy) + \cos d(Uy, T_i x) \sin(1-t)d(Ux, Uy)) \\
&= \left(\alpha \cos d(Ux, u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Ux, T_i x)\right) \sin td(Ux, Uy) \\
&\quad + \left(\alpha \cos d(Uy, u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Uy, T_i x)\right) \sin(1-t)d(Ux, Uy).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\left(\alpha \cos d(Ux, u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Ux, T_i x)\right) \frac{\sin d(Ux, Uy) - \sin td(Ux, Uy)}{\sin(1-t)d(Ux, Uy)} \\
&\geq \alpha \cos d(Uy, u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Uy, T_i x).
\end{aligned}$$

Letting $t \to 1$, we have

$$\begin{aligned}
&\left(\alpha \cos d(Ux, u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Ux, T_i x)\right) \cos d(Ux, Uy) \\
&\geq \alpha \cos d(Uy, u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Uy, T_i x).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\left(\alpha \cos d(Uy, u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Uy, T_i y)\right) \cos d(Ux, Uy) \\
&\geq \alpha \cos d(Ux, u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Ux, T_i y).
\end{aligned}$$

Therefore, we get

$$\left(\alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Ux,T_ix)\right.$$
$$\left. +\ \alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_iy)\right) \cos d(Ux,Uy)$$
$$\geq \alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_ix)$$
$$+ \alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Ux,T_iy),$$

Therefore, we get

$$\left(\alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_ix)\right.$$
$$\left. +\ \alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Ux,T_iy)\right) (1-\cos d(Ux,Uy))$$
$$\leq (1-\alpha)\cos d(Ux,Uy)\sum_{i=1}^{k}\beta^i \left((1-\cos d(Uy,T_ix)) + (1-\cos d(Ux,T_iy)\,)\right.$$
$$\left. -(1-\cos d(Ux,T_ix)) - (1-\cos d(Uy,T_iy))\right).$$

By Lemma 3.4, we obtain

$$\left(\alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_ix)\right.$$
$$\left. +\alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Ux,T_iy)\right) (1-\cos d(Ux,Uy))$$
$$\leq (1-\alpha)\cos d(Ux,Uy)\sum_{i=1}^{k}\beta^i((1-\cos d(Ux,Uy)) + (1-\cos d(T_ix,T_iy)))$$
$$\leq (1-\alpha)((1-\cos d(Ux,Uy)) + (1-\cos d(x,y))).$$

Thus, we get

$$\left(\alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Uy,T_ix)\right.$$
$$\left. +\alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k}\beta^i \cos d(Ux,T_iy) - (1-\alpha)\right) (1-\cos d(Ux,Uy))$$
$$\leq (1-\alpha)(1-\cos d(x,y)).$$

Since $\frac{1}{2} < \cos d(u,v) \le 1$ for any $u,v \in X$, we get

$$\begin{aligned}
&\alpha \cos d(Ux,u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Ux,Tx) \\
&\quad + \alpha \cos d(Uy,u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(Uy,Ty) - (1-\alpha) \\
&> 2\left(\frac{1}{2}\alpha + \frac{1}{2}(1-\alpha)\right) - 1 + \alpha = \alpha,
\end{aligned}$$

that is, we get

$$\alpha(1-\cos d(Ux,Uy)) \le (1-\alpha)(1-\cos d(x,y)).$$

and thus

$$1-\cos d(Ux,Uy) \le \frac{1-\alpha}{\alpha}(1-\cos d(x,y)).$$

Since $\frac{1}{2} < \alpha < 1$, we have $0 < \frac{1-\alpha}{\alpha} < 1$. Hence U is a contraction by Lemma 4.1. □

In Theorem 4.2, U has a unique fixed point $x \in X$. That is, it satisfies that

$$x = Ux = \operatorname*{argmax}_{z\in X}\left\{\alpha \cos d(z,u) + (1-\alpha)\sum_{i=1}^{k} \beta^i \cos d(z,T_i x)\right\}.$$

Thus we can define a sequence $\{x_n\}$ by

$$x_n = \operatorname*{argmax}_{z\in X}\left\{\alpha_n \cos d(z,u) + (1-\alpha_n)\sum_{i=1}^{k} \beta_n^i \cos d(z,T_i x_n)\right\}$$

where $\alpha_n \in]\frac{1}{2},1[$ and $\beta_n^i \in]0,1[$ for $i=1,2,\dots,k$ in a CAT(1) space.

Remark 4.3. By Theorem 4.2, since $\{\alpha_n\}$ cannot tend to 0, we cannot show the Browder's type convergence theorem in CAT(1) spaces for this mapping.

On the other hand, we can show the following convergence theorem.

4.1 A convergence theorem of Xu and Ori type

In this section, we prove a convergence theorem of Xu and Ori type in CAT(1) spaces.

Theorem 4.4. *Let X be a complete* CAT(1) *space with $d(u,v) < \frac{\pi}{3}$ for any $u,v \in X$. For $i=1,2,\dots,k$, let $T_i\colon X \to X$ be nonexpansive mappings with $\bigcap_{i=1}^k F(T_i) \ne \varnothing$. Suppose $\{\alpha_n\} \subset \mathbb{R}$ and $a \in \mathbb{R}$ such that $\frac{1}{2} < \alpha_n \le a < 1$ for $n \in \mathbb{N}$. Let $\{\beta_n^i\} \subset]0,1[$ for $i=1,2,\dots,k$ such that $\sum_{i=1}^k \beta_n^i = 1$. Let $x_1 \in X$ and generate $\{x_n\}$ as follows:*
For $n \in \mathbb{N}$ and given $x_n \in X$, let x_{n+1} be a unique point in X satisfying that

$$x_{n+1} = \operatorname*{argmax}_{z\in X}\left\{\alpha_n \cos d(z,x_n) + (1-\alpha_n)\sum_{i=1}^{k} \beta_n^i \cos d(z,T_i x_{n+1})\right\}.$$

Then, $\{x_n\}$ is well-defined and Δ-convergent to some $x_0 \in \bigcap_{i=1}^k F(T_i)$.

Proof. Define a mapping $V_n \colon X \to X$ by

$$V_n x = \operatorname*{argmax}_{z \in X} \left\{ \alpha_n \cos d(z, x_n) + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(z, T_i x) \right\}$$

for every $x \in X$. Then, V_n is well-defined as a single-valued mapping on X by Theorem 2.3. We can show that V_n is a contraction in the same way of Theorem 4.2 and thus it has a unique fixed point $x_{n+1} \in X$. That is, it satisfies that

$$x_{n+1} = V_n x_{n+1} = \operatorname*{argmax}_{z \in X} \left\{ \alpha_n \cos d(z, x_n) + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(z, T_i x_{n+1}) \right\}.$$

This implies that x_{n+1} satisfying this equation exists uniquely, and hence $\{x_n\}$ is well-defined. Next, we show $\{x_n\}$ is Δ-convergent to some $x_0 \in \bigcap_{i=1}^k F(T_i)$. Let $p \in \bigcap_{i=1}^k F(T_i)$ and $t \in]0,1[$. Then, we have

$$\begin{aligned}
&\left(\alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(T_i x_{n+1}, x_{n+1}) \right) \sin d(x_{n+1}, p) \\
&= \left(\alpha_n \cos d(x_n, V_n x_{n+1}) + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(T_i x_{n+1}, V_n x_{n+1}) \right) \sin d(x_{n+1}, p) \\
&\geq \alpha_n \cos d(x_n, t x_{n+1} \oplus (1-t)p) \sin d(x_{n+1}, p) \\
&\quad + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(T_i x_{n+1}, t x_{n+1} \oplus (1-t)p) \sin d(x_{n+1}, p) \\
&\geq \alpha_n (\cos d(x_n, x_{n+1}) \sin t d(x_{n+1}, p) + \cos d(x_n, p) \sin(1-t) d(x_{n+1}, p)) \\
&\quad + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i (\cos d(T_i x_{n+1}, x_{n+1}) \sin t d(x_{n+1}, p) \\
&\qquad + \cos d(T_i x_{n+1}, p) \sin d(1-t)(x_{n+1}, p)) \\
&= \left(\alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(T_i x_{n+1}, x_{n+1}) \right) \sin t d(x_{n+1}, p) \\
&\quad + \left(\alpha_n \cos d(x_n, p) + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(T_i x_{n+1}, p) \right) \sin(1-t) d(x_{n+1}, p).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\left(\alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(T_i x_{n+1}, x_{n+1}) \right) \frac{\sin d(x_{n+1}, p) - \sin t d(x_{n+1}, p)}{\sin(1-t) d(x_{n+1}, p)} \\
&\geq \alpha_n \cos d(x_n, p) + (1-\alpha_n) \cos d(T x_{n+1}, p).
\end{aligned}$$

Letting $t \to 1$, we have

$$\begin{aligned}
\cos d(x_{n+1}, p) &\geq \left(\alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(T_i x_{n+1}, x_{n+1}) \right) \cos d(x_{n+1}, p) \\
&\geq \alpha_n \cos d(x_n, p) + (1-\alpha_n) \cos d(x_{n+1}, p).
\end{aligned}$$

Therefore, since $\{\alpha_n\} \subset]\frac{1}{2}, a]$, we obtain $\cos d(x_{n+1}, p) \geq \cos d(x_n, p)$. This implies $d(x_n, p) \geq d(x_{n+1}, p)$. Since the real sequence $\{d(x_n, p)\}$ is nonincreasing, there exists

$$\lim_{n\to\infty} d(x_n, p) = c_p \in \left[0, \frac{\pi}{2}\right[.$$

Then, we have

$$\begin{aligned}
1 &\geq \alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(T_i x_{n+1}, x_{n+1}) \\
&\geq (\alpha_n \cos d(x_n, p) + (1-\alpha_n) \cos d(x_{n+1}, p)) \frac{1}{\cos d(x_{n+1}, p)} \\
&\geq \frac{\alpha_n(\cos d(x_n, p) - \cos d(x_{n+1}, p))}{\cos d(x_{n+1}, p)} + 1 \\
&\to 1
\end{aligned}$$

as $n \to \infty$. This implies

$$\lim_{n\to\infty} \left(\alpha_n \cos d(x_n, x_{n+1}) + (1-\alpha_n) \sum_{i=1}^{k} \beta_n^i \cos d(T_i x_{n+1}, x_{n+1}) \right) = 1.$$

Then we obtain

$$\lim_{n\to\infty} \cos d(x_n, x_{n+1}) = \lim_{n\to\infty} \cos d(T_i x_{n+1}, x_{n+1}) = 1$$

for $i = 1, 2, \ldots, k$. In fact, we assume $\{\cos d(x_n, x_{n+1})\}$ does not converge to 1. Then, there exist $\varepsilon > 0$ and a subsequence $\{\cos d(x_{n_j}, x_{n_j+1})\}$ of $\{\cos d(x_n, x_{n+1})\}$ such that $\cos d(x_{n_j}, x_{n_j+1}) \leq 1 - \varepsilon$ for $j \in \mathbb{N}$. Furthermore, since $\{\alpha_{n_j}\} \subset]\frac{1}{2}, a]$, we may assume $\alpha_{n_j} \to \alpha_0 \in [\frac{1}{2}, a]$ without loss of generality. Then we have

$$\begin{aligned}
1 &\leq \limsup_{j\to\infty} \left(\alpha_{n_j} \cos d(x_{n_j}, x_{n_j+1}) + (1-\alpha_{n_j}) \sum_{j=1}^{k} \beta_{n_j}^i \cos d(T_i x_{n_j}, x_{n_j+1}) \right) \\
&\leq \alpha_0 \limsup_{j\to\infty} \cos d(x_{n_j}, x_{n_j+1}) + (1-\alpha_0) \sum_{i=1}^{k} \beta_{n_j}^i \limsup_{j\to\infty} \cos d(T_i x_{n_j+1}, x_{n_j+1}) \\
&\leq \alpha_0(1-\varepsilon) + (1-\alpha_0) = 1 - \alpha_0 \varepsilon < 1.
\end{aligned}$$

This is a contradiction. Thus we have $\lim_{n\to\infty} \cos d(x_n, x_{n+1}) = 1$, and similarly we get $\lim_{n\to\infty} \cos d(T_i x_{n+1}, x_{n+1}) = 1$ for $i = 1, 2, \ldots, k$. Hence we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(T_i x_{n+1}, x_{n+1}) = 0$$

for $i = 1, 2, \ldots, k$. Let $x_0 \in X$ be a unique asymptotic center of a sequence $\{x_n\}$ and let $u \in X$ be an asymptotic center of any subsequence $\{x_{n_j}\}$ of $\{x_n\}$. We will show $u = x_0$. From the definition of asymptotic center, we have

$$\begin{aligned} r(\{x_{n_j}\}) &= \limsup_{j\to\infty} d(x_{n_j}, u) \\ &\le \limsup_{j\to\infty} d(x_{n_j}, T_i u) \\ &\le \limsup_{j\to\infty} (d(x_{n_j}, T_i x_{n_j}) + d(T_i x_{n_j}, T_i u)) \\ &= \limsup_{j\to\infty} d(T_i x_{n_j}, T_i u) \\ &\le \limsup_{j\to\infty} d(x_{n_j}, u) = r(\{x_{n_j}\}). \end{aligned}$$

for $i = 1, 2, \dots, k$. This implies $T_i u \in \mathrm{AC}(\{x_{n_j}\})$ for $i = 1, 2, \dots, k$. From the uniqueness of an asymptotic center, we get $u = T_i u$ for $i = 1, 2, \dots, k$, that is, $u \in \bigcap_{i=1}^k F(T_i)$. It follows that $\{d(x_n, u)\}$ is convergent to c_u. Therefore, we obtain

$$\begin{aligned} r(\{x_n\}) &= \limsup_{n\to\infty} d(x_n, x_0) \\ &\le \limsup_{n\to\infty} d(x_n, u) = c_u = \lim_{j\to\infty} d(x_{n_j}, u) \\ &\le \limsup_{j\to\infty} d(x_{n_j}, x_0) \\ &\le \limsup_{n\to\infty} d(x_n, x_0) = r(\{x_n\}). \end{aligned}$$

Thus $u \in \mathrm{AC}(\{x_n\})$. From the uniqueness of an asymptotic center, we get $u = x_0$. Hence, $\{x_n\}$ is Δ-convergent to $x_0 \in \bigcap_{i=1}^k F(T_i)$. This is the desired result. □

We get the next proposition from Theorem 4.4.

Corollary 4.5. *Let X be a complete $\mathrm{CAT}(1)$ space with $d(u,v) < \frac{\pi}{3}$ for any $u, v \in X$ and $T\colon X \to X$ nonexpansive with $F(T) \neq \varnothing$. Suppose $\{\alpha_n\} \subset \mathbb{R}$ and $a \in \mathbb{R}$ such that $\frac{1}{2} < \alpha_n \le a < 1$ for $n \in \mathbb{N}$. Let $x_1 \in X$ and generate $\{x_n\}$ as follows:*
For $n \in \mathbb{N}$ and given $x_n \in X$, let x_{n+1} be a unique point in X satisfying that

$$x_{n+1} = \operatorname*{argmax}_{z\in X} \{\alpha_n \cos d(z, x_n) + (1-\alpha_n)\cos d(z, T x_{n+1})\} = \alpha_n x_n \overset{1}{\oplus} (1-\alpha_n) T x_{n+1}$$

Then, $\{x_n\}$ is well-defined and Δ-convergent to some $x_0 \in F(T)$.

Chapter 5

Convergence theorems in CAT(-1) spaces

In this chapter, we consider some convergence results of Browder and Xu-Ori type sequence with (-1)-convex combination in CAT(-1) spaces. To prove our main result, we first show the following lemmas.

Lemma 5.1. *Let $\alpha \in]0, \frac{1}{2}]$. Define $f\colon \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = x \tanh^{-1} \frac{(1-\alpha)\sinh x}{\alpha + (1-\alpha)\cosh x}$$

for $x \in \mathbb{R}$. Then, f is increasing.

Lemma 5.2. *Let $d > 0$. Define $f\colon]0, \infty[\to \mathbb{R}$, by*

$$f(t) = \frac{\sinh td}{t}$$

for $t \in]0, \infty[$. Then, f is strictly increasing.

Proof. We have

$$f'(t) = \frac{td \cosh td - \sinh td}{t^2} = \frac{1}{t^2} \int_0^{td} x \sinh x \, dx > 0.$$

Thus we obtain the result. $\square$

Theorem 5.3. *Let $d > 0$. For any $\alpha \in]0, \frac{1}{2}[$, let*

$$\sigma = \frac{1}{d} \tanh^{-1} \frac{\alpha \sinh d}{1 - \alpha + \alpha \cosh d}.$$

Then $\alpha < \sigma$.

Proof. Define $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = \alpha \cosh(1-t)d + (1-\alpha)\cosh td$$

for $t \in [0,1]$. Then, $g''(t) = d^2 g(t) > 0$ and hence g is strictly convex. Thus, since $g'(\sigma) = 0$, we have $\sigma = \operatorname{argmin}_{t \in [0,1]} g(t)$. By Lemma 5.2, we have

$$\begin{aligned} g'(\alpha) &= -\alpha d \sinh(1-\alpha)d + (1-\alpha)\sinh \alpha d \\ &= d\alpha(1-\alpha)\left(-\frac{\sinh(1-\alpha)d}{1-\alpha} + \frac{\sinh \alpha d}{\alpha}\right) < 0. \end{aligned}$$

Therefore, we get $\alpha < \sigma$ and it is the desired result. $\square$

Let X be a CAT(0) space. Let $x, y, z \in X$ space and $t \in]0,1[$, we have the following inequality:

$$d(tx \oplus (1-t)y, z)^2 \leq td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2.$$

Since any CAT(0) space is a CAT(-1) space, the above inequality also holds for any CAT(-1) spaces.

Theorem 5.4. *Let X be a* CAT(-1) *space, $T\colon X \to X$ a nonexpansive mapping. Let $u \in X$, $\alpha \in]0, \frac{1}{2}]$. Define $U\colon X \to X$ by*

$$Ux = \alpha u \overset{-1}{\oplus} (1-\alpha)Tx$$

for $x \in X$. Then, U is a contraction.

Proof. Let $x, y \in X$. If $d(Ux, Uy) = 0$, then U obviously satisfies that there exists $\beta \in [0,1[$ such that

$$d(Ux, Uy) \leq \beta d(x,y).$$

Thus, we consider the case of $d(Ux, Uy) \neq 0$. Then, we have

$$\begin{aligned}
&d(Ux, Uy)^2 \\
&= d(\alpha u \overset{-1}{\oplus} (1-\alpha)Tx, \alpha u \overset{-1}{\oplus} (1-\alpha)Ty)^2 \\
&= d(\sigma_1 u \oplus (1-\sigma_1)Tx, \sigma_2 u \oplus (1-\sigma_2)Ty)^2 \\
&\leq \sigma_1 d(u, \sigma_2 u \oplus (1-\sigma_2)Ty)^2 + (1-\sigma_1)d(Tx, \sigma_2 u \oplus (1-\sigma_2)Ty)^2 \\
&\quad - \sigma_1(1-\sigma_1)d(u,Tx)^2 \\
&\leq \sigma_1(1-\sigma_2)^2 d(u,Ty)^2 + (1-\sigma_1)(\sigma_2 d(u,Tx)^2 + (1-\sigma_2)d(Tx,Ty)^2 \\
&\quad - \sigma_2(1-\sigma_2)d(u,Ty)^2) - \sigma_1(1-\sigma_1)d(u,Tx)^2 \\
&= (\sigma_1 - \sigma_2)((1-\sigma_2)d(u,Ty)^2 - (1-\sigma_1)d(u,Tx)^2) + (1-\sigma_1)(1-\sigma_2)d(Tx,Ty)^2,
\end{aligned}$$

where

$$\sigma_1 = \frac{1}{d(u,Tx)} \tanh^{-1} \frac{\alpha \sinh d(u,Tx)}{1 - \alpha + \alpha \cosh d(u,Tx)},$$

$$\sigma_2 = \frac{1}{d(u,Ty)} \tanh^{-1} \frac{\alpha \sinh d(u,Ty)}{1 - \alpha + \alpha \cosh d(u,Ty)}.$$

We consider the following cases:

(i) $\sigma_1 \geq \sigma_2$,
(ii) $\sigma_2 \geq \sigma_1$.

First, we consider the case (i). Since

$$(1-\sigma_2)d(u,Ty)^2 = d(u,Ty) \tanh^{-1} \frac{(1-\alpha)\sinh d(u,Ty)}{\alpha + (1-\alpha)\cosh d(u,Ty)},$$

using Lemma 2.5 and Lemma 5.1, we get

$$(1-\sigma_2)d(u,Ty)^2 \leq (1-\sigma_1)d(u,Tx)^2.$$

Similarly, we consider the case (ii) and we get

$$(1-\sigma_1)d(u,Tx)^2 \le (1-\sigma_2)d(u,Ty)^2.$$

Therefore, in any case of (i) and (ii), we have

$$d(Ux,Uy)^2 \le (1-\sigma_1)(1-\sigma_2)d(Tx,Ty)^2.$$

By Theorem 5.3, we have

$$(1-\sigma_1)(1-\sigma_2) \le (1-\alpha)^2,$$

and thus we obtain

$$d(Ux,Uy)^2 \le (1-\alpha)^2 d(Tx,Ty)^2 \le (1-\alpha)^2 d(x,y)^2.$$

Therefore, we get

$$d(Ux,Uy) \le (1-\alpha)d(x,y),$$

and hence we have U is a contraction. □

Remark 5.5. From Remark 3.13, since we do not have a quadrilateral inequality (v) in CAT(-1) spaces corresponding to inequality (5) in CAT(1) spaces in Chapter 3, we cannot prove the following claim.

Claim 5.6. Let X be a CAT(-1) space and $T_i : X \to X$ nonexpansive mappings for $i = 1, 2, \dots, k$. Let $u \in X$ and $\alpha \in]0, \frac{1}{2}]$. let $\beta_i \in]0,1[$ such that $\sum_{i=1}^{k} \beta^i = 1$. Define $U : X \to X$ by

$$Ux = \operatorname*{argmin}_{z\in X} \left\{ \alpha \cosh d(z,u) + (1-\alpha) \sum_{i=1}^{k} \beta^i \cosh d(z, T_i x) \right\}.$$

for $x \in X$. Then, U is a contraction.

5.1 A convergence theorem of Browder type

In this section, we consider a convergence theorem of Browder type sequence with (-1)-convex combination in CAT(-1) spaces.

Theorem 5.7. *Let X be a complete* CAT(-1) *space, $T\colon X \to X$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $u \in X$ and $\{\alpha_n\} \subset]0, \frac{1}{2}]$ such that $\alpha_n \to 0$ as $n \to \infty$. Define $\{x_n\} \subset X$ by*

$$x_n = \alpha_n u \overset{-1}{\oplus} (1-\alpha_n)Tx_n.$$

Then, $\{x_n\}$ is well-defined and convergent to $P_{F(T)}u$.

Proof. We know that Theorem 5.4 implies the well-definedness of x_n for every $n \in \mathbb{N}$. Let $p = P_{F(T)}u$. Then

$$d(p,u) = \inf_{y\in F(T)} d(y,u).$$

By Lemma 2.10, we have

$$\begin{aligned}\cosh d(x_n,p) &= \cosh d(\alpha_n u \overset{-1}{\oplus} (1-\alpha_n)Tx_n, p)\\ &\le \alpha_n \cosh d(u,p) + (1-\alpha_n)\cosh d(Tx_n,p)\\ &\le \alpha_n \cosh d(u,p) + (1-\alpha_n)\cosh d(x_n,p).\end{aligned}$$

Thus, we have

$$\cosh d(x_n, p) \le \cosh d(u, p),$$

and hence we get

$$d(x_n, p) \le d(u, p)$$

for any $n \in \mathbb{N}$. It implies that $\{x_n\}$ and $\{Tx_n\}$ are bounded. Since

$$d(x_n, Tx_n) \le d(x_n, p) + d(p, Tx_n),$$

we have $\{d(x_n, Tx_n)\}$ is bounded. From the definition of x_n, we have

$$\begin{aligned}
&(\alpha_n \cosh d(x_n, u) + (1 - \alpha_n) \cosh d(x_n, Tx_n)) \sinh d(x_n, p) \\
&\le (\alpha_n \cosh d(tx_n \oplus (1-t)p, u) + (1 - \alpha_n) \cosh d(tx_n \oplus (1-t)p, Tx_n)) \sinh d(x_n, p) \\
&\le \alpha_n (\cosh d(x_n, u) \sinh td(x_n, p) + \cosh d(p, u) \sinh(1-t)d(x_n, p)) \\
&\quad + (1 - \alpha_n)(\cosh d(x_n, Tx_n) \sinh td(x_n, p) + \cosh d(p, Tx_n) \sinh(1-t)d(x_n, p)) \\
&= (\alpha_n \cosh d(x_n, u) + (1 - \alpha_n) \cosh d(x_n, Tx_n)) \sinh td(x_n, p) \\
&\quad + (\alpha_n \cosh d(p, u) + (1 - \alpha_n) \cosh d(p, Tx_n)) \sinh(1-t)d(x_n, p)
\end{aligned}$$

for $t \in]0, 1[$. Thus we have

$$\begin{aligned}
&(\alpha_n \cosh d(x_n, u) + (1 - \alpha_n) \cosh d(x_n, Tx_n)) \frac{\sinh d(x_n, p) - \sinh td(x_n, p)}{\sinh(1-t)d(x_n, p)} \\
&\le \alpha_n \cosh d(p, u) + (1 - \alpha_n) \cosh d(p, Tx_n).
\end{aligned}$$

Letting $t \to 1$, we get

$$\begin{aligned}
&(\alpha_n \cosh d(x_n, u) + (1 - \alpha_n) \cosh d(x_n, Tx_n)) \cosh d(x_n, p) \\
&\le \alpha_n \cosh d(p, u) + (1 - \alpha_n) \cosh d(Tx_n, p) \\
&\le \alpha_n \cosh d(p, u) + (1 - \alpha_n) \cosh d(x_n, p).
\end{aligned}$$

Thus, we have

$$\alpha_n \cosh d(x_n, u) + (1 - \alpha_n) \cosh d(x_n, Tx_n) \le \alpha_n \frac{\cosh d(p, u)}{\cosh d(x_n, p)} + (1 - \alpha_n). \tag{5.1}$$

Since $\alpha_n \to 0$ and $\{x_n\}$ is bounded, we get $\limsup_{n\to\infty} \cosh d(x_n, Tx_n) \le 1$, and hence we obtain

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

From (5.1), we obtain

$$\begin{aligned}
\alpha_n \cosh d(x_n, u) &\le \alpha_n \cosh d(x_n, u) + (1 - \alpha_n)(\cosh d(x_n, Tx_n) - 1) \\
&\le \alpha_n \frac{\cosh d(p, u)}{\cosh d(x_n, p)} \le \alpha_n \cosh d(p, u).
\end{aligned}$$

Therefore, we get $\cosh d(x_n, u) \le \cosh d(p, u)$ and thus

$$d(x_n, u) \le d(p, u). \tag{5.2}$$

To show that $\{x_n\}$ is Δ-convergent to p, we take $\{x_{n_i}\} \subset \{x_n\}$ arbitrarily and let v be an element of the asymptotic center of $\{x_{n_i}\}$. Taking subsequence repeatedly, we can find $\{x'_j\} \subset \{x_{n_i}\}$ such that

$$\limsup_{j\to\infty} d(x'_j, p) = \limsup_{i\to\infty} d(x_{n_i}, p)$$

and there exists $q \in X$ such that $x'_j \xrightarrow{\Delta} q$. We show $q = p$. Since T is nonexpansive, we have

$$\begin{aligned}\limsup_{j\to\infty} d(x'_j, Tq) &\leq \limsup_{j\to\infty}(d(x'_j, Tx'_j) + d(Tx'_j, Tq)) \\ &\leq \limsup_{j\to\infty} d(x'_j, Tx'_j) + \limsup_{j\to\infty} d(Tx'_j, Tq) \\ &\leq \limsup_{j\to\infty} d(x'_j, q).\end{aligned}$$

From the uniqueness of the element of $\mathrm{AC}(\{x'_j\})$, we get $q \in F(T)$. Since $x'_j \xrightarrow{\Delta} q$ and $d(x_n, u) \leq d(p, u)$, we have

$$d(q, u) \leq \liminf_{j\to\infty} d(x'_j, u) \leq d(p, u).$$

Since p is the unique nearest point of u on $F(T)$, the above inequality implies that $q = p$ and $p \in \mathrm{AC}(\{x'_j\})$. From the assumptions of $\{x'_j\}$, we have

$$\limsup_{i\to\infty} d(x_{n_i}, p) = \lim_{j\to\infty} d(x'_j, p) \leq \limsup_{j\to\infty} d(x'_j, v) \leq \limsup_{i\to\infty} d(x_{n_i}, v).$$

Hence $p \in \mathrm{AC}(\{x_{n_i}\})$ and it implies that $v = p$. Since v is an asymptotic center of $\{x_{n_i}\} \subset \{x_n\}$, which is arbitrarily chosen, and it coincides with p, $\{x_n\}$ is Δ-convergent to p. We finally show that the convergence of $\{x_n\}$ to p. Since $\{x_n\}$ is Δ-convergent to p and from (5.2), we have

$$d(p, u) \leq \liminf_{n\to\infty} d(x_n, u) \leq \limsup_{n\to\infty} d(x_n, u) \leq d(p, u),$$

and hence we get

$$\lim_{n\to\infty} d(x_n, u) = d(p, u).$$

Therefore, we get $x_n \to p$ by Lemma 2.12, which is the desired result. $\square$

5.2 A convergence theorem of Xu and Ori type

In this section, we consider a convergence theorem of Xu-Ori type sequence with (-1)-convex combination in $\mathrm{CAT}(-1)$ spaces.

Theorem 5.8. *Let X be a complete $\mathrm{CAT}(-1)$ space and $T\colon X \to X$ nonexpansive with $F(T) \neq \emptyset$. Suppose $\{\alpha_n\} \subset \mathbb{R}$ and $a \in \mathbb{R}$ such that $0 < a \leq \alpha_n \leq \frac{1}{2}$ for $n \in \mathbb{N}$. Let $x_1 \in X$ and generate $\{x_n\}$ as follows:*
For $n \in \mathbb{N}$ and given $x_n \in X$, let x_{n+1} be a unique point in X satisfying that

$$x_{n+1} = \alpha_n x_n \overset{-1}{\oplus} (1 - \alpha_n) T x_{n+1}.$$

Then, $\{x_n\}$ is well-defined and Δ-convergent to some $x_0 \in F(T)$.

Proof. Define a mapping $V_n \colon X \to X$ by

$$V_n x = \operatorname*{argmin}_{y \in X} \left(\alpha_n \cosh d(y, x_n) + (1-\alpha_n) \cosh d(y, Tx)\right)$$

for $x \in X$. In the same way of Theorem 5.4, we get V_n is a contraction and thus it has a unique fixed point $x_{n+1} \in X$. That is, it satisfies that

$$x_{n+1} = V_n x_{n+1} = \operatorname*{argmin}_{y \in X} \left(\alpha_n \cosh d(y, x_n) + (1-\alpha_n) \cosh d(y, Tx_{n+1})\right),$$

and hence $\{x_n\}$ is well-defined. Next, we show $\{x_n\}$ is Δ-convergent to some element in $F(T)$. Let $p \in F(T)$ and $t \in \,]0,1[$. Then, we have

$$\begin{aligned}
&(\alpha_n \cosh d(x_n, x_{n+1}) + (1-\alpha_n) \cosh d(Tx_{n+1}, x_{n+1})) \sinh d(x_{n+1}, p) \\
&= (\alpha_n \cosh d(x_n, V_n x_{n+1}) + (1-\alpha_n) \cosh d(Tx_{n+1}, V_n x_{n+1})) \sinh d(x_{n+1}, p) \\
&\le \alpha_n \cosh d(x_n, tx_{n+1} \oplus (1-t)p) \sinh d(x_{n+1}, p) \\
&\quad + (1-\alpha_n) \cosh d(Tx_{n+1}, tx_{n+1} \oplus (1-t)p) \sinh d(x_{n+1}, p) \\
&\le \alpha_n (\cosh d(x_n, x_{n+1}) \sinh td(x_{n+1}, p) + \cosh d(x_n, p) \sinh (1-t) d(x_{n+1}, p)) \\
&\quad + (1-\alpha_n)(\cosh d(Tx_{n+1}, x_{n+1}) \sinh td(x_{n+1}, p) \\
&\qquad + \cosh d(Tx_{n+1}, p) \sinh(1-t) d(x_{n+1}, p)) \\
&= (\alpha_n \cosh d(x_n, x_{n+1}) + (1-\alpha_n) \cosh d(Tx_{n+1}, x_{n+1})) \sinh td(x_{n+1}, p) \\
&\quad + (\alpha_n \cosh d(x_n, p) + (1-\alpha_n) \cosh d(Tx_{n+1}, p)) \sinh(1-t) d(x_{n+1}, p).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&(\alpha_n \cosh d(x_n, x_{n+1}) + (1-\alpha_n) \cosh d(Tx_{n+1}, x_{n+1})) \frac{\sinh d(x_{n+1}, p) - \sinh td(x_{n+1}, p)}{\sinh(1-t) d(x_{n+1}, p)} \\
&\le \alpha_n \cosh d(x_n, p) + (1-\alpha_n) \cosh d(Tx_{n+1}, p).
\end{aligned}$$

Letting $t \to 1$, we get

$$\begin{aligned}
&(\alpha_n \cosh d(x_n, x_{n+1}) + (1-\alpha_n) \cosh d(Tx_{n+1}, x_{n+1})) \cosh d(x_{n+1}, p) \\
&\le \alpha_n \cosh d(x_n, p) + (1-\alpha_n) \cosh d(Tx_{n+1}, p) \\
&\le \alpha_n \cosh d(x_n, p) + (1-\alpha_n) \cosh d(x_{n+1}, p).
\end{aligned}$$

Thus, we get

$$\cosh d(x_{n+1}, p) \le \alpha_n \cosh d(x_n, p) + (1-\alpha_n) \cosh d(x_{n+1}, p).$$

Therefore, since $\{\alpha_n\} \subset \,]0, \frac{1}{2}]$, we obtain

$$\cosh d(x_{n+1}, p) \le \cosh d(x_n, p).$$

This implies $d(x_{n+1}, p) \le d(x_n, p)$. Since the real sequence $\{d(x_n, p)\}$ is nonincreasing and bounded below, there exists

$$\lim_{n \to \infty} d(x_n, p) = c_p \in \mathbb{R}.$$

Then, we have

$$\begin{aligned}
1 &\leq \alpha_n \cosh d(x_n, x_{n+1}) + (1-\alpha_n)\cosh d(Tx_{n+1}, x_{n+1}) \\
&\leq (\alpha_n \cosh d(x_n, p) + (1-\alpha_n)\cosh d(x_{n+1}, p))\frac{1}{\cosh d(x_{n+1}, p)} \\
&\leq \frac{\alpha_n(\cosh d(x_n, p) - \cosh d(x_{n+1}, p))}{\cosh d(x_{n+1}, p)} + 1 \\
&\to 1
\end{aligned}$$

as $n \to \infty$. This implies

$$\lim_{n\to\infty}(\alpha_n \cosh d(x_n, x_{n+1}) + (1-\alpha_n)\cosh d(Tx_{n+1}, x_{n+1})) = 1.$$

Then we get

$$\lim_{n\to\infty}\cosh d(x_n, x_{n+1}) = \lim_{n\to\infty}\cosh d(Tx_{n+1}, x_{n+1}) = 1.$$

Indeed, we assume $\{\cosh d(x_n, x_{n+1})\}$ does not converge to 1. Then, there exist $\varepsilon > 0$ and a subsequence $\{\cosh d(x_{n_i}, x_{n_i+1})\}$ of $\{\cosh d(x_n, x_{n+1})\}$ such that $\cosh d(x_{n_i}, x_{n_i+1}) \geq 1+\varepsilon$ for $i \in \mathbb{N}$. Furthermore, since $\{\alpha_{n_i}\} \subset [a, \frac{1}{2}]$, we may assume $\alpha_{n_i} \to \alpha_0 \in [a, \frac{1}{2}]$ without loss of generality. Then we have

$$\begin{aligned}
1 &= \lim_{i\to\infty}(\alpha_{n_i}\cosh d(x_{n_i}, x_{n_i+1}) + (1-\alpha_{n_i})\cosh d(Tx_{n_i}, x_{n_i+1})) \\
&\geq \alpha_0 \liminf_{i\to\infty}\cosh d(x_{n_i}, x_{n_i+1}) + (1-\alpha_0)\liminf_{i\to\infty}\cosh d(Tx_{n_i+1}, x_{n_i+1}) \\
&\geq \alpha_0(1+\varepsilon) + (1-\alpha_0) = 1 + \alpha_0\varepsilon > 1.
\end{aligned}$$

This is a contradiction. Thus we have $\lim_{n\to\infty}\cosh d(x_n, x_{n+1}) = 1$, and similarly we get $\lim_{n\to\infty}\cosh d(Tx_{n+1}, x_{n+1}) = 1$. Hence we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(Tx_{n+1}, x_{n+1}) = 0.$$

Let $x_0 \in X$ be a unique asymptotic center of a sequence $\{x_n\}$ and let $u \in X$ be an asymptotic center of any subsequence $\{x_{n_i}\}$ of $\{x_n\}$. We will show $u = x_0$. From the definition of asymptotic center, we have

$$\begin{aligned}
r(\{x_{n_i}\}) &= \limsup_{i\to\infty} d(x_{n_i}, u) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, Tu) \\
&\leq \limsup_{i\to\infty}(d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tu)) \\
&= \limsup_{i\to\infty} d(Tx_{n_i}, Tu) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, u) = r(\{x_{n_i}\}).
\end{aligned}$$

This implies $Tu \in \mathrm{AC}(\{x_{n_i}\})$. From the uniqueness of an asymptotic center, we get $u \in F(T)$. It follows that $\{d(x_n, u)\}$ is convergent to c_u. Therefore, we obtain

$$\begin{aligned}
r(\{x_n\}) &= \limsup_{n\to\infty} d(x_n, x_0) \\
&\leq \limsup_{n\to\infty} d(x_n, u) = c_u = \lim_{i\to\infty} d(x_{n_i}, u) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, x_0) \\
&\leq \limsup_{n\to\infty} d(x_n, x_0) = r(\{x_n\}).
\end{aligned}$$

Thus $u \in \mathrm{AC}(\{x_n\})$. From the uniqueness of an asymptotic center, we get $u = x_0$. Hence, $\{x_n\}$ is Δ-convergent to $x_0 \in F(T)$. This is the desired result. □

Finally, We consider the existence of fixed points of Xu-Ori type sequence in CAT(-1) spaces.

Theorem 5.9. *Let X be a complete* CAT(-1) *space and $T\colon X \to X$ a nonexpansive mapping. Suppose $\{\alpha_n\} \subset]0, \frac{1}{2}[$ such that $\alpha_n \to 0$ for $n \in \mathbb{N}$. Let $x_1 \in X$ and generate $\{x_n\}$ as follows: For $n \in \mathbb{N}$ and given $x_n \in X$, let x_{n+1} be a unique point in X satisfying that*

$$x_{n+1} = \alpha_n x_n \overset{-1}{\oplus} (1-\alpha_n) T x_{n+1}.$$

Then the following assertions are equivalent:

(1) *$F(T) \neq \varnothing$;*
(2) *$\{x_n\}$ and $\{Tx_n\}$ are bounded.*

Proof. By Theorem 5.4, we know $\{x_n\}$ is well-defined. We assume $F(T) \neq \varnothing$ and let $p \in F(T)$. Then, we get

$$d(x_{n+1}, p) \geq d(x_n, p)$$

as in the proof of Theorem 5.8. Thus $\{d(x_n, p)\}$ is bounded and hence we get $\{x_n\}$ is bounded. Moreover, since T is nonexpansive, we get $d(Tx_n, p) \leq d(x_n, p)$ and hence $\{Tx_n\}$ is bounded. Conversely, we assume $\{x_n\}$ and $\{Tx_n\}$ are bounded. Then we have

$$d(x_{n+1}, Tx_{n+1}) = d\left(\alpha_n x_n \overset{-1}{\oplus} (1-\alpha_n) T x_{n+1}, Tx_{n+1}\right) = \alpha_n d(x_n, Tx_{n+1}),$$

thus we get $d(x_n, Tx_n) \to 0$. Since $\{x_n\}$ is bounded, there exists $\{x_{n_i}\} \subset \{x_n\}$ and $x_0 \in X$ such that $x_{n_i} \overset{\Delta}{\rightharpoonup} x_0$. Then we have

$$\begin{aligned}
\limsup_{i\to\infty} d(x_{n_i}, x_0) &\leq \limsup_{i\to\infty} d(x_{n_i}, Tx_0) \\
&= \limsup_{i\to\infty} (d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tx_0)) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, Tx_{n_i}) + \limsup_{i\to\infty} d(Tx_{n_i}, Tx_0) \\
&= \limsup_{i\to\infty} d(Tx_{n_i}, Tx_0) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, x_0).
\end{aligned}$$

This implies $x_0 = Tx_0 \in \mathrm{AC}(\{x_{n_i}\})$. Since the asymptotic center of any bounded sequence in X is a singleton, we get $x_0 \in F(T)$. This is the desired result. □

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