タイトル	Iterative sequences for finitely many resolvent operators in geodesic spaces
作成者（著者）	笠原, 健吾
公開者	東邦大学
発行日	2021.03.17
掲載情報	東邦大学大学院理学研究科 博士論文.
資料種別	学位論文
内容記述	主査: 木村泰紀
著者版フラグ	ETD
報告番号	32661甲第1007号
学位記番号	甲第164号
学位授与年月日	2021.03.17
学位授与機関	東邦大学
メタデータのＵＲＬ	https://mylibrary.toho-u.ac.jp/webopac/TD28182548

Iterative sequences for finitely many resolvent operators in geodesic spaces

7518001 Kengo Kasahara
Department of Information Science
Toho University

Doctor thesis
January 2021

Acknowledgements

I am grateful to Professor Yasunori Kimura for teaching me many things about mathematics. I was so happy that I could study with a lot of my colleagues. I could not write this paper very well without the professor's help and my time with the colleagues.

Contents

Chapter 1

Introduction

In the study of nonlinear convex analysis, we often attempt to find a convex minimizer of a family of convex functions. Namely, we consider the following problem: Let $\{f^k\}$ be a family of convex functions on a geodesic metric space X and find $x_0 \in X$ such that

$$f^k(x_0) = \inf_{x \in X} f^k(x)$$

for all k. We focus on iterative schemes which are effective to find a common minimizer of a finite number of convex functions. We especially pay attention to Halpern's [6] and Mann's [23] iterative scheme. We introduce some results which use these schemes in Banach and Hilbert spaces. Reich [26] proved weak convergence of the Mann type iteration in a Banach space. Wittmann [30] proved strong convergence of the Halpern type iteration in a Banach space. By using two nonexpansive mappings, Takahashi and Tamura [29] proved weak convergence of Mann type iteration.

We also know many researchers have proved convergence theorems of iterative schemes on geodesic spaces. In a complete CAT(0) space, Dhompongsa and Panyanak [4] proved Δ-convergence of Mann's iterative scheme, and Saejung [27] also proved convergence of Halpern's iterative scheme. We also know a large number of results by using Mann's and Halpern's iterative schemes in a CAT(1) space. Piątek [25] considered Halpern's iterative scheme by using a nonexpansive mapping in a complete CAT(1) space. Kimura, Saejung and Yotkaew [19] proved Δ-convergence of Mann's iterative scheme by using a strongly quasinonexpansive and Δ-demiclosed mapping in a complete CAT(1) space. Kimura and Satô [22] also proved convergence of Halpern's schemes under the same setting. Considering these results, the author focus on Halpern's and Mann's iterative scheme by using a finite number of mappings. Moreover, Kimura and Kohsaka [17] proved convergence of Mann and Halpern types of iterative schemes with a sequence of resolvent operators for a single proper lower semicontinuous convex function. The author proves convergence of Halpern's and Mann's iterative schemes by using a finite number of resolvent operators in complete geodesic spaces.

We introduce resolvent operators for convex functions and balanced mappings. They are important in this thesis. First, we recall the resolvent operator for convex function. In a Hilbert space, we know the resolvent operator J_f is defined as follows: Let f be a proper lower semicontinuous convex function from a Hilbert space H to $]-\infty, \infty]$. Then J_f is defined by

$$J_f x = \operatorname*{argmin}_{y \in H} \{f(y) + \frac{1}{2} \|y - x\|^2\}$$

for every $x \in X$. We know that the resolvent J_f is a single-valued mapping from H to H and it is nonexpansive. In a complete CAT(0) space, Jost [10] and Mayer [24] defined the resolvent Q_f as follows: Let f be a proper lower semicontinuous convex function from a complete

CAT(0) space X into $]-\infty, \infty]$. Then Q_f is defined by

$$Q_f x = \underset{y \in X}{\operatorname{argmin}} \{f(y) + \frac{1}{2} d(y, x)^2\}$$

for all $x \in X$. We also know the resolvent Q_f is a single-valued mapping from X to X and it is nonexpansive. In a complete CAT(1) space, Kimura and Kohsaka [16, 17] defined the resolvent R_f as follows: Let f be a proper lower semicontinuous convex function from a complete CAT(1) space X into $]-\infty, \infty]$. Then R_f is defined by

$$R_f x = \underset{y \in X}{\operatorname{argmin}} \{f(y) + \tan d(y, x) \sin d(y, x)\}$$

for all $x \in X$. We also know the resolvent R_f is a single-valued mapping from X to X and it is strongly quasinonexpansive and Δ-demiclosed. In this paper, we use the resolvent operators in complete CAT(0) and CAT(1) spaces.

Next, we introduce a balanced mapping in geodesic spaces. Hasegawa and Kimura [7] defined it in a complete CAT(0) space.

Theorem 1.1. (Hasegawa and Kimura [7]) *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \dots, N$. Let $\alpha^k \in [0, 1]$ for every $k = 1, 2, \dots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let x be a point of X. Then the set*

$$\underset{y \in X}{\operatorname{argmin}} \sum_{k=1}^{N} \alpha^k d(T^k x, y)^2$$

consists of one point.

Moreover, they prove convergence of Mann and Halpern types of iterative schemes by using a balanced mapping in a complete CAT(0) space. The following theorems are convergence theorems of Mann and Halpern types iterative schemes proved by them.

Theorem 1.2. (Hasegawa and Kimura [7]) *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \dots, N$ such that $F = \bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\left\{\alpha_n^k\right\}, \{\beta_n\} \subset [a, 1-a]$ for every $k = 1, 2, \dots, N$ and $n \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_n^k = 1$. Define a mapping U_n from X to X by*

$$U_n x = \underset{y \in X}{\operatorname{argmin}} \sum_{k=1}^{N} \alpha_n^k d(T^k x, y)^2$$

for every $x \in X$ and $n \in \mathbb{N}$. For a given point $x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by

$$x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point in F.

Theorem 1.3. (Hasegawa-Kimura [8]) *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \dots, N$ such that $F = \bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\beta_n\} \subset]0, 1[$, $\left\{\alpha_n^k\right\} \subset [a, 1-a]$ for every $k = 1, 2, \dots, N$ and $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{k=1}^{N} \alpha_n^k = 1$*

and $\sum_{n=1}^{\infty}\sum_{k=1}^{N}\left|\alpha_{n+1}^{k}-\alpha_{n}^{k}\right|<\infty$*. Define a mapping* U_n *from* X *to* X *by*

$$U_n x = \operatorname*{argmin}_{y\in X}\sum_{k=1}^{N}\alpha_n^k d(T^k x, y)^2$$

for every $x \in X$ *and* $n \in \mathbb{N}$*. For given points* $u, x_1 \in X$*, let* $\{x_n\}$ *be a sequence in* X *generated by*

$$x_{n+1} = \beta_n u \oplus (1-\beta_n)U_n x_n$$

for every $n \in \mathbb{N}$*. Then* $\{x_n\}$ *converges to* $P_F u$*, where* P_F *is the metric projection from* X *to* F*.*

We define a balanced mapping in a complete CAT(1) space and apply it to generate an iterative sequence converging to a common minimizer of convex functions.

This thesis is organized in the following order. In Chapter 2, we show some necessary tools for our results. In Chapter 3, we [12] proved convergence of Mann's iterative scheme by using two resolvent operators in a complete CAT(1) space (Theorem 3.1). This theorem was proved by using the technique proposed in [18]. In Chapter 4, we show convergence of Halpern's and Mann's iterative scheme by using iterative sequences for finitely many resolvent operators dependent on the order in a complete CAT(1) space. In Chapter 5, we show convergence of Mann type iteration by using a balanced mapping for a finite number of resolvent operators in a complete CAT(0) space. In Chapter 6, we define a balanced mapping in a complete CAT(1) space and we proved Δ-convergence of Mann type iteration by using it. In Chapter 7, we conclude in this paper.

Chapter 2

Preliminaries

Let X be a metric space and $\{x_n\}$ a sequence in X. An element $z \in X$ is called an asymptotic center of $\{x_n\} \subset X$ if

$$\limsup_{n\to\infty} d(x_n, z) = \inf_{x\in X} \limsup_{n\to\infty} d(x_n, x).$$

Moreover, we say $\{x_n\}$ Δ-converges to a Δ-limit z if z is the unique asymptotic center of any subsequences of $\{x_n\}$. For $x, y \in X$, a mapping $c : [0, l] \to X$ is called a geodesic joining x and y if c satisfies

$$c(0) = x, c(l) = y, \text{ and } d(c(u), c(v)) = |u - v|$$

for every $u, v \in [0, l]$. An image $[x, y]$ of c is called a geodesic segment joining x and y. For $r > 0$, X is called an r-geodesic space if for every $x, y \in X$ with $d(x, y) < r$, there exists a geodesic c joining x and y. Moreover, if such a geodesic segment is unique for each pair of points, then X is called a uniquely r-geodesic space.

We denote a two-dimensional model space with curvature $\kappa \in \mathbb{R}$ by M^2_κ, where

$$M^2_\kappa = \begin{cases} \dfrac{1}{\sqrt{\kappa}}\mathbb{S}^2 & (\kappa > 0), \\ \mathbb{R}^2 & (\kappa = 0), \\ \dfrac{1}{\sqrt{-\kappa}}\mathbb{H}^2 & (\kappa < 0). \end{cases}$$

The diameter of M^2_κ is denoted by D_κ, that is,

$$D_\kappa = \begin{cases} \dfrac{\pi}{\sqrt{\kappa}} & (\kappa > 0), \\ \infty & (\kappa \le 0). \end{cases}$$

Let X be a uniquely D_κ-geodesic space. For a triangle $\triangle(x, y, z) \subset X$ such that $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, let a comparison triangle $\triangle(\bar{x}, \bar{y}, \bar{z})$ in the two-dimensional model space M^2_κ be such that each corresponding edge has the same length as that of the original triangle. X is called a CAT(κ) space if for any $x, y, z \in X$ every $p, q \in \triangle(x, y, z)$ and their corresponding points $\bar{p}, \bar{q} \in \triangle(\bar{x}, \bar{y}, \bar{z})$ satisfy that

$$d(p, q) \le d_{M^2_\kappa}(\bar{p}, \bar{q}),$$

where $d_{M^2_\kappa}$ is the metric on M^2_κ.

We show some tools and their properties in CAT(0) and CAT(1) spaces. First, we show them in a CAT(0) space.

Let X be a CAT(0) space. For every $x, y \in X$ and $\alpha \in [0,1]$, if $z \in [x,y]$ satisfies that $d(y,z) = \alpha d(x,y)$ and $d(x,z) = (1-\alpha)d(x,y)$, then we denote z by $z = \alpha x \oplus (1-\alpha)y$.

Let X be a CAT(0) space and let T be a mapping from X to X. If $d(Tx, Ty) \leq d(x,y)$ for every $x, y \in X$, then we call T a nonexpansive mapping. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{z \in X : z = Tz\}$.

Let X be a complete CAT(0) space and let C be a nonempty closed convex subset of X. Then for every $x \in X$, there exists a unique point $x_0 \in C$ satisfying

$$d(x, x_0) = \inf_{y \in C} d(x, y).$$

We define the metric projection P_C from X onto C by $P_C x = x_0$. We know that the metric projection P_C is a nonexpansive mapping such that $F(P_C) = C$.

Let X be a complete CAT(0) space. Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. For $\lambda > 0$, the resolvent $Q_{\lambda f}$ of λf is defined by

$$Q_{\lambda f} x = \operatorname*{argmin}_{y \in X} \{\lambda f(y) + d(y,x)^2\}$$

for all $x \in X$ [10, 24]. We know that $Q_{\lambda f}$ is a single-valued mapping from X to X. We also know that the resolvent $Q_{\lambda f}$ is nonexpansive such that $F(Q_{\lambda f}) = \operatorname{argmin}_{x \in X} f$.

Let X be a complete CAT(0) space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$. Let $\{\alpha^k\} \subset]0,1[$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Hasegawa and Kimura [7] define a balanced mapping U from X to X by

$$Ux = \operatorname*{argmin}_{y \in X} \sum_{k=1}^N d(T^k x, y)^2$$

for every $x \in X$. They find that this mapping U is defined as a single-valued mapping, has nonexpansiveness and $F(U) = \bigcap_{k=1}^N F(T^k)$. We show some lemmas in a CAT(0) space which is used in Chapter 5.

Lemma 2.1. (Hasegawa and Kimura [7]) *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$. Let $\{\alpha^k\} \subset]0,1[$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Define a balanced mapping $U : X \to X$ by $Ux = \operatorname{argmin}_{y \in X} \sum_{k=1}^N \alpha^k d(T^k x, y)^2$ for every $x \in X$. Then*

$$\sum_{k=1}^N \alpha^k d(T^k x, Ux)^2 \leq \sum_{k=1}^N \alpha^k d(T^k x, Uy)^2 - \sum_{k=1}^N \alpha^k d(Uy, Ux)^2$$

for every $x, y \in X$.

Lemma 2.2. (Hasegawa and Kimura [8]) *Let X be a complete* CAT(0) *space. Let U be a nonexpansive mapping from X to X. Suppose $\{x_n\} \subset X$ is Δ-convergent to $x_0 \in X$ and $\{d(x_n, Ux_n)\}$ is convergent to* 0. *Then $x_0 \in F(U)$.*

Lemma 2.3. (Kimura and Kohsaka [16]) *Let X be a complete* CAT(0) *space. Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Let $\lambda, \mu > 0$, and $R_{\lambda f}, R_{\mu f}$ be the resolvent of $\lambda f, \mu f$. Then*

$$(\lambda + \mu) d(R_{\lambda f} x, R_{\mu f} x)^2 + \mu d(R_{\lambda f} x, x)^2 + \lambda d(R_{\mu f} x, x)^2 \leq \lambda d(R_{\lambda f} x, x)^2 + \mu d(R_{\mu f} x, x)^2$$

for every $x \in X$.

Next, we show tools and their properties in a CAT(1) space.

Let X be a CAT(1) space. For every $x, y \in X$ with $d(x,y) < \pi$ and $\alpha \in [0,1]$, if $z \in [x,y]$ satisfies that $d(y,z) = \alpha d(x,y)$ and $d(x,z) = (1-\alpha)d(x,y)$, then we denote z by $z = \alpha x \oplus (1-\alpha)y$. A subset $C \subset X$ is said to be π-convex if $\alpha x \oplus (1-\alpha)y \in C$ for every $x, y \in C$ with $d(x,y) < \pi$ and $\alpha \in [0,1]$.

Let X be a CAT(1) space and let T be a mapping from X to X such that the set $F(T) = \{z \in X : z = Tz\}$ of fixed points of T is not empty. If $d(Tx, p) \le d(x,p)$ for every $x \in X$ and $p \in F(T)$, then we call T a quasinonexpansive mapping.

T is called a strongly quasinonexpansive mapping if T is a quasinonexpansive mapping, and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ whenever $\{x_n\} \subset X$ satisfies $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n,p)/\cos d(Tx_n,p)) = 1$ for every $p \in F(T)$.

Let X be a CAT(1) space and let T be a mapping from X to X such that $F(T) \neq \emptyset$. T is called a Δ-demiclosed mapping if $z \in F(T)$ whenever $\{x_n\}$ Δ-converges to z and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Following [1], we define the notions of a strongly quasinonexpansive sequence and a Δ-demiclosed sequence on CAT(1) spaces as follows: Let $\{T_n\}$ be a sequence of mappings from X to X. $\{T_n\}$ is said to be a strongly quasinonexpansive sequence if each T_n is quasinonexpansive and $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$ whenever $\sup_{n\in\mathbb{N}} d(x_n,p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n,p)/\cos d(T_n x_n, p)) = 1$ for every $p \in \bigcap_{n=1}^\infty F(T_n)$. $\{T_n\}$ is said to be a Δ-demiclosed sequence if $z \in \bigcap_{n=1}^\infty F(T_n)$ whenever $\{x_n\}$ is Δ-convergent to z and $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$.

Let X be a complete CAT(1) space and let $C \subset X$ be a nonempty closed π-convex subset such that $d(x, C) = \inf_{y\in C} d(x,y) < \pi/2$ for every $x \in X$. Then for every $x \in X$, there exists a unique point $x_0 \in C$ satisfying

$$d(x, x_0) = \inf_{y\in C} d(x,y).$$

We define the metric projection P_C from X onto C by $P_C x = x_0$. We know that the metric projection P_C is a strongly quasinonexpansive and Δ-demiclosed mapping such that $F(P_C) = C$.

Let X be a complete CAT(1) space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $f\colon X \to\,]-\infty, \infty]$ be a proper lower semicontinuous convex function. The resolvent R_f of f is defined by a unique minimizer of a function $f + \tan d(\cdot, x)\sin d(\cdot, x)$, that is,

$$R_f x = \operatorname*{argmin}_{y\in X}(f(y) + \tan d(y,x)\sin d(y,x))$$

for all $x \in X$; see [16]. We know that R_f is a single-valued mapping from X to X. We also know that the resolvent R_f is strongly quasinonexpansive and Δ-demiclosed such that $F(R_f) = \operatorname{argmin}_{x\in X} f$ [16, 17]. we show some lemmas in a CAT(1) space.

Lemma 2.4. (Espínola and Fernández-León [5]) *Let X be a complete* CAT(1) *space. Let $\{x_n\}$ be a sequence in X. If $r(\{x_n\}) < \pi/2$, then the following hold:*

(a) *$AC(\{x_n\})$ consists of exactly one point;*
(b) *$\{x_n\}$ has a Δ-convergent subsequence.*

Lemma 2.5. (He, Fang, López and Li [9]) *Let X be a complete* CAT(1) *space and $p \in X$. If a sequence $\{x_n\}$ in X satisfies that* $\limsup_{n\to\infty} d(x_n, p) < \pi/2$ *and that $\{x_n\}$ is Δ-convergent to $x \in X$, then $d(x,p) \le \liminf_{n\to\infty} d(x_n,p)$.*

Lemma 2.6. (Kimura and Satô [20]) *Let X be a* CAT(1) *space. For every $x, y, z \in X$ with*

$d(x,y)+d(y,z)+d(z,x)<2\pi$ *and* $\alpha\in[0,1]$, *the following inequality holds:*

$$\cos d(x,w)\sin d(y,z)\geq\cos d(x,y)\sin(\alpha d(y,z))+\cos d(x,z)\sin((1-\alpha)d(y,z)),$$

where $w=\alpha y\oplus(1-\alpha)z$.

Lemma 2.7. (Kimura and Satô [21]) *Let* X *be a complete* CAT(1) *space such that* $d(u,v)<\pi/2$ *for all* $u,v\in X$. *Let* S,T *be quasinonexpansive mappings from* X *to* X *with* $F(S)\cap F(T)\neq\emptyset$. *Then, for every* $\alpha\in\,]0,1[$, $F(S)\cap F(T)=F(\alpha S\oplus(1-\alpha)T)$ *and the mapping* $\alpha S\oplus(1-\alpha)T$ *is quasinonexpansive.*

Lemma 2.8. (Kimura and Satô [22]) *Let* X *be a* CAT(1) *space. For every* $x,y,z\in X$ *with* $d(x,y)+d(y,z)+d(z,x)<2\pi$ *and* $\alpha\in[0,1]$, *the following inequality holds:*

$$\cos d(x,w)\geq\alpha\cos d(x,y)+(1-\alpha)\cos d(x,z),$$

where $w=\alpha y\oplus(1-\alpha)z$.

Lemma 2.9. (Kimura and Satô [22]) *Let* X *be a* CAT(1) *space and* y_0,y_1 *and* y *elements of* X *such that* $d(y_0,y)+d(y_1,y)+d(y_0,y_1)<2\pi$. *Then*

$$\cos d\left(\frac{1}{2}y_0\oplus\frac{1}{2}y_1,y\right)\cos\frac{d(y_0,y_1)}{2}\geq\min\{\cos d(y_0,y),\cos d(y_1,y)\}.$$

Lemma 2.10. (Kimura and Satô [22]) *Let* X *be a* CAT(1) *space such that* $d(v,v')<\pi$ *for every* $v,v'\in X$. *Let* $\alpha\in[0,1]$ *and* $u,y,z\in X$. *Then*

$$\begin{aligned}&1-\cos d(\alpha u\oplus(1-\alpha)y,z)\\&\qquad\leq(1-\beta)(1-\cos d(y,z))+\beta\left(1-\frac{\cos d(u,z)}{\sin d(u,y)\tan(\frac{\alpha}{2}d(u,y))+\cos d(u,y)}\right),\end{aligned}$$

where

$$\beta=\begin{cases}1-\dfrac{\sin((1-\alpha)d(u,y))}{\sin d(u,y)} & (u\neq y),\\ \alpha & (u=y).\end{cases}$$

In addition, we show some lemmas which are related to real sequences.

Lemma 2.11. (Aoyama, Kimura, Takahashi and Toyoda [3]) *Let* $\{s_n\},\{u_n\}\subset\,]0,\infty[$, $\{t_n\}\subset\mathbb{R}$ *and* $\{\alpha_n\}\subset[0,1]$ *such that* $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\sum_{n=1}^{\infty}u_n<\infty$ *and* $\limsup_{n\to\infty}t_n\leq 0$. *Suppose that*

$$s_{n+1}\leq(1-\alpha_n)s_n+\alpha_n t_n+u_n$$

for all $n\in\mathbb{N}$. *Then* $\lim_{n\to\infty}s_n=0$.

Lemma 2.12. *Let* σ *be a real number in* $]-1,0[$ *and* $\{b_n\},\{c_n\}$ *real sequences in* $[\sigma,1]$ *and* $\liminf_{n\to\infty}b_nc_n\geq 1$. *Then* $\lim_{n\to\infty}b_n=\lim_{n\to\infty}c_n=1$.

Lemma 2.13. *Let* s *be a real number in* $]0,\infty[$ *and* $\{b_n\},\{c_n\}$ *bounded real sequences such that* $b_n\leq 0, s<c_n$ *and* $\lim_{n\to\infty}b_n/c_n=0$. *Then* $\lim_{n\to\infty}b_n=0$.

Lemma 2.14. *Let* $\{b_n\},\{c_n\}$ *be bounded real sequences such that* $\lim_{n\to\infty}(b_n-c_n)=0$. *Then* $\liminf_{n\to\infty}b_n=\liminf_{n\to\infty}c_n$.

Chapter 3

Mann iterative sequence by using two resolvent operators

In this chapter, we prove the convergence of Mann type iteration by using the iterative sequences defined by $x_1 \in X$ and

$$x_{n+1} = \alpha_n(\beta_n x_n \oplus (1-\beta_n)R_{\lambda_n f}x_n) \oplus (1-\alpha_n)(\gamma_n x_n \oplus (1-\gamma_n)R_{\mu_n g}x_n)$$

in a complete CAT(1) space. To prove our main theorem, we employ the technique proposed in [18].

Theorem 3.1. (Kasahara and Kimura [12]) *Let X be an admissible complete* CAT(1) *space. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of positive real numbers such that $\inf_n \lambda_n > 0$ and $\inf_n \mu_n > 0$. Let f and g be proper lower semicontinuous convex functions from X into $]-\infty, \infty]$ such that $F = \operatorname{argmin}_X f \cap \operatorname{argmin}_X g \neq \emptyset$. Let $R_{\lambda_n f}$ and $R_{\mu_n g}$ be the resolvents of $\lambda_n f$ and $\mu_n g$, respectively. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a, 1-a]$. Let $\{x_n\}$ be a sequence in X defined by $x_1 \in X$ and*

$$\begin{cases} u_n = \beta_n x_n \oplus (1-\beta_n)R_{\lambda_n f}x_n, \\ v_n = \gamma_n x_n \oplus (1-\gamma_n)R_{\mu_n g}x_n, \\ x_{n+1} = \alpha_n u_n \oplus (1-\alpha_n)v_n \end{cases}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point of F.

Proof. Let $z \in F$. By Lemma 2.8, we have

$$\begin{aligned} \cos d(u_n, z) &\geq \beta_n \cos d(x_n, z) + (1-\beta_n)\cos d(R_{\lambda_n f}x_n, z) \\ &\geq \cos d(x_n, z). \end{aligned}$$

Similarly, we have

$$\cos d(v_n, z) \geq \cos d(x_n, z).$$

By these inequalities, we obtain

$$\begin{aligned} \cos d(x_{n+1}, z) &\geq \alpha_n \cos d(u_n, z) + (1-\alpha_n)\cos d(v_n, z) \\ &\geq \cos d(x_n, z). \end{aligned}$$

Thus, we have $d(x_{n+1}, z) \leq d(x_n, z)$ for all $n \in \mathbb{N}$ and there exists

$$D = \lim_{n\to\infty} d(x_n, z) \leq d(x_1, z) < \frac{\pi}{2}.$$

By Lemma 2.6, we have

$$
\begin{aligned}
&\cos d(u_n, z)\sin d(x_n, R_{\lambda_n f}x_n)\\
&\qquad \geq \cos d(x_n, z)\sin \beta_n d(x_n R_{\lambda_n f}x_n) + \cos d(R_{\lambda_n f}x_n, z)\sin(1-\beta_n)d(x_n, R_{\lambda_n f}x_n)\\
&\qquad \geq 2\cos d(x_n, z)\sin\frac{d(x_n, R_{\lambda_n f}x_n)}{2}\cos\frac{(2\beta_n - 1)d(x_n, R_{\lambda_n f}x_n)}{2}. \qquad (3.1)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&\cos d(v_n, z)\sin d(x_n, R_{\mu_n g}x_n)\\
&\qquad \geq \cos d(x_n, z)\sin \gamma_n d(x_n R_{\lambda_n f}x_n) + \cos d(R_{\mu_n g}x_n, z)\sin(1-\gamma_n)d(x_n, R_{\mu_n g}x_n)\\
&\qquad \geq 2\cos d(x_n, z)\sin\frac{d(x_n, R_{\mu_n g}x_n)}{2}\cos\frac{(2\gamma_n - 1)d(x_n, R_{\mu_n g}x_n)}{2}. \qquad (3.2)
\end{aligned}
$$

Let $d_n = d(x_n, z)$, $f_n = d(x_n, R_{\lambda_n f}x_n)/2$ and $g_n = d(x_n, R_{\mu_n g}x_n)/2$ for all $n \in \mathbb{N}$.

If $f_n \neq 0$ and $g_n = 0$, then we have $v_n = x_n$. From (3.1), (3.2) and Lemma 2.8, we have

$$
\begin{aligned}
2\cos d_{n+1}\sin f_n\cos f_n &= \cos d_{n+1}\sin 2f_n\\
&\geq \alpha_n\cos d(u_n, z)\sin 2f_n + (1-\alpha_n)\cos d(v_n, z)\sin 2f_n\\
&\geq 2\alpha_n\cos d_n\sin f_n\cos(2\beta_n - 1)f_n + 2(1-\alpha_n)\cos d_n\sin f_n\cos f_n
\end{aligned}
$$

Dividing by $2\sin f_n > 0$, we obtain

$$\cos d_{n+1}\cos f_n \geq \alpha_n\cos d_n\cos(2\beta_n - 1)f_n + (1-\alpha_n)\cos d_n\cos f_n. \qquad (3.3)$$

If $f_n = 0$ and $g_n \neq 0$, then we have $u_n = x_n$. Similarly, we have

$$\cos d_{n+1}\cos g_n \geq \alpha_n\cos d_n\cos g_n + (1-\alpha_n)\cos d_n\cos(2\gamma_n - 1)g_n. \qquad (3.4)$$

If $f_n \neq 0$ and $g_n \neq 0$, then from (3.1), (3.2) and Lemma 2.8, we have

$$
\begin{aligned}
&\cos d_{n+1}\sin 2f_n\sin 2g_n\\
&\quad \geq \alpha_n\cos d(u_n, z)\sin 2f_n\sin 2g_n + (1-\alpha_n)\cos d(v_n, z)\sin 2f_n\sin 2g_n\\
&\quad \geq 4\cos d_n\sin f_n\sin g_n(\alpha_n\cos(2\beta_n - 1)f_n\cos g_n + (1-\alpha_n)\cos f_n\cos(2\gamma_n - 1)g_n).
\end{aligned}
$$

Dividing by $4\sin f_n\sin g_n > 0$, we have

$$
\begin{aligned}
&\cos d_{n+1}\cos f_n\cos g_n\\
&\qquad \geq \alpha_n\cos d_n\cos(2\beta_n - 1)f_n\cos g_n + (1-\alpha_n)\cos d_n\cos f_n\cos(2\gamma_n - 1)g_n. \qquad (3.5)
\end{aligned}
$$

If $f_n = 0$ and $g_n = 0$, then we also have the inequality (3.5), and the inequality (3.5) can be reduced to the inequality (3.3), (3.4) for each case. From inequality (3.5), we have

$$\left(\frac{\epsilon_n\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n}\right)\left(\frac{\epsilon_n\cos g_n}{\alpha_n\cos(2\gamma_n - 1)g_n} - \frac{1-\alpha_n}{\alpha_n}\right) \geq 1,$$

where $\epsilon_n = \cos d_{n+1}/\cos d_n$ for $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty}\epsilon_n = \cos D/\cos D = 1$. Since $\{\alpha_n\} \subset [a, 1-a]$ for all $n \in \mathbb{N}$, we obtain

$$\liminf_{n\to\infty}\left(\frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n}\right)\left(\frac{\cos g_n}{\alpha_n\cos(2\gamma_n - 1)g_n} - \frac{1-\alpha_n}{\alpha_n}\right) \geq 1, \qquad (3.6)$$

We show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the following hold:

$$-\frac{1}{2} \leq \frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n} \leq 1 \tag{3.7}$$

and

$$-\frac{1}{2} \leq \frac{\cos g_n}{\alpha_n \cos(2\gamma_n - 1)g_n} - \frac{1-\alpha_n}{\alpha_n} \leq 1. \tag{3.8}$$

We show the second inequality of (3.7). Since $\{\beta_n\} \subset [a, 1-a]$ for all $n \in \mathbb{N}$, we obtain

$$\frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n} \leq \frac{1}{1-\alpha_n} - \frac{\alpha_n}{1-\alpha_n} = 1.$$

Similarly, we show the second inequality of (3.8). Next, we show the first inequality of (3.7). Let

$$\sigma_n = \frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n} \quad \text{and} \quad \theta_n = \frac{\cos g_n}{\alpha_n \cos(2\gamma_n - 1)g_n} - \frac{1-\alpha_n}{\alpha_n}.$$

Assume that the first inequality of (3.7) does not hold. Then we can get a subsequence $\{\sigma_{n_i}\} \subset \{\sigma_n\}$ such that $\sigma_{n_i} < -1/2$ and $\lim_{i\to\infty} \sigma_{n_i} = \sigma \leq -1/2$. Since $\{\alpha_n\}, \{\gamma_n\} \subset [a, 1-a]$ and $\{g_n\} \subset [0, \pi/4[$, we get $\{\theta_n\}$ is bounded. Let $\{\theta_{n_{i_j}}\} \subset \{\theta_{n_i}\} \subset \{\theta_n\}$ such that $\{\theta_{n_{i_j}}\}$ converges to $\theta \in \mathbb{R}$. From the inequality (3.6), we have

$$\sigma\theta = \lim_{j\to\infty} \sigma_{n_{i_j}} \theta_{n_{i_j}} \geq \liminf_{n\to\infty} \sigma_n \theta_n \geq 1.$$

Therefore, we may assume that $\theta_{n_{i_j}} < 0$ for all $j \in \mathbb{N}$. Since $\{f_n\}, \{g_n\} \subset [0, \pi/4[$ and $\{\beta_n\}, \{\gamma_n\} \subset [a, 1-a]$, we have

$$0 < \frac{\sqrt{2}}{2(1-a)} \leq \frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} \tag{3.9}$$

and

$$0 < \frac{\sqrt{2}}{2(1-a)} \leq \frac{\cos g_n}{\alpha_n \cos(2\gamma_n - 1)g_n}. \tag{3.10}$$

Let $\rho \in \mathbb{R}$ such that

$$0 < \rho < \min\left\{\frac{\sqrt{2}}{2(1-a)}, \frac{2a}{1-a}\right\}. \tag{3.11}$$

From the inequalities (3.9), (3.10), we have

$$\rho - \frac{\alpha_{n_{i_j}}}{1-\alpha_{n_{i_j}}} \leq \sigma_{n_{i_j}} < 0 \quad \text{and} \quad \rho - \frac{1-\alpha_{n_{i_j}}}{\alpha_{n_{i_j}}} \leq \theta_{n_{i_j}} < 0. \tag{3.12}$$

From the inequalities $(3.11), (3.12)$, we have

$$\begin{aligned}\sigma_{n_{i_j}}\theta_{n_{i_j}} &\leq \left(\rho - \frac{\alpha_{n_{i_j}}}{1-\alpha_{n_{i_j}}}\right)\left(\rho - \frac{1-\alpha_{n_{i_j}}}{\alpha_{n_{i_j}}}\right) \\ &= \rho^2 - \left(\frac{\alpha_{n_{i_j}}}{1-\alpha_{n_{i_j}}} + \frac{1-\alpha_{n_{i_j}}}{\alpha_{n_{i_j}}}\right)\rho + 1 \\ &\leq \rho^2 - \frac{2a}{1-a}\rho + 1 \\ &= \rho\left(\rho - \frac{2a}{1-a}\right) + 1.\end{aligned}$$

Then, as $j \to \infty$, we obtain

$$1 \leq \sigma\theta \leq \rho\left(\rho - \frac{2a}{1-a}\right) + 1 < 1.$$

This is a contradiction. Similarly, we obtain the left inequality of (3.8). From the inequalities $(3.7), (3.8)$ and Lemma 2.12, we have

$$\lim_{n\to\infty}\left(\frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n-1)f_n} - \frac{\alpha_n}{1-\alpha_n}\right) = \lim_{n\to\infty}\left(\frac{\cos g_n}{\alpha_n\cos(2\gamma_n-1)g_n} - \frac{1-\alpha_n}{\alpha_n}\right) = 1.$$

Hence, we have

$$\lim_{n\to\infty}\frac{\cos f_n - \cos(2\beta_n-1)f_n}{(1-\alpha_n)\cos(2\beta_n-1)f_n} = 0.$$

By Lemma 2.13, we get

$$\lim_{n\to\infty}(\cos f_n - \cos(2\beta_n-1)f_n) = 0.$$

By Lemma 2.14, we have

$$\liminf_{n\to\infty}\cos f_n = \liminf_{n\to\infty}\cos(2\beta_n-1)f_n = \liminf_{n\to\infty}\cos|2\beta_n-1|\,f_n.$$

Hence, we obtain

$$\limsup_{n\to\infty} f_n = \limsup_{n\to\infty}(|2\beta_n-1|\,f_n) \leq \limsup_{n\to\infty}|2\beta_n-1|\limsup_{n\to\infty} f_n.$$

Furthermore, we get

$$\liminf_{n\to\infty}(1-|2\beta_n-1|)\limsup_{n\to\infty} f_n = \left(1 - \limsup_{n\to\infty}|2\beta_n-1|\right)\limsup_{n\to\infty} f_n \leq 0.$$

Since $\{\beta_n\} \subset [a, 1-a]$ for all $n \in \mathbb{N}$, we get $\liminf_{n\to\infty}(1-|2\beta_n-1|) > 0$ and thus $\limsup_{n\to\infty} f_n = 0$. It implies that $d(x_n, R_{\lambda_n f}x_n) \to 0$. Similarly, we get $d(x_n, R_{\mu_n g}x_n) \to 0$.

Next, we show $\{x_n\}$ Δ-converges to a point of F. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Since $r(\{x_n\}) \leq D < \pi/2$ and Theorem 2.4(a), there exists a unique asymptotic center x_0 of $\{x_{n_k}\}$. Since $r(\{x_{n_k}\}) < \pi/2$ and Theorem 2.4(b), there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_l}}\}$ Δ-converges to $x_0' \in X$. Moreover, since $d(x_{n_{k_l}}, R_{\lambda_{n_{k_l}} f}x_{n_{k_l}}) \to 0$, $d(x_{n_{k_l}}, R_{\mu_{n_{k_l}} g}x_{n_{k_l}}) \to 0$ and $\{R_{\lambda_{n_{k_l}} f}\}, \{R_{\mu_{n_{k_l}} g}\}$ are Δ-demiclosed sequences, we obtain $x_0' \in$

$F = \operatorname{argmin}_X f \cap \operatorname{argmin}_X g$. If $x_0 \neq x_0'$, by Theorem 2.4(a) and the uniqueness of the asymptotic centers, we get

$$\begin{aligned}
\limsup_{k\to\infty} d(x_{n_k}, x_0) &< \limsup_{k\to\infty} d(x_{n_k}, x_0') \\
&= \lim_{n\to\infty} d(x_n, x_0') \\
&= \limsup_{l\to\infty} d(x_{n_{k_l}}, x_0') \\
&< \limsup_{l\to\infty} d(x_{n_{k_l}}, x_0) \\
&\leq \limsup_{k\to\infty} d(x_{n_k}, x_0).
\end{aligned}$$

This is a contradiction. Hence, we have $x_0 \in F = \operatorname{argmin}_X f \cap \operatorname{argmin}_X g$. Let $\{u_k\}, \{v_k\}$ be subsequences of $\{x_n\}$, $u_0 \in AC(\{u_k\})$ and $v_0 \in AC(\{v_k\})$. If $u_0 \neq v_0$, then since $u_0 \notin AC(\{v_k\}), v_0 \notin AC(\{u_k\})$ and by Theorem 2.4(a), we have

$$\begin{aligned}
\limsup_{k\to\infty} d(u_k, u_0) &< \limsup_{k\to\infty} d(u_k, v_0) \\
&= \lim_{n\to\infty} d(x_n, v_0) \\
&= \limsup_{k\to\infty} d(v_k, v_0) \\
&< \limsup_{k\to\infty} d(v_k, u_0) \\
&= \lim_{n\to\infty} d(x_n, u_0) \\
&= \limsup_{k\to\infty} d(u_k, u_0).
\end{aligned}$$

This is a contradiction. We obtain $u_0 = v_0$, then we have $\{x_n\}$ Δ-converges to a point of $F = \operatorname{argmin}_X f \cap \operatorname{argmin}_X g$. This completes the proof. □

Chapter 4

Iterative sequences for finitely many resolvent operators dependent of the order

In this chapter, we prove convergence theorems for a finitely many resolvent operators for convex functions which is dependent on their order of taking convex combination. We note that Theorem 4.9 is not a generalization of Theorem 3.1. Before we prove Theorems 4.9 and 4.10, we show some lemmas by using a finite number of resolvent operators for iterative schemes.

Lemma 4.1. (Kasahara and Kimura [14]) *Let X be a* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\sigma \in [a, 1-a]$. For given points $y, y^0, y^1 \in X$, define $w \in X$ by*

$$w = \sigma y^0 \oplus (1-\sigma) y^1.$$

Then

$$\cos d(w, y) \cos(ad(y^0, y^1)) \geq \min\{\cos d(y^0, y), \cos d(y^1, y)\}.$$

Proof. If $y^0 = y^1$, it is obvious. Otherwise, by Lemma 2.6, we have

$$\begin{aligned}
&\cos d(w, y) \sin d(y^0, y^1) \\
&\quad \geq \cos d(y^0, y) \sin(\sigma d(y^0, y^1)) + \cos d(y^1, y) \sin((1-\sigma) d(y^0, y^1)) \\
&\quad \geq \min\{\cos d(y^0, y), \cos d(y^1, y)\}(\sin(\sigma d(y^0, y^1)) + \sin((1-\sigma) d(y^0, y^1))) \\
&\quad = 2 \min\{\cos d(y^0, y), \cos d(y^1, y)\} \sin \frac{d(y^0, y^1)}{2} \cos \frac{(2\sigma - 1) d(y^0, y^1)}{2}.
\end{aligned}$$

Dividing above by $2\sin(d(y^0, y^1)/2)$, we have

$$\begin{aligned}
&\cos d(w, y) \cos \frac{d(y^0, y^1)}{2} \\
&\qquad \geq \min\{\cos d(y^0, y), \cos d(y^1, y)\} \cos \frac{(2\sigma - 1) d(y^0, y^1)}{2} \\
&\qquad \geq \min\{\cos d(y^0, y), \cos d(y^1, y)\} \cos \frac{(1 - 2a) d(y^0, y^1)}{2}.
\end{aligned}$$

Moreover, dividing above by $\cos((1-2a)d(y^0, y^1)/2)$, we have

$$\begin{aligned}
&\min\{\cos d(y^0, y), \cos d(y^1, y)\} \\
&\le \cos d(w, y) \frac{\cos \dfrac{(1-2a)d(y^0, y^1)}{2} \cos(ad(y^0, y^1)) - \sin \dfrac{(1-2a)d(y^0, y^1)}{2} \sin(ad(y^0, y^1))}{\cos \dfrac{(1-2a)d(y^0, y^1)}{2}} \\
&\le \cos d(w, y) \cos(ad(y^0, y^1)).
\end{aligned}$$

This completes the proof. □

Lemma 4.2. (Kasahara and Kimura [14]) *Let X be a* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\sigma^l \in [a, 1-a]$ for every $l = 0, 1, \dots, N-1$. For given points $y, y^k \in X$ for every $k = 0, 1, \dots, N$, define $w^l \in X$ by*

$$w^N = y^N \text{ and } w^l = \sigma^l y^l \oplus (1-\sigma^l) w^{l+1}$$

for every $l = 0, 1, \dots, N-1$. Then

$$\cos d(w^0, y) \cos(ad(y^0, w^1)) \ge \min_{k \in \{0,1,\dots,N\}} \cos d(y^k, y).$$

Proof. By Lemma 4.1,

$$\cos d(w^0, y) \cos(ad(y^0, w^1)) \ge \min\{\cos d(y^0, y), \cos d(w^1, y)\}.$$

We also have

$$\begin{aligned}
\cos d(w^l, y) &\ge \cos d(w^l, y) \cos(ad(y^l, w^{l+1})) \\
&\ge \min\{\cos d(y^l, y), \cos d(w^{l+1}, y)\}
\end{aligned}$$

for $l = 1, 2, \dots, N-1$. Hence $\cos d(w^0, y) \cos(ad(y^0, w^1)) \ge \min_{k \in \{0,1,\dots,N\}} \cos d(y^k, y)$. This completes the proof. □

Corollary 4.3. (Kasahara and Kimura [14]) *Let X be a* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let T^k be a quasinonexpansive mapping from X to X for every $k = 0, 1, \dots, N$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\sigma^l \in [a, 1-a]$ for every $l = 0, 1, \dots, N-1$. Define $U^l : X \to X$ by*

$$U^N = T^N \text{ and } U^l = \sigma^l T^l \oplus (1-\sigma^l) U^{l+1}$$

for every $l = 0, 1, \dots, N-1$. Let $x \in X$ and $p \in \bigcap_{k=0}^N F(T^k)$. Then

$$\cos d(U^0 x, p) \cos(ad(T^0 x, U^1 x)) \ge \cos d(x, p).$$

Next, we show several properties of a sequence of resolvents. Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that $\operatorname{argmin}_X f \ne \emptyset$ and let $\{\lambda_n\}$ be a real sequence such that $\inf \lambda_n > 0$. Then we know that $\{R_{\lambda_n f}\}$ is a strongly quasinonexpansive sequence and Δ-demiclosed sequence; see [17]. Therefore we obtain the following results by using Lemma 2.7 inductively.

Lemma 4.4. (Kasahara and Kimura [14]) *Let X be a complete* CAT(1) *space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty,\infty]$ for every $k = 0,1,\dots,N$ such that $\bigcap_{k=0}^N \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\sigma^l \in [a, 1-a]$ for every $l = 0,1,\dots,N-1$ and $\lambda^k \in [a,\infty[$ for every $k = 0,1,\dots,N$. Let $R_{\lambda^k f^k}$ be the resolvent of $\lambda^k f^k$ for every $k = 0,1,\dots,N$. Define $U^l : X \to X$ by*

$$U^N = R_{\lambda^N f^N} \text{ and } U^l = \sigma^l R_{\lambda^l f^l} \oplus (1-\sigma^l)U^{l+1}$$

for every $l = 0,1,\dots,N-1$. Then

$$F(U^0) = \bigcap_{k=0}^N \operatorname*{argmin}_X f^k.$$

Lemma 4.5. (Kasahara and Kimura [14]) *Let X be a complete* CAT(1) *space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let $\{T_n\}$ be a strongly quasinonexpansive sequence. Let f be a proper lower semicontinuous convex function from X into $]-\infty,\infty]$ such that $\bigcap_{n=1}^\infty F(T_n) \cap \operatorname{argmin}_X f \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\sigma_n\} \subset [a, 1-a]$ and $\{\lambda_n\} \subset [a,\infty[$. Let $R_{\lambda_n f}$ be the resolvent of $\lambda_n f$ for every $n \in \mathbb{N}$. Then $\{\sigma_n R_{\lambda_n f} \oplus (1-\sigma_n)T_n\}$ is a strongly quasinonexpansive sequence.*

Proof. Let $V_n = \sigma_n R_{\lambda_n f} \oplus (1-\sigma_n)T_n$ for every $n \in \mathbb{N}$. By Lemma 2.7, V_n is a quasinonexpansive mapping for every $n \in \mathbb{N}$. By Corollary 4.3, for $\{x_n\} \subset X$ and $p \in \bigcap_{n=1}^\infty F(T_n) \cap \operatorname{argmin}_X f$ such that $\lim_{n\to\infty} \cos d(x_n,p)/\cos d(V_n x_n, p) = 1$ and $\sup_{n\in\mathbb{N}} d(x_n,p) < \pi/2$, we have

$$\cos d(V_n x_n, p) \cos(a d(R_{\lambda_n f} x_n, T_n x_n)) \geq \cos d(x_n, p)$$

and thus

$$\cos(a d(R_{\lambda_n f} x_n, T_n x_n)) \geq \frac{\cos d(x_n,p)}{\cos d(V_n x_n, p)} \to 1.$$

That is, $\lim_{n\to\infty} d(R_{\lambda_n f} x_n, T_n x_n) = 0$. So we have

$$\lim_{n\to\infty} d(T_n x_n, V_n x_n) = \lim_{n\to\infty} \sigma_n d(R_{\lambda_n f} x_n, T_n x_n) = 0.$$

Since $1 = \lim_{n\to\infty} \cos d(x_n,p)/\cos d(V_n x_n,p) = \lim_{n\to\infty} \cos d(x_n,p)/\cos d(T_n x_n,p)$, we have

$$\lim_{n\to\infty} d(T_n x_n, x_n) = 0.$$

Hence, we obtain

$$d(V_n x_n, x_n) \leq d(V_n x_n, T_n x_n) + d(T_n x_n, x_n) \to 0.$$

This completes the proof. □

Corollary 4.6. (Kasahara and Kimura [14]) *Let X be a complete* CAT(1) *space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty,\infty]$ for every $k = 0,1,\dots,N$ such that $\bigcap_{k=0}^N \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\sigma_n^l\} \subset [a, 1-a]$ for every $l = 0,1,\dots,N-1$ and $\{\lambda_n^k\} \subset [a,\infty[$ for every $k = 0,1,\dots,N$. Let $R_{\lambda_n^k f^k}$ be the resolvent of $\lambda_n^k f^k$ for every $k = 0,1,\dots,N$ and $n \in \mathbb{N}$. Define $U_n^l : X \to X$ by*

$$U_n^N = R_{\lambda_n^N f^N} \text{ and } U_n^l = \sigma_n^l R_{\lambda_n^l f^l} \oplus (1-\sigma_n^l)U_n^{l+1}$$

for every $l = 0,1,\dots,N-1$ and $n \in \mathbb{N}$. Then $\{U_n^0\}$ is a strongly quasinonexpansive sequence.

Lemma 4.7. (Kasahara and Kimura [14]) *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $\{T_n\}$ be a quasinonexpansive and Δ-demiclosed sequence. Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ such that $\bigcap_{n=1}^{\infty} F(T_n) \cap \operatorname{argmin}_X f \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\sigma_n\} \subset [a, 1-a]$ and $\{\lambda_n\} \subset [a, \infty[$. Let $R_{\lambda_n f}$ be the resolvent of $\lambda_n f$ for every $n \in \mathbb{N}$. Then $\{\sigma_n R_{\lambda_n f} \oplus (1-\sigma_n) T_n\}$ is a Δ-demiclosed sequence.*

Proof. Let $V_n = \sigma_n R_{\lambda_n f} \oplus (1-\sigma_n) T_n$ for every $n \in \mathbb{N}$. Let $p \in \bigcap_{n=1}^{\infty} F(T_n) \cap \operatorname{argmin}_X f$, $\{x_n\} \subset X$, and $z \in X$ such that $\lim_{n\to\infty} d(V_n x_n, x_n) = 0$ and suppose that $\{x_n\}$ is Δ-convergent to z. Then

$$\cos d(V_n x_n, p) \cos(a d(R_{\lambda_n f} x_n, T_n x_n)) \geq \cos d(x_n, p)$$

and thus

$$\begin{aligned} 1 \geq \cos(a d(R_{\lambda_n f} x_n, T_n x_n)) &\geq \frac{\cos d(x_n, p)}{\cos d(V_n x_n, p)} \\ &\geq \frac{\cos(d(x_n, V_n x_n) + d(V_n x_n, p))}{\cos d(V_n x_n, p)} \to 1. \end{aligned}$$

Hence $\lim_{n\to\infty} d(R_{\lambda_n f} x_n, T_n x_n) = 0$. Thus we have

$$\begin{aligned} d(R_{\lambda_n f} x_n, V_n x_n) &= (1-\sigma_n) d(R_{\lambda_n f} x_n, T_n x_n) \\ &\leq (1-a) d(R_{\lambda_n f} x_n, T_n x_n) \to 0. \end{aligned}$$

Since $R_{\lambda_n f}$ is a Δ-demiclosed sequence, we have $R_{\lambda_n f} z = z$. Similarly,

$$\begin{aligned} d(T_n x_n, V_n x_n) &= \sigma_n d(R_{\lambda_n f} x_n, T_n x_n) \\ &\leq (1-a) d(R_{\lambda_n f} x_n, T_n x_n) \to 0. \end{aligned}$$

Since $\{T_n\}$ is a Δ-demiclosed sequence, we have $T_n z = z$. Hence $V_n z = z$. This completes the proof. ☐

Corollary 4.8. (Kasahara and Kimura [14]) *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 0, 1, \ldots, N$ such that $\bigcap_{k=0}^{N} \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\sigma_n^l\} \subset [a, 1-a]$ for every $l = 0, 1, \ldots, N-1$ and $\{\lambda_n^k\} \subset [a, \infty[$ for every $k = 0, 1, \ldots, N$. Let $R_{\lambda_n^k f^k}$ be the resolvent of $\lambda_n^k f^k$ for every $k = 0, 1, \ldots, N$ and $n \in \mathbb{N}$. Define $U_n^l : X \to X$ by*

$$U_n^N = R_{\lambda_n^N f^N} \text{ and } U_n^l = \sigma_n^l R_{\lambda_n^l f^l} \oplus (1-\sigma_n^l) U_n^{l+1}$$

for every $l = 0, 1, \ldots, N-1$ and $n \in \mathbb{N}$. Then $\{U_n^0\}$ is a Δ-demiclosed sequence.

By using these lemmas, we obtain the following result. Note that this result has been proved in [11]. For the sake of completeness, we give the proof.

Theorem 4.9. (Kasahara and Kimura [14]) *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 0, 1, \ldots, N$ such that $F = \bigcap_{k=0}^{N} \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\sigma_n^l\} \subset [a, 1-a]$ for every $l = 0, 1, \ldots, N-1$ and $\{\lambda_n^k\} \subset$*

$[a, \infty[$ *for every* $k = 0, 1, \ldots, N$. *Let* $R_{\lambda_n^k f^k}$ *be the resolvent of* $\lambda_n^k f^k$ *for every* $k = 0, 1, \ldots, N$ *and* $n \in \mathbb{N}$. *Define* $U_n^l : X \to X$ *by*

$$U_n^N = R_{\lambda_n^N f^N} \text{ and } U_n^l = \sigma_n^l R_{\lambda_n^l f^l} \oplus (1 - \sigma_n^l) U_n^{l+1}$$

for every $l = 0, 1, \ldots, N-1$ *and* $n \in \mathbb{N}$. *Let* $\{\alpha_n\}$ *be a real sequence in* $]0, 1[$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *For given points* $u, x_1 \in X$, *let* $\{x_n\}$ *be the sequence in* X *generated by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) U_n^0 x_n$$

for $n \in \mathbb{N}$. *Suppose that one of the following conditions holds:*

(a) $\sup_{v, v' \in X} d(v, v') < \pi/2$;
(b) $d(u, P_F u) < \pi/4$ *and* $d(u, P_F u) + d(x_0, P_F u) < \pi/2$;
(c) $\sum_{n=0}^{\infty} \alpha_n^2 = \infty$.

Then $\{x_n\}$ *converges to* $P_F u$.

To prove this theorem, we also employ the technique proposed in [22]. Note that $F = \bigcap_{k=0}^{N} \operatorname{argmin}_X f^k$.

Proof. Let $p = P_F u$ and let

$$\begin{aligned}
s_n &= 1 - \cos d(x_n, p), \\
t_n &= 1 - \frac{\cos d(u, p)}{\sin d(u, U_n^0 x_n) \tan(\frac{\alpha_n}{2} d(u, U_n^0 x_n)) + \cos d(u, U_n^0 x_n)}, \\
\beta_n &= \begin{cases} 1 - \dfrac{\sin((1 - \alpha_n) d(u, U_n^0 x_n))}{\sin d(u, U_n^0 x_n)} & (u \neq U_n^0 x_n), \\ \alpha_n & (u = U_n^0 x_n) \end{cases}
\end{aligned}$$

for $n \in \mathbb{N}$. Since U_n^0 is a quasinonexpansive mapping, it follows from Lemma 2.10 that

$$s_{n+1} \le (1 - \beta_n)(1 - \cos d(U_n^0 x_n, p)) + \beta_n t_n \le (1 - \beta_n) s_n + \beta_n t_n$$

for $n \in \mathbb{N}$. By Lemma 2.8, we have

$$\begin{aligned}
\cos d(x_{n+1}, p) &= \cos d(\alpha_n u \oplus (1 - \alpha_n) U_n^0 x_n, p) \\
&\ge \alpha_n \cos d(u, p) + (1 - \alpha_n) \cos d(U_n^0 x_n, p) \\
&\ge \alpha_n \cos d(u, p) + (1 - \alpha_n) \cos d(x_n, p) \\
&\ge \min\{\cos d(u, p), \cos d(x_n, p)\}
\end{aligned}$$

for $n \in \mathbb{N}$. So we have

$$\cos d(x_n, p) \ge \min\{\cos d(u, p), \cos d(x_0, p)\} = \cos \max\{d(u, p), d(x_0, p)\} > 0$$

for $n \in \mathbb{N}$. Hence $\sup_{n\in\mathbb{N}} d(x_n, p) \le \max\{d(u, p), d(x_0, p)\} < \pi/2$. Next, we will show each of the conditions (a),(b) and (c) implies that $\sum_{n=0}^{\infty} \beta_n = \infty$. For the conditions (a) and (b), let $M = \sup_{n\in\mathbb{N}} d(u, U_n^0 x_n)$. Thus we will show $M < \pi/2$. In the case of (a), it is obvious. In the case of (b), since $\sup_{n\in\mathbb{N}} d(x_n, p) \le \max\{d(u, p), d(x_0, p)\}$, we have

$$\begin{aligned}
M &\le \sup_{n\in\mathbb{N}} (d(u, p) + d(U_n^0 x_n, p)) \\
&\le \sup_{n\in\mathbb{N}} (d(u, p) + d(x_n, p)) \\
&\le \max\{2d(u, p), d(u, p) + d(x_0, p)\} < \pi/2.
\end{aligned}$$

Thus in the cases of (a) and (b), we have

$$\begin{aligned}\beta_n &\geq 1 - \frac{\sin((1-\alpha_n)M)}{\sin M}\\ &= \frac{2}{\sin M}\sin\left(\frac{\alpha_n}{2}M\right)\cos\left(\left(1-\frac{\alpha_n}{2}\right)M\right)\\ &\geq \alpha_n \cos M\end{aligned}$$

for $n \in \mathbb{N}$. Since $\sum_{n=0}^{\infty}\alpha_n = \infty$, each of the conditions (a) and (b) implies that $\sum_{n=0}^{\infty}\beta_n = \infty$. In the case of (c), we have

$$\beta_n \geq 1 - \sin\frac{(1-\alpha_n)\pi}{2} = 1 - \cos\frac{\alpha_n}{2} \geq \frac{\alpha_n^2\pi^2}{16}$$

for $n \in \mathbb{N}$. Hence the condition (c) also implies that $\sum_{n=0}^{\infty}\beta_n = \infty$. For $\{s_{n_i}\} \subset \{s_n\}$ with a nondecreasing real sequence $\{n_i\} \subset \mathbb{N}$ such that $\liminf_{i\to\infty}(s_{n_i+1} - s_{n_i}) \geq 0$, we have

$$\begin{aligned}0 &\leq \liminf_{i\to\infty}(s_{n_i+1} - s_{n_i})\\ &= \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(x_{n_i+1}, p))\\ &\leq \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - (\alpha_{n_i}\cos d(u,p) + (1-\alpha_{n_i})\cos d(U_{n_i}^0 x_{n_i}, p)))\\ &= \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(U_{n_i}^0 x_{n_i}, p))\\ &\leq \limsup_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(U_{n_i}^0 x_{n_i}, p)) \leq 0.\end{aligned}$$

Hence $\lim_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(U_{n_i}^0 x_{n_i}, p)) = 0$. Since $\sup_{n\in\mathbb{N}} d(U_n^0 x_n, p) < \pi/2$, we have $\lim_{i\to\infty}(\cos d(x_{n_i}, p)/\cos d(U_{n_i}^0 x_{n_i}, p)) = 1$. Since $\{U_{n_i}^0\}$ is a strongly quasinonexpansive sequence, it follows that $\lim_{i\to\infty} d(x_{n_i}, U_{n_i}^0 x_{n_i}) = 0$. Let $\{x_{n_j}\} \subset \{x_{n_i}\}$ be a Δ-convergent subsequence such that $\lim_{j\to\infty} d(u, x_{n_j}) = \liminf_{i\to\infty} d(u, x_{n_i})$. Since $\{U_n^0\}$ is a Δ-demiclosed sequence, $\{U_{n_j}^0\}$ is also a Δ-demiclosed sequence. $\{U_{n_j}^0\}$ is a Δ-demiclosed sequence and $\lim_{j\to\infty} d(x_{n_j}, U_{n_j}^0 x_{n_j}) = 0$, the Δ-limit $z \in \{x_{n_j}\}$ belongs to F. By Lemma 2.5, we have

$$\liminf_{i\to\infty} d(u, U_{n_i}^0 x_{n_i}) = \liminf_{i\to\infty} d(u, x_{n_i}) = \lim_{j\to\infty} d(u, x_{n_j}) \geq d(u,z) \geq d(u,p).$$

Hence

$$\begin{aligned}\limsup_{i\to\infty} t_{n_i} &= \limsup_{i\to\infty}\left(1 - \frac{\cos d(u,p)}{\sin d(u, U_{n_i}^0 x_{n_i})\tan(\frac{\alpha_{n_i}}{2} d(u, U_{n_i}^0 x_{n_i}) + \cos d(u, U_{n_i}^0 x_{n_i})}\right)\\ &= \limsup_{i\to\infty}\left(1 - \frac{\cos d(u,p)}{\cos d(u, U_{n_i}^0 x_{n_i})}\right) \leq 0.\end{aligned}$$

From Lemma 2.14, we have $\lim_{n\to\infty} s_n = 0$. Therefore $\{x_n\}$ converges to p. This completes the proof. □

Motivated by this result, we consider the following theorem, which uses a similar iterative scheme to that in the previous theorem.

Theorem 4.10. (Kasahara and Kimura [14]) *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 0, 1, \ldots, N$ such that $F = \bigcap_{k=0}^{N} \operatorname{argmin}_X f^k \neq \emptyset$. For a* *given real number* $a \in \left]0, \frac{1}{2}\right]$, *let* $\{\sigma_n^l\} \subset [a, 1-a]$ *for every* $l = 0, 1, \dots, N-1$ *and* $\{\lambda_n^k\} \subset [a, \infty[$ *for every* $k = 0, 1, \dots, N$. *Let* $R_{\lambda_n^k f^k}$ *be the resolvent of* $\lambda_n^k f^k$ *for every* $k = 0, 1, \dots, N$ *and* $n \in \mathbb{N}$. *Define* $U_n^l : X \to X$ *by*

$$U_n^N = R_{\lambda_n^N f^N} \text{ and } U_n^l = \sigma_n^l R_{\lambda_n^l f^l} \oplus (1 - \sigma_n^l) U_n^{l+1}$$

for every $l = 0, 1, \dots, N-1$ *and* $n \in \mathbb{N}$. *Let* $\{\alpha_n\}$ *be a real sequence in* $[a, 1-a]$. *For a given point* $x_1 \in X$, *let* $\{x_n\}$ *be the sequence in* X *generated by*

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) U_n^0 x_n$$

for $n \in \mathbb{N}$. *Then* $\{x_n\}$ Δ*-converges to a point of* F.

Proof. Let $z \in F$. Since U_n^0 is a quasinonexpansive mapping, it follows from Lemma 2.8 that

$$\begin{aligned} \cos d(x_{n+1}, z) &\geq \alpha_n \cos d(x_n, z) + (1 - \alpha_n) \cos d(U_n^0 x_n, z) \\ &\geq \cos d(x_n, z). \end{aligned}$$

Thus we have $d(x_{n+1}, z) \leq d(x_n, z)$ for $n \in \mathbb{N}$. There exists $D = \lim_{n\to\infty} d(x_n, z) \leq d(x_1, z) < \pi/2$. From Lemma 2.6, we get

$$\begin{aligned} &\cos d(x_{n+1}, z) \sin d(x_n, U_n^0 x_n) \\ &\quad \geq \cos d(x_n, z) \sin \alpha_n d(x_n, U_n^0 x_n) + \cos d(U_n^0 x_n, z) \sin(1 - \alpha_n) d(x_n, U_n^0 x_n) \\ &\quad \geq 2 \cos d(x_n, z) \sin \frac{d(x_n, U_n^0 x_n)}{2} \cos \frac{(2\alpha_n - 1) d(x_n, U_n^0 x_n)}{2}. \end{aligned}$$

If $d(x_n, U_n^0 x_n) \neq 0$, we obtain

$$\cos d(x_{n+1}, z) \cos \frac{d(x_n, U_n^0 x_n)}{2} \geq \cos d(x_n, z) \cos \frac{(2\alpha_n - 1) d(x_n, U_n^0 x_n)}{2}.$$

Since $\{\alpha_n\} \subset [a, 1-a]$, we get

$$1 > \frac{\cos \frac{d(x_n, U_n^0 x_n)}{2}}{\cos \frac{(1-2a) d(x_n, U_n^0 x_n)}{2}} \geq \frac{\cos \frac{d(x_n, U_n^0 x_n)}{2}}{\cos \frac{(2\alpha_n - 1) d(x_n, U_n^0 x_n)}{2}} \geq \frac{\cos d(x_n, z)}{\cos d(x_{n+1}, z)}.$$

Since $D = \lim_{n\to\infty} d(x_n, z) \leq d(x_1, z) < \pi/2$, we have

$$\lim_{n\to\infty} \frac{\cos \frac{d(x_n, U_n^0 x_n)}{2}}{\cos \frac{(1-2a) d(x_n, U_n^0 x_n)}{2}} = 1$$

and hence $\lim_{n\to\infty} d(x_n, U_n^0 x_n) = 0$. Let x_0 be an asymptotic center of $\{x_n\}$ and y an asymptotic center of any subsequence $\{x_{n_k}\} \subset \{x_n\}$. There exists $\{x_{n_{k_l}}\} \subset \{x_{n_k}\}$ such that $\{x_{n_{k_l}}\}$ Δ-converges to w. Since $\{U_{n_{k_l}}^0\}$ is a Δ-demiclosed sequence and $\lim_{n\to\infty} d(U_{n_{k_l}}^0 x_{n_{k_l}}, x_{n_{k_l}}) = 0$, we obtain $w \in F$. Since there exists $\lim_{n\to\infty} d(x_{n_k}, w)$, we have

$$\begin{aligned} \limsup_{k\to\infty} d(x_{n_k}, w) &= \lim_{k\to\infty} d(x_{n_k}, w) \\ &= \lim_{l\to\infty} d(x_{n_{k_l}}, w) \\ &\leq \limsup_{l\to\infty} d(x_{n_{k_l}}, y) \\ &\leq \limsup_{k\to\infty} d(x_{n_k}, y). \end{aligned}$$

Hence we obtain $y = w \in F$. Similarly, we get $x_0 = y$. Therefore $\{x_n\}$ Δ-converges to $x_0 \in F$. $\square$

Chapter 5

Mann iterative sequence with a balanced mapping for finitely many resolvent operators in CAT(0) spaces

In a complete CAT(0) space, Kimura and Hasegawa [7] define a balanced mapping. The following is the definition and properties of a balanced mapping in a complete CAT(0) space.

Let X be a complete CAT(0) space. Let T^k be a nonexpansive mapping from X to X for $k = 1, 2, \ldots, N$. Let $\{\alpha^k\} \subset \,]0, 1[$ for $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. A balanced mapping U from X to X is defined by

$$Ux = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} d(T^k x, y)^2$$

for every $x \in X$. They find that this mapping U is defined as a single-valued mapping, has nonexpansiveness and $F(U) = \bigcap_{k=1}^{N} F(T^k)$. Moreover, they proved the convergence theorems by using a balanced mapping for a finite number of nonexpansive mappings in a complete CAT(0) space.

In this chapter, we prove the convergence of Mann type iteration by using a balanced mapping for a finite number of resolvent operators in a complete CAT(0) space. In Chapter 4, we showed the convergence theorem which uses iterative sequences generated by finitely many resolvent operators dependent on the order of convex combination. On the other hand, in this chapter, we use a balanced mapping generated by finitely many mappings without regard of order.

Theorem 5.1. (Kasahara and Kimura [13]) *Let X be a complete* CAT(0) *space. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 1, 2, \ldots, N$ such that $F = \bigcap_{k=1}^{N} \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\left\{\alpha_n^k\right\}, \{\beta_n\} \subset [a, 1-a]$ and $\left\{\lambda_n^k\right\} \subset [a, \infty[$ for every $k = 1, 2, \ldots, N$ and $n \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_n^k = 1$. Let $Q_{\lambda_n^k f^k}$ be the resolvent of $\lambda_n^k f^k$ for every $k = 1, 2, \ldots, N$ and $n \in \mathbb{N}$. Define a mapping U_n from X to X by*

$$U_n x = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, y\right)^2$$

for every $x \in X$ and $n \in \mathbb{N}$. For a given point $x_1 \in X$, let $\{x_n\}$ be a sequence in X generated *by*

$$x_{n+1} = \beta_n x_n \oplus (1-\beta_n) U_n x_n$$

for every $n \in \mathbb{N}$. *Then* $\{x_n\}$ Δ*-converges to a point in* F.

Proof. Let $z \in F$. Then we have

$$\begin{aligned} d(x_{n+1}, z)^2 &= d(\beta_n x_n \oplus (1-\beta_n) U_n x_n, z)^2 \\ &\leq \beta_n d(x_n, z)^2 + (1-\beta_n) d(U_n x_n, z)^2 - \beta_n (1-\beta_n) d(U_n x_n, x_n)^2 \\ &\leq d(x_n, z)^2 - \beta_n (1-\beta_n) d(U_n x_n, x_n)^2 \\ &\leq d(x_n, z)^2. \end{aligned}$$

Thus, we obtain $d(x_{n+1}, z) \leq d(x_n, z)$ for all $n \in \mathbb{N}$ and there exists

$$D = \lim_{n\to\infty} d(x_n, z) \leq d(x_1, z).$$

Since $0 < a^2 \leq \beta_n(1-\beta_n)$, we have $\lim_{n\to\infty} d(U_n x_n, x_n) = 0$. It follows that

$$\begin{aligned} \lim_{n\to\infty} d(x_n, z) &\leq \lim_{n\to\infty} \left(d(x_n, U_n x_n) + d(U_n x_n, z)\right) \\ &= \lim_{n\to\infty} d(U_n x_n, z) \\ &= \lim_{n\to\infty} d(U_n x_n, U_n z) \\ &\leq \lim_{n\to\infty} d(x_n, z). \end{aligned}$$

Thus we get $\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(U_n x_n, z) = D$. By Lemma 2.1,

$$\begin{aligned} \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x_n, U_n x_n\right)^2 &\leq \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x_n, z\right)^2 - d(z, U_n x_n)^2 \\ &\leq \sum_{k=1}^{N} \alpha_n^k d(x_n, z)^2 - d(z, U_n x_n)^2 \\ &= d(x_n, z)^2 - d(z, U_n x_n)^2. \end{aligned}$$

Since $0 < a \leq \alpha_n^k$, we obtain $\lim_{n\to\infty} d(Q_{\lambda_n^k f^k} x_n, U_n x_n) = 0$ for every $k = 1, 2, \dots, N$. Since $\lim_{n\to\infty} d(U_n x_n, x_n) = 0$, we also get $\lim_{n\to\infty} d(Q_{\lambda_n^k f^k} x_n, x_n) = 0$ for every $k = 1, 2, \dots, N$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ which Δ-converges to a point $x_0 \in X$. Assume $x_0 \notin \operatorname{argmin}_X f^1$. Then we get

$$\begin{aligned} \limsup_{r\to\infty} d\left(x_{n_r}, x_0\right) &< \limsup_{r\to\infty} d\left(x_{n_r}, Q_{\lambda_n^1 f^1} x_0\right) \\ &\leq \limsup_{r\to\infty} \left(d\left(x_{n_r}, Q_{\lambda_n^1 f^1} x_{n_r}\right) + d\left(Q_{\lambda_n^1 f^1} x_{n_r}, Q_{\lambda_n^1 f^1} x_0\right)\right) \\ &\leq \limsup_{r\to\infty} d\left(x_{n_r}, x_0\right). \end{aligned}$$

We obtain a contradiction and $x_0 \in \operatorname{argmin}_X f^1$. Similarly, we can show $x_0 \in \operatorname{argmin}_X f^k$ for all $k = 1, 2, \dots, N$. Suppose that there are two subsequences $\{u_i\}$ and $\{v_i\}$ of $\{x_n\}$ which

Δ-converges to u_0 and v_0, respectively. Then we obtain that $u_0, v_0 \in \bigcap_{k=1}^N \operatorname{argmin}_X f^k$ and both $\{d(x_n, u_0)\}$ and $\{d(x_n, v_0)\}$ have limits. Assume that $u_0 \neq v_0$, then we obtain

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, u_0) &= \lim_{i\to\infty} d(u_i, u_0) \\
&< \lim_{i\to\infty} d(u_i, v_0) \\
&= \lim_{n\to\infty} d(x_n, v_0) \\
&= \lim_{i\to\infty} d(v_i, v_0) \\
&< \lim_{i\to\infty} d(v_i, u_0) \\
&= \lim_{n\to\infty} d(x_n, u_0)
\end{aligned}$$

It is a contradiction and thus $u_0 = v_0$. Hence we obtain $\{x_n\}$ Δ-converges to $x_0 \in F$. □

Theorem 5.2. (Kasahara and Kimura [13]) *Let X be a complete* CAT(0) *space. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 1, 2, \ldots, N$ such that $F = \bigcap_{k=1}^N \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\beta_n\} \subset]0, 1[$, $\left\{\alpha_n^k\right\} \subset [a, 1-a]$ and $\left\{\lambda_n^k\right\} \subset [a, \infty[$ for every $k = 1, 2, \ldots, N$ and $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^\infty \beta_n = \infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$, $\sum_{k=1}^N \alpha_n^k = 1$, $\sum_{n=1}^\infty \sum_{k=1}^N \left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$ and $\sum_{n=1}^\infty \sum_{k=1}^N \left|\lambda_{n+1}^k - \lambda_n^k\right| < \infty$. Let $Q_{\lambda_n^k f^k}$ be the resolvent of $\lambda_n^k f^k$ for every $k = 1, 2, \ldots, N$ and $n \in \mathbb{N}$. Define a mapping U_n from X to X by*

$$U_n x = \operatorname*{argmin}_{y\in X} \sum_{k=1}^N \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, y\right)^2$$

for every $x \in X$ and $n \in \mathbb{N}$. For given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by

$$x_{n+1} = \beta_n u \oplus (1 - \beta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_F u$.

Proof. We show boundedness of $\{x_n\}$ and $\{U_n x_n\}$. Let $z \in F$. Then we have

$$\begin{aligned}
d(x_{n+1}, z) &= d(\beta_n u \oplus (1 - \beta_n) U_n x_n, z) \\
&\leq \beta_n d(u, z) + (1 - \beta_n) d(U_n x_n, z) \\
&\leq \beta_n d(u, z) + (1 - \beta_n) d(x_n, z) \\
&\leq \max\{d(u, z), d(x_n, z)\} \\
&\leq \max\{d(u, z), d(x_1, z)\}.
\end{aligned}$$

Thus we obtain $\{x_n\}$ and $\{U_n x_n\}$ are bounded. We also have

$$\begin{aligned}
d(x_{n+2}, x_{n+1}) &\le d(\beta_{n+1} u \oplus (1-\beta_{n+1}) U_{n+1} x_{n+1}, \beta_n u \oplus (1-\beta_n) U_n x_n) \\
&\le d(\beta_{n+1} u \oplus (1-\beta_{n+1}) U_{n+1} x_{n+1}, \beta_n u \oplus (1-\beta_n) U_{n+1} x_{n+1}) \\
&\quad + d(\beta_n u \oplus (1-\beta_n) U_{n+1} x_{n+1}, \beta_n u \oplus (1-\beta_n) U_n x_n) \\
&\le |\beta_{n+1} - \beta_n|\, d(U_{n+1} x_{n+1}, u) + (1-\beta_n) d(U_{n+1} x_{n+1}, U_n x_n) \\
&\le |\beta_{n+1} - \beta_n|\, d(U_{n+1} x_{n+1}, u) \\
&\quad + (1-\beta_n)(d(U_{n+1} x_{n+1}, U_n x_{n+1}) + d(U_n x_{n+1}, U_n x_n)) \\
&\le (1-\beta_n) d(x_{n+1}, x_n) \\
&\quad + |\beta_{n+1} - \beta_n|\, d(U_{n+1} x_{n+1}, u) + d(U_{n+1} x_{n+1}, U_n x_{n+1}).
\end{aligned}$$

We show $\sum_{n=1}^{\infty} d(U_{n+1} x_{n+1}, U_n x_{n+1}) < \infty$. Let $t \in \,]0,1[$. For all $x \in X$, we have

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, U_n x\right)^2 \\
&\quad \le \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, t U_n x \oplus (1-t) U_{n+1} x\right)^2 \\
&\quad \le t \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, U_n x\right)^2 + (1-t) \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, U_{n+1} x\right)^2 \\
&\qquad - t(1-t) \sum_{k=1}^{N} \alpha_n^k d(U_n x, U_{n+1} x)^2 \\
&\quad = t \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, U_n x\right)^2 + (1-t) \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, U_{n+1} x\right)^2 \\
&\qquad - t(1-t) d(U_n x, U_{n+1} x)^2.
\end{aligned}$$

Since $1 - t > 0$, we obtain

$$t d(U_{n+1} x, U_n x)^2 \le \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, U_{n+1} x\right)^2 - \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, U_n x\right)^2.$$

Letting $t \to 1$, we have

$$d(U_{n+1} x, U_n x)^2 \le \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, U_{n+1} x\right)^2 - \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, U_n x\right)^2.$$

Similarly, we have

$$d(U_{n+1} x, U_n x)^2 \le \sum_{k=1}^{N} \alpha_{n+1}^k d\left(Q_{\lambda_{n+1}^k f^k} x, U_n x\right)^2 - \sum_{k=1}^{N} \alpha_{n+1}^k d\left(Q_{\lambda_{n+1}^k f^k} x, U_{n+1} x\right)^2.$$

From the above two inequalities, we get

$$d(U_{n+1}x,U_nx)^2 \le \frac{1}{2}\sum_{k=1}^{N}\Big(\alpha_n^k d\left(Q_{\lambda_n^k f^k}x,U_{n+1}x\right)^2-\alpha_n^k d\left(Q_{\lambda_n^k f^k}x,U_nx\right)^2 + \alpha_{n+1}^k d\left(Q_{\lambda_{n+1}^k f^k}x,U_nx\right)^2-\alpha_{n+1}^k d\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)^2\Big).$$

Put $D=d(Q_{\lambda_n^k f^k}x,Q_{\lambda_{n+1}^k f^k}x)$. We obtain

$$\begin{aligned}
&\alpha_n^k d\left(Q_{\lambda_n^k f^k}x,U_{n+1}x\right)^2-\alpha_{n+1}^k d\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)^2\\
&\le \alpha_n^k\left(D+d\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)\right)^2-\alpha_{n+1}^k d\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)^2\\
&=\alpha_n^k\left(D^2+2Dd\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)+d\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)^2\right)\\
&\quad -\alpha_{n+1}^k d\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)^2\\
&\le \alpha_n^k\left(D^2+2Dd\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)\right)+\left|\alpha_{n+1}^k-\alpha_n^k\right| d\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)^2.
\end{aligned}$$

Summarizing above inequalities, we get

$$\begin{aligned}
&d(U_{n+1}x,U_nx)^2\\
&\le \frac{1}{2}\sum_{k=1}^{N}\Big(\left|\alpha_{n+1}^k-\alpha_n^k\right|\left(d\left(Q_{\lambda_n^k f^k}x,U_nx\right)^2+d\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)^2\right)\\
&\quad+\alpha_n^k\left(D^2+2Dd\left(Q_{\lambda_{n+1}^k f^k}x,U_{n+1}x\right)\right)+\alpha_{n+1}^k\left(D^2+2Dd\left(Q_{\lambda_n^k f^k}x,U_nx\right)\right)\Big)\\
&\le \sum_{k=1}^{N}\left(4\left|\alpha_{n+1}^k-\alpha_n^k\right| d(x,z)^2+D^2+4Dd(x,z)\right).
\end{aligned}$$

On the other hand, by Lemma 2.3, we have

$$\begin{aligned}
&d\left(Q_{\lambda_{n+1}^k f^k}x,Q_{\lambda_n^k f^k}x\right)^2\\
&\le \frac{\lambda_{n+1}^k-\lambda_n^k}{\lambda_{n+1}^k+\lambda_n^k}\left(d\left(Q_{\lambda_{n+1}^k f^k}x,x\right)^2-d\left(Q_{\lambda_n^k f^k}x,x\right)^2\right)\\
&\le \frac{\left|\lambda_{n+1}^k-\lambda_n^k\right|}{2a}\left(d\left(Q_{\lambda_{n+1}^k f^k}x,x\right)+d\left(Q_{\lambda_n^k f^k}x,x\right)\right)\left|d\left(Q_{\lambda_{n+1}^k f^k}x,x\right)-d\left(Q_{\lambda_n^k f^k}x,x\right)\right|\\
&\le \frac{\left|\lambda_{n+1}^k-\lambda_n^k\right|}{2a}\left(d\left(Q_{\lambda_{n+1}^k f^k}x,x\right)+d\left(Q_{\lambda_n^k f^k}x,x\right)\right)d\left(Q_{\lambda_{n+1}^k f^k}x,Q_{\lambda_n^k f^k}x\right)\\
&\le \frac{\left|\lambda_{n+1}^k-\lambda_n^k\right|}{2a}\left(d\left(Q_{\lambda_{n+1}^k f^k}x,z\right)+d\left(Q_{\lambda_n^k f^k}x,z\right)+2d(x,z)\right)d\left(Q_{\lambda_{n+1}^k f^k}x,Q_{\lambda_n^k f^k}x\right)\\
&\le \frac{\left|\lambda_{n+1}^k-\lambda_n^k\right|}{2a}\cdot 4d(x,z)d\left(Q_{\lambda_{n+1}^k f^k}x,Q_{\lambda_n^k f^k}x\right)
\end{aligned}$$

Then we get

$$D=d\left(Q_{\lambda_{n+1}^k f^k}x,Q_{\lambda_n^k f^k}x\right)\le \frac{2\left|\lambda_{n+1}^k-\lambda_n^k\right|}{a}d(x,z).$$

By the above inequality, we have

$$d(U_{n+1}x, U_nx)^2 \leq 4d(x,z)^2 \sum_{k=1}^{N} \left(\left|\alpha_{n+1}^k - \alpha_n^k\right| + \frac{(\lambda_{n+1}^k - \lambda_n^k)^2}{a^2} + \frac{2\left|\lambda_{n+1}^k - \lambda_n^k\right|}{a} \right).$$

Since $\sum_{n=1}^{\infty}\sum_{k=1}^{N}\left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$, $\sum_{n=1}^{\infty}\sum_{k=1}^{N}\left|\lambda_{n+1}^k - \lambda_n^k\right| < \infty$ and boundedness of $\{x_n\}$, we obtain $\sum_{n=1}^{\infty} d(U_{n+1}x_{n+1}, U_nx_{n+1}) < \infty$. By Lemma 2.4, we have $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Furthermore,

$$\begin{aligned}
d(U_nx_n, x_n) &\leq d(U_nx_n, x_{n+1}) + d(x_{n+1}, x_n)\\
&= d(U_nx_n, \beta_n u \oplus (1-\beta_n)U_nx_n) + d(x_{n+1}, x_n)\\
&\leq \beta_n d(U_nx_n, u) + d(x_{n+1}, x_n).
\end{aligned}$$

Since $\lim_{n\to\infty}\beta_n = 0$ and $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$, we get $\lim_{n\to\infty} d(U_nx_n, x_n) = 0$. We show $\limsup_{n\to\infty}\left(d(u, P_Fu)^2 - (1-\beta_n)d(u, U_nx_n)^2\right) \leq 0$. We have

$$\begin{aligned}
&\left|\left(d(u,P_Fu)^2 - (1-\beta_n)d(u,U_nx_n)^2\right) - \left(d(u,P_Fu)^2 - d(u,x_n)^2\right)\right|\\
&\quad= \left|d(u,x_n)^2 - d(u,U_nx_n)^2 + \beta_n d(u,U_nx_n)^2\right|\\
&\quad\leq \left|d(u,x_n)^2 - d(u,U_nx_n)^2\right| + \beta_n d(u,U_nx_n)^2\\
&\quad= \left|(d(u,x_n) + d(u,U_nx_n))(d(u,x_n) - d(u,U_nx_n))\right| + \beta_n d(u,U_nx_n)^2\\
&\quad= \left|d(u,x_n) + d(u,U_nx_n)\right|\left|d(u,x_n) - d(u,U_nx_n)\right| + \beta_n d(u,U_nx_n)^2\\
&\quad\leq \left|d(u,x_n) + d(u,U_nx_n)\right| d(U_nx_n,x_n) + \beta_n d(u,U_nx_n)^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\beta_n = 0$, $\lim_{n\to\infty} d(U_nx_n, x_n) = 0$ and boundedness of $\{x_n\}, \{U_nx_n\}$ we get $\lim_{n\to\infty}\left|\left(d(u,P_Fu)^2 - (1-\beta_n)d(u,U_nx_n)^2\right) - \left(d(u,P_Fu)^2 - d(u,x_n)^2\right)\right| = 0$. From boundedness of $\{x_n\}$, we can take a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ Δ-converges to x_0 and $\liminf_{n\to\infty} d(u, x_n) = \lim_{i\to\infty} d(u, x_{n_i})$. Thus we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\left(d(u,P_Fu)^2 - (1-\beta_n)d(u,U_nx_n)^2\right) &= \limsup_{n\to\infty}\left(d(u,P_Fu)^2 - d(u,x_n)^2\right)\\
&= d(u,P_Fu)^2 - \liminf_{i\to\infty} d(u,x_{n_i})^2\\
&\leq d(u,P_Fu)^2 - d(u,x_0)^2.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\sum_{k=1}^{N}\left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$ and $\sum_{n=1}^{\infty}\sum_{k=1}^{N}\left|\lambda_{n+1}^k - \lambda_n^k\right| < \infty$, we obtain $\left\{\alpha_n^k\right\}$ converges to $\alpha^k \in [a, 1-a]$ and $\left\{\lambda_n^k\right\}$ converges to $\lambda^k \in [a, \infty[$ for every $k = 1, 2, \ldots, N$. Let $Ux = \operatorname{argmin}_{y\in X}\sum_{k=1}^{N}\alpha^k d\left(Q_{\lambda^k f^k}x, y\right)^2$. Then we have

$$\begin{aligned}
&\lim_{n\to\infty} d(U_nx_n, Ux_n)\\
&\leq 2\lim_{n\to\infty} d(x_n, z)\sqrt{\sum_{k=1}^{N}\left(|\alpha_n^k - \alpha^k| + \frac{(\lambda_n^k - \lambda^k)^2}{a^2} + \frac{2\left|\lambda_n^k - \lambda^k\right|}{a}\right)}\\
&= 0.
\end{aligned}$$

Therefore, we obtain $\lim_{n\to\infty} d(Ux_n, x_n) = 0$. Since $F = \bigcap_{k=1}^{N}\operatorname{argmin}_X f^k$ and Lemma 2.2, we have $x_0 \in F$. Therefore, we get $d(u, P_Fu) \leq d(u, x_0)$. We also obtain

$\limsup_{n\to\infty}(d(u, P_F u)^2 - (1-\beta_n)d(u, U_n x_n)^2) \leq 0$. From this inequality, we have

$$\begin{aligned} d(x_{n+1}, P_F u)^2 &\leq d(\beta_n u \oplus (1-\beta_n)U_n x_n, P_F u)^2 \\ &\leq \beta_n d(u, P_F u)^2 + (1-\beta_n)d(U_n x_n, P_F u)^2 - \beta_n(1-\beta_n)d(u, U_n x_n)^2 \\ &\leq (1-\beta_n)d(x_n, P_F u)^2 + \beta_n(d(u, P_F u)^2 - (1-\beta_n)d(u, U_n x_n)^2). \end{aligned}$$

By Lemma 2.11, we have $\lim_{n\to\infty} d(x_n, P_F u) = 0$. □

Chapter 6

Balanced mappings in CAT(1) spaces

In this chapter, we define a balanced mapping and find its fundamental properties in a complete CAT(1) space. We begin with the following theorem which guarantees that the balanced mapping can be defined as a single-valued mapping.

Theorem 6.1. (Kajimura, Kasahara, Kimura and Nakagawa [15]) *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let x^k be a point of X for every $k = 1, 2, \dots, N$. Let $\alpha^k \in [0, 1]$ for every $k = 1, 2, \dots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Then the set*

$$\operatorname*{argmax}_{y\in X} \sum_{k=1}^{N} \alpha^k \cos d(x^k, y)$$

consists of one point.

Proof. Let $D = \sup_{y\in X} \sum_{k=1}^N \alpha^k \cos d(x^k, y)$ and $\{y_n\}$ a sequence in X such that $\lim_{n\to\infty} \sum_{k=1}^N \alpha^k \cos d(x^k, y_n) = D$. For $m, n \in \mathbb{N}$, from Lemma 2.9, we have

$$\sum_{k=1}^{N} \alpha^k \cos d\left(x^k, \frac{1}{2}y_n \oplus \frac{1}{2}y_m\right) \cos \frac{d(y_n, y_m)}{2} \geq \sum_{k=1}^{N} \alpha^k \min\{\cos d(y_n, x^k), \cos d(y_m, x^k)\}.$$

Thus we get

$$\cos \frac{d(y_n, y_m)}{2} \geq \frac{\sum_{k=1}^N \alpha^k \min\{\cos d(y_n, x^k), \cos d(y_m, x^k)\}}{D}.$$

Hence we obtain $\{y_n\}$ is a Cauchy sequence. By the completeness of X, there exists $u = \lim_{n\to\infty} y_n$. From the continuity of the metric, we get $\sum_{k=1}^N \alpha^k \cos d(x^k, u) = \sup_{y\in X} \sum_{k=1}^N \alpha^k \cos d(x^k, y)$. Hence $\operatorname{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(x^k, y)$ is nonempty. Let $u, v \in \operatorname{argmax}_{y\in X} \sum_{k=1}^N \cos d(x^k, y)$ and suppose $u \neq v$. By Lemma 2.6, we have

$$\begin{aligned}\sum_{k=1}^{N} \alpha^k \cos d(x^k, u) \sin d(u, v) &\geq \sum_{k=1}^{N} \alpha^k \cos d\left(x^k, \frac{1}{2}u \oplus \frac{1}{2}v\right) \sin d(u, v)\\ &\geq \sin \frac{d(u, v)}{2} \sum_{k=1}^{N} \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)).\end{aligned}$$

Dividing by $\sin(d(u, v)/2)$, we get

$$2 \cos \frac{d(u, v)}{2} \sum_{k=1}^{N} \alpha^k \cos d(x^k, u) \geq \sum_{k=1}^{N} \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)).$$

Similarly, we get

$$2\cos\frac{d(u,v)}{2}\sum_{k=1}^{N}\alpha^k\cos d(x^k,v)\geq\sum_{k=1}^{N}\alpha^k(\cos d(x^k,u)+\cos d(x^k,v)).$$

Therefore, we obtain

$$2\cos\frac{d(u,v)}{2}\sum_{k=1}^{N}\alpha^k(\cos d(x^k,u)+\cos d(x^k,v))\geq 2\sum_{k=1}^{N}\alpha^k(\cos d(x^k,u)+\cos d(x^k,v)).$$

Then we have

$$1>\cos\frac{d(u,v)}{2}\geq 1,$$

which is a contradiction. Hence we get $u=v$. □

By Theorem 6.1, we know the set $\operatorname{argmax}_{y\in X}\sum_{k=1}^{N}\alpha^k\cos d(x^k,y)$ is a singleton. In what follows, a balanced mapping U from X to X for a sequence $\alpha^1,\alpha^2,\dots,\alpha^N\in[0,1]$ and mappings $T^1,T^2,\dots,T^N$ is defined by

$$Ux=\operatorname*{argmax}_{y\in X}\sum_{k=1}^{N}\alpha^k\cos d(T^kx,y)$$

for every $x\in X$. We prove some basic properties of balanced mappings in this section.

Theorem 6.2. (Kajimura, Kasahara, Kimura and Nakagawa [15]) *Let X be a complete CAT(1) space such that $d(x,y)<\pi/2$ for every $x,y\in X$. Let T^k be a quasinonexpansive mapping from X to X for every $k=1,2,\dots,N$ such that $\bigcap_{k=1}^{N}F(T^k)\neq\emptyset$. Let $\alpha^k\in\,]0,1[$ for every $k=1,2,\dots,N$ such that $\sum_{k=1}^{N}\alpha^k=1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then $F(U)=\bigcap_{k=1}^{N}F(T^k)$.*

Proof. Let $z\in\bigcap_{k=1}^{N}F(T^k)$. Then we have

$$\begin{aligned}Uz&=\operatorname*{argmax}_{y\in X}\sum_{k=1}^{N}\alpha^k\cos d(T^kz,y)\\&=\operatorname*{argmax}_{y\in X}\sum_{k=1}^{N}\alpha^k\cos d(z,y)\\&=\operatorname*{argmax}_{y\in X}\cos d(z,y)\\&=z.\end{aligned}$$

Hence we get $z\in F(U)$. Let $z\in F(U), w\in\bigcap_{k=1}^{N}F(T^k)$ and $t\in\,]0,1[$. We may assume that

$z \neq w$. From Lemma 2.6, we have

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha^k \cos d(T^k z, z) \sin d(z, w) \\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k z, tz \oplus (1-t)w) \sin d(z, w) \\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k z, z) \sin td(z, w) + \sum_{k=1}^{N} \alpha^k \cos d(T^k z, w) \sin(1-t)d(z, w) \\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k z, z) \sin td(z, w) + \cos d(z, w) \sin(1-t)d(z, w).
\end{aligned}$$

Hence we get

$$2\sum_{k=1}^{N} \alpha^k \cos d(T^k z, z)(\sin d(z, w) - \sin td(z, w)) \geq \cos d(z, w) \sin(1-t)d(z, w),$$

and it implies that

$$\begin{aligned}
&2\sum_{k=1}^{N} \alpha^k \cos d(T^k z, z) \sin \frac{(1-t)d(z, w)}{2} \cos \frac{(1+t)d(z, w)}{2} \\
&\quad \geq 2\cos d(z, w) \sin \frac{(1-t)d(z, w)}{2} \cos \frac{(1-t)d(z, w)}{2}.
\end{aligned}$$

Dividing by $2\sin((1-t)d(z, w)/2) \cos d(z, w)$ and letting $t \to 1$, we get

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k z, z) \geq 1.$$

Therefore we have $\cos d(T^k z, z) = 1$ for every $k = 1, 2, \ldots, N$. Hence we get $z \in \bigcap_{k=1}^{N} F(T^k)$, and then we have $F(U) = \bigcap_{k=1}^{N} F(T^k)$. □

Lemma 6.3. (Kajimura, Kasahara, Kimura and Nakagawa [15]) *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. Let $\alpha^k \in [0, 1]$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then we have*

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \geq \cos d(x, z)$$

for every $x \in X$ and $z \in \bigcap_{k=1}^{N} F(T^k)$.

Proof. Let $z \in \bigcap_{k=1}^{N} F(T^k)$ and $t \in]0, 1[$. Then, from Theorem 6.2, we have $z \in F(U)$. We may assume that $Ux \neq z$ since if $Ux = z$, the inequality is obvious true. By Lemma 2.6, we get

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin d(Ux, z)\\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, tUx \oplus (1-t)z) \sin d(Ux, z)\\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin td(Ux, z) + \sum_{k=1}^{N} \alpha^k \cos d(T^k x, z) \sin(1-t)d(Ux, z)\\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin td(Ux, z) + \sum_{k=1}^{N} \alpha^k \cos d(x, z) \sin(1-t)d(Ux, z)\\
&\quad = \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin td(Ux, z) + \cos d(x, z) \sin(1-t)d(Ux, z).
\end{aligned}$$

Hence we obtain

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux)(\sin d(Ux, z) - \sin td(Ux, z)) \geq \cos d(x, z) \sin(1-t)d(Ux, z),$$

and it implies that

$$\begin{aligned}
&2\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin \frac{(1-t)d(Ux, z)}{2} \cos \frac{(1+t)d(Ux, z)}{2}\\
&\quad \geq 2\cos d(x, z) \sin \frac{(1-t)d(Ux, z)}{2} \cos \frac{(1-t)d(Ux, z)}{2}.
\end{aligned}$$

Dividing by $2\sin((1-t)d(Ux, z)/2)$ and tending $t \to 1$, we get

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \geq \cos d(x, z)$$

for $x \in X$. □

Theorem 6.4. (Kajimura, Kasahara, Kimura and Nakagawa [15]) *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive mapping from X to X for every $k = 1, 2, \dots, N$ such that $\bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. Let $\alpha^k \in \,]0, 1[$ for every $k = 1, 2, \dots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then U is a quasinonexpansive mapping.*

Proof. From Theorem 6.2, let $z \in F(U) = \bigcap_{k=1}^{N} F(T^k)$. By Lemma 6.3, we have

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \geq \cos d(x, z)$$

for $x \in X$. Since $\cos d(T^k x, Ux) \leq 1$, we get

$$\begin{aligned}
\cos d(Ux, z) &\geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z)\\
&\geq \cos d(x, z).
\end{aligned}$$

Thus, we obtain

$$d(Ux, z) \leq d(x, z).$$

Hence U is a quasinonexpansive mapping. $\square$

Theorem 6.5. (Kajimura, Kasahara, Kimura and Nakagawa [15]) *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive and Δ-demiclosed mapping from X to X for every $k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^N F(T^k) \neq \emptyset$. Let $\alpha^k \in]0, 1[$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then U is a Δ-demiclosed mapping.*

Proof. From Theorem 6.2, let $z \in F(U) = \bigcap_{k=1}^N F(T^k)$. Let $\{x_n\} \subset X$ satisfying $d(Ux_n, x_n) \to 0$ and $\{x_n\}$ Δ-converges to $x_0 \in X$. By Lemma 6.3, we have

$$\sum_{k=1}^N \alpha^k \cos d(T^k x_n, Ux_n) \cos d(Ux_n, z) \geq \cos d(x_n, z).$$

Then we get

$$\sum_{k=1}^N \alpha^k \cos d(T^k x_n, Ux_n) \geq \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)}.$$

Since $\lim_{n\to\infty}(\cos d(x_n, z)/\cos d(Ux_n, z)) = 1$, we obtain

$$\lim_{n\to\infty} \sum_{k=1}^N \alpha^k \cos d(T^k x_n, Ux_n) = 1.$$

Hence we get $\lim_{n\to\infty} d(T^k x_n, Ux_n) = 0$ for every $k = 1, 2, \ldots, N$. Then we have $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$. Since T^k is a Δ-demiclosed mapping for every $k = 1, 2, \ldots, N$, we obtain $x_0 \in F$. $\square$

Theorem 6.6. (Kajimura, Kasahara, Kimura and Nakagawa [15]) *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a strongly quasinonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^N F(T^k) \neq \emptyset$. Let $\alpha^k \in]0, 1[$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then U is a strongly quasinonexpansive mapping.*

Proof. From Theorem 6.2, let $z \in F(U) = \bigcap_{k=1}^N F(T^k)$. Let $\{x_n\} \subset X$ satisfying $\limsup_{n\to\infty} d(x_n, z) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n, z)/\cos d(Ux_n, z)) = 1$. By Lemma 6.3, we have

$$\sum_{k=1}^N \alpha^k \cos d(T^k x_n, Ux_n) \cos d(Ux_n, z) \geq \cos d(x_n, z).$$

Then we get

$$\sum_{k=1}^N \alpha^k \cos d(T^k x_n, Ux_n) \geq \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)}.$$

Since $\lim_{n\to\infty}(\cos d(x_n,z)/\cos d(Ux_n,z)) = 1$, we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{N}\alpha^k \cos d(T^k x_n, Ux_n) = 1.$$

Hence we get $\lim_{n\to\infty} d(T^k x_n, Ux_n) = 0$ for every $k = 1, 2, \ldots, N$. For any $k = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{\cos d(x_n,z)}{\cos d(Ux_n,z)} &= \liminf_{n\to\infty}\frac{\cos d(x_n,z)}{\cos d(Ux_n,z)} \\
&\le \liminf_{n\to\infty}\frac{\cos d(x_n,z)}{\cos(d(Ux_n,T^k x_n)+d(T^k x_n,z))} \\
&= \liminf_{n\to\infty}\frac{\cos d(x_n,z)}{\cos d(T^k x_n,z)} \\
&\le \limsup_{n\to\infty}\frac{\cos d(x_n,z)}{\cos d(T^k x_n,z)} \\
&\le \limsup_{n\to\infty}\frac{\cos d(x_n,z)}{\cos(d(T^k x_n,Ux_n)+d(Ux_n,z))} \\
&= \limsup_{n\to\infty}\frac{\cos d(x_n,z)}{\cos d(Ux_n,z)} \\
&= \lim_{n\to\infty}\frac{\cos d(x_n,z)}{\cos d(Ux_n,z)}.
\end{aligned}$$

Thus we obtain $\lim_{n\to\infty}(\cos d(x_n,z)/\cos d(T^k x_n,z)) = 1$. Since T^k is a strongly quasinonexpansive mapping for every $k = 1, 2, \ldots, N$, we get $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$. Since $\lim_{n\to\infty} d(T^k x_n, Ux_n) = 0$ and $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$, we obtain $\lim_{n\to\infty} d(Ux_n, x_n) = 0$. □

Lemma 6.7. (Kajimura, Kasahara, Kimura and Nakagawa [15]) *Let X be a complete* CAT(1) *space such that $d(x,y) < \pi/2$ for every $x, y \in X$. Let T^k be a mapping from X to X for every $k = 1, 2, \ldots, N$. Let $\alpha^k \in [0,1]$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N}\alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then we have*

$$\sum_{k=1}^{N}\alpha^k \cos d(T^k x, Ux) \ge \frac{\sum_{k=1}^{N}\alpha^k \cos d(T^k x, Uy)}{\cos d(Ux, Uy)}$$

for every $x, y \in X$.

Proof. Let $t \in\,]0,1[$. We may assume $Ux \ne Uy$. By Lemma 2.6, we have

$$\begin{aligned}
&\sum_{k=1}^{N}\alpha^k \cos d(T^k x, Ux)\sin d(Ux,Uy) \\
&\quad\ge \sum_{k=1}^{N}\alpha^k \cos d(T^k x, tUx \oplus (1-t)Uy)\sin d(Ux,Uy) \\
&\quad\ge \sum_{k=1}^{N}\alpha^k (\cos d(T^k x, Ux)\sin td(Ux,Uy) + \cos d(T^k x, Uy)\sin(1-t)d(Ux,Uy)).
\end{aligned}$$

Then we get

$$
\begin{aligned}
&2\sum_{k=1}^{N}\alpha^k \cos d(T^k x, Ux)\cos\frac{(1+t)d(Ux,Uy)}{2}\sin\frac{(1-t)d(Ux,Uy)}{2} \\
&\qquad\geq \sum_{k=1}^{N}\alpha^k \cos d(T^k x, Uy)\sin(1-t)d(Ux,Uy).
\end{aligned}
$$

Dividing by $2\cos((1+t)d(Ux,Uy)/2)\sin((1-t)d(Ux,Uy)/2)$, we get

$$
\sum_{k=1}^{N}\alpha^k \cos d(T^k x, Ux) \geq \sum_{k=1}^{N}\alpha^k \cos d(T^k x, Uy)\frac{\cos\dfrac{(1-t)d(Ux,Uy)}{2}}{\cos\dfrac{(1+t)d(Ux,Uy)}{2}}.
$$

Letting $t \to 1$, we obtain the desired result. □

We prove a convergence theorem of a Mann iterative sequence by using a balanced mapping for quasinonexpansive and Δ-demiclosed mappings in a complete CAT(1) space.

Theorem 6.8. (Kajimura, Kasahara, Kimura and Nakagawa [15]) *Let X be a complete* CAT(1) *space such that $d(x,y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive and Δ-demiclosed mapping from X to X for every $k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. For a given real number $a \in]0, 1/2]$, let $\{\alpha_n^k\}, \{\delta_n\} \subset [a, 1-a]$ for every $k = 1, 2, \ldots, N$ and $n \in \mathbb{N}$ such that $\sum_{k=1}^{N}\alpha_n^k = 1$. Let U_n be a balanced mapping for $\{\alpha_n^k\}$ and $\{T^k\}$. For a given point $x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$
x_{n+1} = \delta_n x_n \oplus (1-\delta_n)U_n x_n
$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point in $\bigcap_{k=1}^{N} F(T^k)$.

Proof. From Theorem 6.2, we know that $F(U_n) = \bigcap_{k=1}^{N} F(T^k)$ for every $n \in \mathbb{N}$. Let $z \in F = F(U_n) = \bigcap_{k=1}^{N} F(T^k)$. From Lemmas 2.8 and Theorem 6.4, we have

$$
\begin{aligned}
\cos d(x_{n+1}, z) &= \cos d(\delta_n x_n \oplus (1-\delta_n)U_n x_n, z) \\
&\geq \delta_n \cos d(x_n, z) + (1-\delta_n)\cos d(U_n x_n, z) \\
&\geq \cos d(x_n, z).
\end{aligned}
$$

Thus, we obtain $d(x_{n+1}, z) \leq d(x_n, z)$ for all $n \in \mathbb{N}$ and there exists

$$
D = \lim_{n\to\infty} d(x_n, z) \leq d(x_1, z) < \frac{\pi}{2}.
$$

Since $\{\delta_n\} \subset [a, 1-a]$, from Lemma 2.6, we get

$$
\begin{aligned}
&\cos d(x_{n+1}, z)\sin d(x_n, U_n x_n) \\
&\quad = \cos d(\delta_n x_n \oplus (1-\delta_n)U_n x_n, z)\sin d(x_n, U_n x_n) \\
&\quad \geq \cos d(x_n, z)\sin \delta_n d(x_n, U_n x_n) + \cos d(U_n x_n, z)\sin(1-\delta_n)d(x_n, U_n x_n) \\
&\quad \geq \cos d(x_n, z)(\sin \delta_n d(x_n, U_n x_n) + \sin(1-\delta_n)d(x_n, U_n x_n)) \\
&\quad \geq 2\cos d(x_n, z)\sin a d(x_n, U_n x_n).
\end{aligned}
$$

Putting $E = \lim_{n\to\infty} d(x_n, U_n x_n)$ and letting $n \to \infty$, we get

$$\cos D \sin E \geq 2 \cos D \sin aE.$$

Using elementary calculation, we have $E = 0$, that is,

$$\lim_{n\to\infty} d(x_n, U_n x_n) = 0.$$

We show $\lim_{n\to\infty} d(x_n, T^k x_n) = 0$ for all $k = 1, 2, \ldots, N$. Since $\{x_n\}$ is bounded, it follows that

$$\begin{aligned} D = \lim_{n\to\infty} d(x_n, z) &\leq \lim_{n\to\infty} (d(x_n, U_n x_n) + d(U_n x_n, z)) \\ &= \lim_{n\to\infty} d(U_n x_n, z) \\ &\leq \lim_{n\to\infty} d(x_n, z) = D. \end{aligned}$$

Thus we get $\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(U_n x_n, z) = D$. By Lemma 6.7, we have

$$\begin{aligned} \sum_{k=1}^{N} \alpha_n^k \cos d(T^k x_n, U_n x_n) &\geq \frac{\sum_{k=1}^{N} \alpha_n^k \cos d(T^k x_n, z)}{\cos d(U_n x_n, z)} \\ &\geq \frac{\sum_{k=1}^{N} \alpha_n^k \cos d(x_n, z)}{\cos d(U_n x_n, z)} \\ &\geq \frac{\cos d(x_n, z)}{\cos d(U_n x_n, z)}. \end{aligned}$$

Since $0 < a \leq \alpha_n^k$ and $\sum_{k=1}^{N} \alpha_n^k = 1$, we obtain $\lim_{n\to\infty} d(T^k x_n, U_n x_n) = 0$ for every $k = 1, 2, \ldots, N$. Since $\lim_{n\to\infty} d(U_n x_n, x_n) = 0$, we also get $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$. Let x_0 be an asymptotic center of $\{x_n\}$ and for every $\{x_{n_k}\} \subset \{x_n\}$, let y be an asymptotic center of $\{x_{n_k}\}$. There exists $\{x_{n_{k_l}}\} \subset \{x_{n_k}\}$ satisfying that $\{x_{n_{k_l}}\}$ Δ-converges to w. Since T^k is Δ-demiclosed and $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$, we get $w \in F$. Since there exists $\lim_{n\to\infty} d(x_{n_k}, w)$, we have

$$\begin{aligned} \limsup_{k\to\infty} d(x_{n_k}, w) &= \lim_{k\to\infty} d(x_{n_k}, w) \\ &= \lim_{l\to\infty} d(x_{n_{k_l}}, w) \\ &\leq \limsup_{l\to\infty} d(x_{n_{k_l}}, y) \\ &\leq \limsup_{k\to\infty} d(x_{n_k}, y). \end{aligned}$$

Since y is an asymptotic center of $\{x_{n_k}\}$, we obtain $y = w$. Then we have $y \in F$. Hence we get

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, y) &= \lim_{n\to\infty} d(x_n, y) \\ &= \lim_{k\to\infty} d(x_{n_k}, y) \\ &\leq \limsup_{k\to\infty} d(x_{n_k}, x_0) \\ &\leq \limsup_{n\to\infty} d(x_n, x_0). \end{aligned}$$

Since x_0 is an asymptotic center of $\{x_n\}$, we obtain $x_0 = y$. Therefore we obtain $\{x_n\}$ Δ-converges to $x_0 \in F$. □

We can apply this result to the problem of finding a common minimizer of finitely many convex functions defined on a complete CAT(1) space as follows:

Corollary 6.9. *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 1, 2, \dots, N$ such that $F = \bigcap_{k=1}^{N} \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\left\{\alpha_n^k\right\}, \{\beta_n\} \subset [a, 1-a]$ and $\lambda^k \in [a, \infty[$ for every $k = 1, 2, \dots, N$ and $n \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_n^k = 1$. Let $R_{\lambda^k f^k}$ be the resolvent of $\lambda^k f^k$ for every $k = 1, 2, \dots, N$. Let U_n be a balanced mapping for $\{\alpha_n^k\}$ and $\{R_{\lambda^k f^k}\}$. For a given point $x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \delta_n x_n \oplus (1 - \delta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point in F.

Chapter 7

Conclusion

In this thesis, we proved some convergence theorems by using a finite number of resolvent operators. These convergence theorems can be applied to a convex optimization problem for a finite family of convex functions. We summarize our main results Theorems 3.1, 4.10, 4.9, 5.1, 5.2 and 6.9 as Theorems 7.1–7.6.

Theorem 7.1. *Let X be an admissible complete* $\mathrm{CAT}(1)$ *space. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of positive real numbers such that $\inf_n \lambda_n > 0$ and $\inf_n \mu_n > 0$. Let f and g be proper lower semicontinuous convex functions from X into $]-\infty, \infty]$ such that $F = \operatorname{argmin}_X f \cap \operatorname{argmin}_X g \neq \emptyset$. Let $R_{\lambda_n f}$ and $R_{\mu_n g}$ be the resolvents of $\lambda_n f$ and $\mu_n g$, respectively. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a, 1-a]$. Let $\{x_n\}$ be a sequence in X defined by $x_1 \in X$ and*

$$\begin{cases} u_n = \beta_n x_n \oplus (1-\beta_n) R_{\lambda_n f} x_n, \\ v_n = \gamma_n x_n \oplus (1-\gamma_n) R_{\mu_n g} x_n, \\ x_{n+1} = \alpha_n u_n \oplus (1-\alpha_n) v_n \end{cases}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point of F.

Theorem 7.2. *Let X be a complete* $\mathrm{CAT}(1)$ *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 0, 1, \dots, N$ such that $F = \bigcap_{k=0}^{N} \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\sigma_n^l\} \subset [a, 1-a]$ for every $l = 0, 1, \dots, N-1$ and $\{\lambda_n^k\} \subset [a, \infty[$ for every $k = 0, 1, \dots, N$. Let $R_{\lambda_n^k f^k}$ be the resolvent of $\lambda_n^k f^k$ for every $k = 0, 1, \dots, N$ and $n \in \mathbb{N}$. Define $U_n^l : X \to X$ by*

$$U_n^N = R_{\lambda_n^N f^N} \text{ and } U_n^l = \sigma_n^l R_{\lambda_n^l f^l} \oplus (1-\sigma_n^l) U_n^{l+1}$$

for every $l = 0, 1, \dots, N-1$ and $n \in \mathbb{N}$. Let $\{\alpha_n\}$ be a real sequence in $]0, 1[$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. For given points $u, x_1 \in X$, let $\{x_n\}$ be the sequence in X generated by

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n) U_n^0 x_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v,v' \in X} d(v, v') < \pi/2$;
(b) $d(u, P_F u) < \pi/4$ *and* $d(u, P_F u) + d(x_0, P_F u) < \pi/2$;
(c) $\sum_{n=0}^{\infty} \alpha_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_F u$.

Theorem 7.3. *Let X be a complete* $\mathrm{CAT}(1)$ *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every*

$k = 0, 1, \ldots, N$ such that $F = \bigcap_{k=0}^{N} \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\sigma_n^l\} \subset [a, 1-a]$ for every $l = 0, 1, \ldots, N-1$ and $\{\lambda_n^k\} \subset [a, \infty[$ for every $k = 0, 1, \ldots, N$. Let $R_{\lambda_n^k f^k}$ be the resolvent of $\lambda_n^k f^k$ for every $k = 0, 1, \ldots, N$ and $n \in \mathbb{N}$. Define $U_n^l : X \to X$ by

$$U_n^N = R_{\lambda_n^N f^N} \text{ and } U_n^l = \sigma_n^l R_{\lambda_n^l f^l} \oplus (1 - \sigma_n^l) U_n^{l+1}$$

for every $l = 0, 1, \ldots, N-1$ and $n \in \mathbb{N}$. Let $\{\alpha_n\}$ be a real sequence in $[a, 1-a]$. For a given point $x_1 \in X$, let $\{x_n\}$ be the sequence in X generated by

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) U_n^0 x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point of F.

Theorem 7.4. *Let X be a complete $\mathrm{CAT}(0)$ space. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 1, 2, \ldots, N$ such that $F = \bigcap_{k=1}^{N} \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\left\{\alpha_n^k\right\}, \{\beta_n\} \subset [a, 1-a]$ and $\left\{\lambda_n^k\right\} \subset [a, \infty[$ for every $k = 1, 2, \ldots, N$ and $n \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_n^k = 1$. Let $Q_{\lambda_n^k f^k}$ be the resolvent of $\lambda_n^k f^k$ for every $k = 1, 2, \ldots, N$ and $n \in \mathbb{N}$. Define U_n be a mapping from X to X by*

$$U_n x = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, y\right)^2$$

for every $x \in X$ and $n \in \mathbb{N}$. For a given point $x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by

$$x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point in F.

Theorem 7.5. *Let X be a complete $\mathrm{CAT}(0)$ space. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 1, 2, \ldots, N$ such that $F = \bigcap_{k=1}^{N} \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\beta_n\} \subset]0, 1[$, $\left\{\alpha_n^k\right\} \subset [a, 1-a]$ and $\left\{\lambda_n^k\right\} \subset [a, \infty[$ for every $k = 1, 2, \ldots, N$ and $n \in \mathbb{N}$ such that $\lim_{n \to \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{k=1}^{N} \alpha_n^k = 1$, $\sum_{n=1}^{\infty} \sum_{k=1}^{N} \left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{N} \left|\lambda_{n+1}^k - \lambda_n^k\right| < \infty$. Let $Q_{\lambda_n^k f^k}$ be the resolvent of $\lambda_n^k f^k$ for every $k = 1, 2, \ldots, N$ and $n \in \mathbb{N}$. Define U_n be a mapping from X to X by*

$$U_n x = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha_n^k d\left(Q_{\lambda_n^k f^k} x, y\right)^2$$

for every $x \in X$ and $n \in \mathbb{N}$. For given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by

$$x_{n+1} = \beta_n u \oplus (1 - \beta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_F u$.

Theorem 7.6. *Let X be a complete $\mathrm{CAT}(1)$ space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 1, 2, \ldots, N$ such that $F = \bigcap_{k=1}^{N} \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$,* *let* $\{\alpha_n^k\}, \{\beta_n\} \subset [a, 1-a]$ *and* $\lambda^k \in [a, \infty[$ *for every* $k = 1, 2, \ldots, N$ *and* $n \in \mathbb{N}$ *such that* $\sum_{k=1}^{N} \alpha_n^k = 1$. *Let* $R_{\lambda^k f^k}$ *be the resolvent of* $\lambda^k f^k$ *for every* $k = 1, 2, \ldots, N$. *Let* U_n *be a balanced mapping for* $\{\alpha_n^k\}$ *and* $\{R_{\lambda^k f^k}\}$. *For a given point* $x_1 \in X$, *let* $\{x_n\}$ *be a sequence in* X *generated by*

$$x_{n+1} = \delta_n x_n \oplus (1 - \delta_n) U_n x_n$$

for every $n \in \mathbb{N}$. *Then* $\{x_n\}$ Δ*-converges to a point in* F.

Bibliography

[1] K. Aoyama and Y. Kimura, *Strong convergence theorems for strongly nonexpansive sequences*, Appl. Math. Comput. **217** (2011), 7537–7545.

[2] K. Aoyama, Y. Kimura and F. Kohsaka, *Strong convergence theorems for strongly relatively nonexpansive sequences and applications*, J. Nonlinear Anal. Optim. **3** (2012), 67–77.

[3] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mapping in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[4] S. Dhompongsa and B. Panyanak, *On Δ-convergence theorems in CAT(0) spaces*, Comput. Math. Appl. **56**(10) (2008), 2572–2579.

[5] R. Espínola and A. Fernández-León, *CAT(k)-spaces, weak convergence and fixed points*, J. Math. Anal. Appl. **353** (2009), 410–427.

[6] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Am. Math. Soc. **73** (1967), 957–961.

[7] T. Hasegawa and Y. Kimura, *Convergence to a fixed point of a balanced mapping by the Mann algorithm in a Hadamard space*, Linear Nonlinear Anal. **4** (2018), 405–412.

[8] T. Hasegawa and Y. Kimura, *Convergence to a fixed point of a balanced mapping in a Hadamard space*, preprint.

[9] J.S. He, D.H. Fang, G. López and C. Li, *Mann's algorithm for nonexpansive mappings in CAT(κ) spaces*, Nonlinear Anal. **75** (2012), 445–452.

[10] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv. **70** (1995), 659–673.

[11] K. Kasahara, *Iterative sequences for a finite number of mapping on a complete geodesic space*, Master Thesis, Toho University, 2018.

[12] K. Kasahara and Y. Kimura, *Iterative sequences for two resolvent operators in a CAT(1) space (Japanese)*, Hokkaido University Technical Report Series in Mathematics **176** (2019), 449–456.

[13] K. Kasahara and Y. Kimura, *Iterative sequences for a balanced mapping of resolvents*, submitted.

[14] K. Kasahara and Y. Kimura, *Iterative sequences for a finite number of resolvent operators on complete geodesic spaces*, submitted.

[15] T. Kajimura, K. Kasahara, Y. Kimura and K. Nakagawa, *Balanced mappings and an iterative scheme in complete geodesic spaces*, submitted.

[16] Y. Kimura and F. Kohsaka, *Spherical nonspreadingness of resolvents of convex functions in geodesic spaces*, J. Fixed Point Theory Appl. **18** (2016), 93–115.

[17] Y. Kimura and F. Kohsaka, *Two modified proximal point algorithms in geodesic spaces with curvature bounded above*, Rend. Circ. Mat. Pallemo, II. Ser, **68** (2019), 83–104.

[18] Y. Kimura and K. Nakagawa, *Another type of Mann iterative scheme for two mappings in a complete geodesic space*, J. Inequal. Appl. **2014** (2014), 9 pages.

[19] Y. Kimura, S. Saejung and P. Yotkaew, *The Mann algorithm in a complete geodesic space with curvature bounded above*, Fixed Point Theory Appl. **2013** (2013), 13 pages.

[20] Y. Kimura and K. Satô, *Convergence of subsets of a complete geodesic space with curvature*

bounded above, Nonlinear Anal. **75** (2012), 5079–5085.

[21] Y. Kimura and K. Satô, *Image recovery problem on a geodesic space with curvature bound above by one*, Proceedings of the Third Asian Conference on Nonlinear Analysis and Optimization, (Matsue, Japan, 2012), (S. Akashi, T. Tanaka, and W. Takahashi eds.), pp. 165–172.

[22] Y. Kimura and K. Satô, *Halpern iteration for strongly quasinonexpansive mappings on a geodesic space with curvature bounded above by one*, Fixed Point Theory Appl. **2013** (2013), 14 pages.

[23] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[24] U.F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom. **6** (1998), 199–253.

[25] B. Piątek, *Halpern iteration in CAT(κ) spaces*, Acta Math. Sin. Engl. Ser. **27** (2011), 635–646.

[26] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.

[27] S. Saejung, *Halpern's iteration in CAT(0) spaces*, Fixed Point Theory Appl. **2010** (2010), 13 pages.

[28] S. Saejung and P. Yotkaew, *Approximation of zeros of inverse strongly monotone operators in Banach spaces*, Nonlinear Anal. **75** (2012), 742–750.

[29] W. Takahashi and T. Tamura, *Convergence theorems for a pair of nonexpansive mappings*, J. Convex Anal. **5** (1998), 45–56.

[30] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486–491.