タイトル	Approximation of a fixed point in a complete geodesic space and its convergence rate
作成者（著者）	中川, 光一
公開者	東邦大学
発行日	2020.09.23
掲載情報	東邦大学大学院理学研究科 博士論文.
資料種別	学位論文
内容記述	主査: 木村 泰紀 /
著者版フラグ	ETD
報告番号	32661甲第975号
学位記番号	甲第159号
学位授与年月日	2020.09.23
学位授与機関	東邦大学
メタデータのＵＲＬ	https://mylibrary.toho-u.ac.jp/webopac/TD28176045

東邦大学審査学位論文(博士)

Approximation of a fixed point in a complete geodesic space and its convergence rate

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Doctor thesis
July 2020

Contents

Chapter 1

Introduction

The theory of fixed point approximation has been studied in a variety of ways and its results are useful for the other studies. In 1967, Halpern [8] introduced an iteration procedure for approximating fixed points of a nonexpansive mapping. In 1992, Wittmann [27] obtained that a Halpern iteration with nonexpansive mapping converges strongly to a fixed point in a Hilbert space. Later, this iteration method has been extended to Banach spaces and Hilbert spaces by many mathematicians; see [22, 25] for instance. The first result considering a Halpern iteration on a metric space with convex structure was due to Aoyama, Eshita, and Takahashi [1]. In 2010, Saejung [23] introduced this iteration in CAT(0) spaces. In 2011, Piątek [21] proposed strong convergence theorems with single mapping in CAT(1) spaces. In 2013, Kimura and Satô [15] supplemented its proof and completed it. They also proved the convergence theorems in CAT(1) spaces with two mappings.

On the other hand, in 2016, Wada [26] proved the strong convergence theorem for two kinds of mappings in a complete CAT(0) space:

Theorem 1.1 (Wada [26])**.** *Let X be a complete* CAT(0) *space. Let R be a nonexpansive mapping from X into itself and S, T strongly quasinonexpansive and Δ-demiclosed mappings from X into itself with $F = F(R) \cap F(S) \cap F(T) \neq \emptyset$. For $r \in]0, 1/2[$, let $\{\alpha_n\} \subset [r, 1-r] \subset]0, 1[$, $\{\beta_n\}, \{\gamma_n\} \subset]0, 1[$ be real sequences satisfying $\beta_n \to \beta \in]0, 1[$, $\gamma_n \to 0$, and $\sum_{n=1}^{\infty} \gamma_n = \infty$. Define $\{x_n\} \subset X$ by $x_1, u \in X$ and*

$$\begin{cases} s_n = \gamma_n u \oplus (1-\gamma_n) S x_n, \\ t_n = \gamma_n u \oplus (1-\gamma_n) T x_n, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) R(\beta_n s_n \oplus (1-\beta_n) t_n) \end{cases}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to a point in F.

In Chapter 3, we show this result in the setting of complete CAT(1) spaces.

Furthermore, in a Hilbert space, we know that the sequence generated by Mann type and by Halpern type is convergent weakly and strongly to a fixed point, respectively. These results are different but there is a common property such that

$$\lim_{k\to\infty} \|x_k - Tx_k\| = 0, \qquad (*)$$

where T is a nonexpansive mapping from a Hilbert space into itself.

In geodesic spaces, there are also convergence theorems for nonexpansive mappings; see [5] and [23].

In Chapter 4, we focus on a convergence rate of $(*)$ generated by Mann type iteration defined as follows:

$$x_{k+1} = (1-\alpha_k) x_k \oplus \alpha_k T x_k, \quad k = 1, 2, 3, \ldots,$$

where $\{\alpha_k\} \subset]0,1[$ is a sequence of coefficients of a convex combination between x_k and Tx_k.

Our motivation is based on Matsushita's result [18]. He proved its convergence rate is equal to $o(1/\sqrt{\sigma_k})$ in a Hilbert space.

It is known that the definition of CAT(κ) spaces depends on its curvature κ and if $\kappa < \kappa'$, the class of CAT(κ) spaces is included by that of CAT(κ') space. Further, CAT(κ) space has different structure for each κ. Thus, there may exist different theorems of convergence rate for each curvature κ. That is the reason why we consider it in a geodesic space with real curvature to summarize their proofs.

In Chapter 5, we consider the convergence rate of the approximate sequence of Halpern type based on Leuştean's result [16] and get an analogous theorem.

Chapter 2

Preliminaries

In this chapter, we introduce some definitions and notations for the main results. Let X_κ be a metric space for any $\kappa \in \mathbb{R}$. For $x, y \in X_\kappa$, a mapping $c : [0, l] \to X_\kappa$ is said to be a geodesic if c satisfies $c(0) = x, c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, l]$. An image $[x, y]$ of c is called a geodesic segment joining x and y. For $\kappa \in \mathbb{R}$, we define a diameter D_κ as follows:

$$D_\kappa = \begin{cases} \dfrac{\pi}{\sqrt{\kappa}} & (\kappa > 0), \\ \infty & (\kappa \leq 0). \end{cases}$$

X_κ is called a D_κ-geodesic metric space if, for any $x, y \in X_\kappa$ with $d(x, y) < D_\kappa$, there exists a geodesic segment $[x, y]$. Further, if a segment $[x, y]$ is unique for any $x, y \in X_\kappa$ with $d(x, y) < D_\kappa$, then X is called a uniquely D_κ-geodesic metric space. Moreover, X_κ is said to be admissible if $d(x, y) < D_\kappa/2$ for any $x, y \in X_\kappa$. For more general cases, see [3].

Let X_κ be an admissible uniquely geodesic metric space for $\kappa \in \mathbb{R}$. A geodesic triangle is defined by $\triangle(x, y, z) = [x, y] \cup [y, z] \cup [z, x]$. Let M_κ be a two-dimensional model space of X_κ as follows:

$$M_\kappa = \begin{cases} \dfrac{1}{\sqrt{\kappa}} \mathbb{S}^2 & (\kappa > 0), \\ \mathbb{R}^2 & (\kappa = 0), \\ \dfrac{1}{\sqrt{-\kappa}} \mathbb{H}^2 & (\kappa < 0). \end{cases}$$

For $\bar{x}, \bar{y}, \bar{z} \in M_\kappa$, a triangle $\triangle(\bar{x}, \bar{y}, \bar{z}) \subset M_\kappa$ is called a comparison triangle of $\triangle(x, y, z)$ if $d(x, y) = d_{M_\kappa}(\bar{x}, \bar{y})$, $d(y, z) = d_{M_\kappa}(\bar{y}, \bar{z})$, $d(z, x) = d_{M_\kappa}(\bar{z}, \bar{x})$. Further, for any $x, y \in X_\kappa$ and $t \in \,]0, 1[$, if $z \in [x, y]$ satisfies $d(x, z) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y)$, then z is denoted by $z = tx \oplus (1 - t)y$. A point $\bar{z} \in [\bar{x}, \bar{y}]$ is called a comparison point of $z \in [x, y]$ if $d(x, z) = d_{M_\kappa}(\bar{x}, \bar{z})$. X_κ is called a CAT(κ) space if, for any $x, y, z \in X_\kappa$, $p, q \in \triangle(x, y, z) \subset X_\kappa$ and their comparison points $\bar{p}, \bar{q} \in \triangle(\bar{x}, \bar{y}, \bar{z}) \subset M$, the inequality $d(p, q) \leq d_{M_\kappa}(\bar{p}, \bar{q})$ holds.

Let X be a geodesic metric space and $\{x_n\}$ a bounded sequence of X. For $x \in X$, we put $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is defined by $r(\{x_n\}) = \inf_{x\in X} r(x, \{x_n\})$. Further, the asymptotic center of $\{x_n\}$ is defined by $AC(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. If, for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $AC(\{x_{n_k}\}) = \{x_0\}$, *i.e.*, their asymptotic center consists of the unique element x_0, then we say $\{x_n\}$ Δ-converges to x_0 and we denote it by $x_n \xrightarrow{\Delta} x_0$.

Let X be a metric space. A mapping $T : X \to X$ is said to be nonexpansive if T satisfies $d(Tx, Ty) \leq d(x, y)$ for any $x, y \in X$. The set of fixed points of T is denoted by $F(T) = \{z \in X : Tz = z\}$. Further, a mapping $T : X \to X$ with $F(T) \neq \emptyset$ is said to be strongly quasinonexpansive if, for any $x \in X$ and $z \in F(T)$, $d(Tx, z) \leq d(x, z)$ and if $d(x_n, Tx_n) \to 0$ for any bounded sequence $\{x_n\} \subset X$ and $z \in F(T)$ satisfying $\lim_{n\to\infty}(d(x_n, z) - d(Tx_n, z)) = 0$. Moreover, T is said to be Δ-demiclosed if $x_0 \in F(T)$ for any bounded sequence $\{x_n\} \subset X$ and $x_0 \in X$ satisfying $d(x_n, Tx_n) \to 0$ and $x_n \xrightarrow{\Delta} x_0$. Let C be a nonempty closed convex subset of X. A mapping P_C is called metric projection from X onto C if P_C satisfies $d(u, P_C u) = \inf_{x\in C} d(u, x)$ for any $u \in X$. If X is an admissible CAT(κ) space, P_C is well-defined as a single-valued mapping.

Chapter 3

Convergence theorem for two kinds of mappings

In this chapter, we consider a convergence theorem for two kinds of mappings in a complete CAT(1) space. It is based on Theorem 1.1. To begin with, we describe some tools which will be used for the main result in this chapter. We start to show the lemmas with respect to real numbers.

Lemma 3.1 (Aoyama, Kimura, and Kohsaka [2], Saejung and Yotkaew [24]). *Let* $\{a_n\} \subset [0, \infty[\,, \{d_n\} \subset \mathbb{R}$ *and* $\{\gamma_n\} \subset \,]0,1[$ *such that* $\sum_{n=1}^{\infty} \gamma_n = \infty$. *Define a set* $\Phi = \{\varphi : \mathbb{N} \to \mathbb{N}, \textit{nondecreasing and } \lim_{i\to\infty} \varphi(i) = \infty\}$. *Suppose that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n d_n$$

for any $n \in \mathbb{N}$. *If* $\overline{\lim}_{i\to\infty} d_{\varphi(i)} \leq 0$ *for any* $\varphi \in \Phi$ *satisfying* $\underline{\lim}_{i\to\infty}(a_{\varphi(i)+1} - a_{\varphi(i)}) \geq 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 3.2. *Let* $\{D_n\} \subset \,]0, \pi/2[$ *and* $t \in \,]0,1[$. *Suppose*

$$\lim_{n\to\infty} \frac{\sin(tD_n) + \sin((1-t)D_n)}{\sin D_n} = 1.$$

Then $\lim_{n\to\infty} D_n = 0$.

Proof. Assume $\{D_n\}$ does not converge to 0, *i.e.*, there exist a subsequence $\{D_{n_i}\} \subset \{D_n\}$ and $D_0 > 0$ such that $D_{n_i} \to D_0$. Then we get

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{\sin(tD_n) + \sin((1-t)D_n)}{\sin D_n} \\ &= \lim_{i\to\infty} \frac{\sin(tD_{n_i}) + \sin((1-t)D_{n_i})}{\sin D_{n_i}} \\ &= \frac{\sin(tD_0) + \sin((1-t)D_0)}{\sin D_0}. \end{aligned}$$

Further, we have

$$\sin D_0 = \sin(tD_0) + \sin((1-t)D_0) = 2\sin\frac{D_0}{2}\cos\frac{(2t-1)D_0}{2} = 2\sin\frac{D_0}{2}\cos\frac{|1-2t|D_0}{2}.$$

Since $\sin D_0 = 2\sin\dfrac{D_0}{2}\cos\dfrac{D_0}{2}$, we obtain

$$\cos\frac{D_0}{2} = \cos\frac{|1-2t|D_0}{2}.$$

Then we get

$$\frac{D_0}{2} = \frac{|1-2t|D_0}{2},$$

and we have

$$(1 - |1-2t|)D_0 = 0.$$

Since $t \in]0,1[$, we have $1 - |1-2t| > 0$ and hence $D_0 = 0$. This is a contradiction. Thus, we obtain $\{D_n\}$ converges to 0. $\square$

The following are easy to see, so we omit their proofs.

Lemma 3.3. *Let $\{s_n\}$ and $\{t_n\} \subset]-\infty, 0]$. Suppose that $\lim_{n\to\infty}(s_n + t_n) = 0$. Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 0$.*

Lemma 3.4. *Let $\{s_n\}$ and $\{t_n\} \subset [0, \infty[$. Then*

$$\overline{\lim_{n\to\infty}}(s_n \cdot t_n) \geq \overline{\lim_{n\to\infty}} s_n \cdot \underline{\lim_{n\to\infty}} t_n.$$

Next we show the theorems and lemmas with respect to CAT(1) spaces.

Theorem 3.5 (Kimura and Satô [14]). *Let $\triangle(x,y,z)$ be a geodesic triangle in a* CAT(1) *space such that $d(x,y) + d(y,z) + d(z,x) < 2\pi$. Let $u = tx \oplus (1-t)y$ for some $t \in [0,1]$. Then*

$$\cos d(u,z) \sin d(x,y) \geq \cos d(x,z) \sin td(x,y) + \cos d(y,z) \sin(1-t)d(x,y).$$

Corollary 3.6 (Kimura and Satô [15]). *Let $\triangle(x,y,z)$ be a geodesic triangle in a* CAT(1) *space such that $d(x,y) + d(y,z) + d(z,x) < 2\pi$. Let $u = tx \oplus (1-t)y$ for some $t \in [0,1]$. Then*

$$\cos d(u,z) \geq t \cos d(x,z) + (1-t) \cos d(y,z).$$

Lemma 3.7 (Kimura and Satô [15]). *Let X be a complete* CAT(1) *space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let $\alpha \in [0,1]$ and $u, y, z \in X$. Then*

$$\begin{aligned} &1 - \cos d(\alpha u \oplus (1-\alpha)y, z) \\ &\qquad \leq (1-\beta)(1 - \cos d(y,z)) + \beta \left(1 - \frac{\cos d(u,z)}{\sin d(u,y) \tan\left(\frac{\alpha}{2} d(u,y)\right) + \cos d(u,y)}\right), \end{aligned}$$

where

$$\beta = \begin{cases} 1 - \dfrac{\sin((1-\alpha)d(u,y))}{\sin d(u,y)} & (u \neq y), \\ \alpha & (u = y). \end{cases}$$

Lemma 3.8 (Kimura and Satô [15]). *Let X be a* CAT(1) *space with $d(u,v) < \pi/2$ for any $u, v \in X$ and $\triangle(z,x,y) \subset X$. For $t \in]0,1[$, put $u = tz \oplus (1-t)x$ and $v = tz \oplus (1-t)y$. Then the following inequality holds:*

$$d(u,v) \leq \frac{\sin(1-t)M}{\sin M} d(x,y),$$

where $M = \sup_{p,q\in X} d(p,q) < \pi/2$.

Theorem 3.9 (Espínola and Fernández-León [6]). *Let X be a complete* CAT(1) *space and $\{x_n\}$ a sequence in X. If $r(\{x_n\}) < \pi/2$, then the following hold.*

(i) $AC(\{x_n\})$ *consists of exactly one point;*
(ii) $\{x_n\}$ *has a* Δ*-convergent subsequence.*

Lemma 3.10 (He, Fang, Lopez, and Li [9]). *Let* X *be a complete* CAT(1) *space and* $u \in X$. *If a sequence* $\{x_n\}$ *in* X *satisfies that* $\overline{\lim}_{n\to\infty} d(u, x_n) < \pi/2$ *and* $x_n \overset{\Delta}{\rightharpoonup} x \in X$, *then*

$$\varliminf_{n\to\infty} d(u, x_n) \geq d(u, x).$$

Lemma 3.11 (Kimura and Satô [15]). *Let* X *be a* CAT(1) *space with* $d(u, v) < \pi/2$ *for any* $u, v \in X$. *Let* S *and* T *be strongly quasinonexpansive and* Δ*-demiclosed mappings from* X *into itself with* $F(S) \cap F(T) \neq \emptyset$. *For* $\beta \in \,]0, 1[$, *put* $U = \beta S \oplus (1-\beta)T$. *Then the following conditions hold;*

- $F(U) = F(S) \cap F(T)$,
- U *is strongly quasinonexpansive,*
- U *is* Δ*-demiclosed.*

The following theorem is the main result in this chapter. We use three mappings; one is a nonexpansive mapping, and other two mappings are strongly quasinonexpansive and Δ-demiclosed mappings. We know that a nonexpansive mapping satisfies Δ-demiclosedness, but it does not satisfy strong quasinonexpansiveness in general.

Theorem 3.12. *Let* X *be a complete* CAT(1) *space with* $M = \sup_{p,q\in X} d(p, q) < \pi/2$. *Let* R *be a nonexpansive mapping from* X *into itself and* S, T *strongly quasinonexpansive and* Δ*-demiclosed mappings from* X *into itself with* $F = F(R) \cap F(S) \cap F(T) \neq \emptyset$. *For* $r \in \,]0, 1/2[$, *let* $\{\alpha_n\} \subset [r, 1-r] \subset \,]0, 1[$, $\{\beta_n\}, \{\gamma_n\} \subset \,]0, 1[$ *be real sequences satisfying* $\beta_n \to \beta \in \,]0, 1[$, $\gamma_n \to 0$, *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$. *Define* $\{x_n\} \subset X$ *by* $x_1, u \in X$ *and*

$$\begin{cases} s_n = \gamma_n u \oplus (1-\gamma_n) S x_n, \\ t_n = \gamma_n u \oplus (1-\gamma_n) T x_n, \\ y_n = \beta_n s_n \oplus (1-\beta_n) t_n, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) R y_n \end{cases}$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges to* $P_F u \in F$.

Proof. Let $p = P_F u \in F$ and put

$$\begin{aligned}
a_n &= 1 - \cos d(x_n, p), \\
b_n &= 1 - \frac{\cos d(u, p)}{\sin d(u, Sx_n) \tan\left(\frac{\gamma_n}{2} d(u, Sx_n)\right) + \cos d(u, Sx_n)}, \\
c_n &= 1 - \frac{\cos d(u, p)}{\sin d(u, Tx_n) \tan\left(\frac{\gamma_n}{2} d(u, Tx_n)\right) + \cos d(u, Tx_n)}, \\
\sigma_n &= \begin{cases} 1 - \dfrac{\sin(1-\gamma_n) d(u, Sx_n)}{\sin d(u, Sx_n)} & (u \neq Sx_n), \\ \gamma_n & (u = Sx_n), \end{cases} \\
\tau_n &= \begin{cases} 1 - \dfrac{\sin(1-\gamma_n) d(u, Tx_n)}{\sin d(u, Tx_n)} & (u \neq Tx_n), \\ \gamma_n & (u = Tx_n) \end{cases}
\end{aligned}$$

for $n \in \mathbb{N}$. Then, by Lemma 3.7 and Corollary 3.6, we have

$$\begin{aligned}
a_{n+1} &= 1 - \cos d(x_{n+1}, p) \\
&\le 1 - (\alpha_n \cos d(x_n, p) + (1-\alpha_n)\cos d(Ry_n, p)) \\
&\le \alpha_n a_n + (1-\alpha_n)(1 - \cos d(y_n, p)) \\
&\le \alpha_n a_n + (1-\alpha_n)(1 - (\beta_n \cos d(s_n, p) + (1-\beta_n)\cos d(t_n, p))) \\
&\le \alpha_n a_n + (1-\alpha_n)(\beta_n((1-\sigma_n)a_n + \sigma_n b_n) + (1-\beta_n)((1-\tau_n)a_n + \tau_n c_n)) \\
&= (\alpha_n + (1-\alpha_n)(\beta_n(1-\sigma_n) + (1-\beta_n)(1-\tau_n)))a_n + (1-\alpha_n)(\beta_n\sigma_n b_n + (1-\beta_n)\tau_n c_n) \\
&= (1 - (1-\alpha_n)(\beta_n\sigma_n + (1-\beta_n)\tau_n))a_n \\
&\quad + (1-\alpha_n)(\beta_n\sigma_n + (1-\beta_n)\tau_n)\frac{\beta_n\sigma_n b_n + (1-\beta_n)\tau_n c_n}{\beta_n\sigma_n + (1-\beta_n)\tau_n}.
\end{aligned}$$

To apply Lemma 3.1 to our sequence, we will show the following:

(i) $\sum_{n=1}^{\infty}(1-\alpha_n)(\beta_n\sigma_n + (1-\beta_n)\tau_n) = \infty$,

(ii) $\overline{\lim}_{i\to\infty} \dfrac{\beta_{\varphi(i)}\sigma_{\varphi(i)}b_{\varphi(i)} + (1-\beta_{\varphi(i)})\tau_{\varphi(i)}c_{\varphi(i)}}{\beta_{\varphi(i)}\sigma_{\varphi(i)} + (1-\beta_{\varphi(i)})\tau_{\varphi(i)}} \le 0$ for any nondecreasing functions $\varphi : \mathbb{N} \to \mathbb{N}$ satisfying $\lim_{i\to\infty}\varphi(i) = \infty$ and $\underline{\lim}_{i\to\infty}(a_{\varphi(i)+1} - a_{\varphi(i)}) \ge 0$.

First, we show (i). If $u = Sx_n$, then we have $\sigma_n = \gamma_n \ge \gamma_n \cos M$. For any $n \in \mathbb{N}$ satisfying $d(u, Sx_n) > 0$, we get

$$\begin{aligned}
\sigma_n &= \frac{\sin d(u, Sx_n) - \sin(1-\gamma_n)d(u, Sx_n)}{\sin d(u, Sx_n)} \\
&= \frac{2}{\sin d(u, Sx_n)} \sin\left(\frac{\gamma_n}{2} d(u, Sx_n)\right) \cos\left(\left(1 - \frac{\gamma_n}{2}\right) d(u, Sx_n)\right) \\
&\ge \gamma_n \cos\left(\left(1 - \frac{\gamma_n}{2}\right) d(u, Sx_n)\right) \\
&\ge \gamma_n \cos M.
\end{aligned}$$

Thus, we have $\sigma_n \ge \gamma_n \cos M$ for any $n \in \mathbb{N}$. We also get $\tau_n \ge \gamma_n \cos M$. Therefore, we obtain

$$\begin{aligned}
(1-\alpha_n)(\beta_n\sigma_n + (1-\beta_n)\tau_n) &\ge r(\beta_n\gamma_n \cos M + (1-\beta_n)\gamma_n \cos M) \\
&\ge \gamma_n \cdot r\cos M
\end{aligned}$$

for any $n \in \mathbb{N}$. Then (i) holds.

Next, we consider (ii). Let $n_i = \varphi(i)$ for any $i \in \mathbb{N}$. For any $i \in \mathbb{N}$ satisfying $\underline{\lim}_{i\to\infty}(a_{n_i+1} - a_{n_i}) \ge 0$, from Corollary 3.6 and the definition of strong quasinonexpansiveness, we have

$$\begin{aligned}
0 &\le \underline{\lim}_{i\to\infty}(a_{n_i+1} - a_{n_i}) \\
&= \underline{\lim}_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(x_{n_i+1}, p)) \\
&\le \underline{\lim}_{i\to\infty}(\cos d(x_{n_i}, p) - (\alpha_{n_i}\cos d(x_{n_i}, p) + (1-\alpha_{n_i})\cos d(Ry_{n_i}, p))) \\
&= \underline{\lim}_{i\to\infty}((1-\alpha_{n_i})(\cos d(x_{n_i}, p) - \cos d(Ry_{n_i}, p)) \\
&\le \underline{\lim}_{i\to\infty}(1-\alpha_{n_i})(\cos d(x_{n_i}, p) - \cos d(y_{n_i}, p))
\end{aligned}$$

$$\begin{aligned}
&\le \varliminf_{i\to\infty} (1-\alpha_{n_i})(\cos d(x_{n_i},p) - (\beta_{n_i}\cos d(s_{n_i},p) + (1-\beta_{n_i})\cos d(t_{n_i},p)))\\
&= \varliminf_{i\to\infty} (1-\alpha_{n_i})(\beta_{n_i}(\cos d(x_{n_i},p) - \cos d(s_{n_i},p)) + (1-\beta_{n_i})(\cos d(x_{n_i},p) - \cos d(t_{n_i},p)))\\
&\le \varliminf_{i\to\infty} (1-\alpha_{n_i})(\beta_{n_i}(\cos d(x_{n_i},p) - (\gamma_{n_i}\cos d(u,p) + (1-\gamma_{n_i})\cos d(Sx_{n_i},p)))\\
&\quad + (1-\beta_{n_i})(\cos d(x_{n_i},p) - (\gamma_{n_i}\cos d(u,p) + (1-\gamma_{n_i})\cos d(Tx_{n_i},p))))\\
&\le \varliminf_{i\to\infty} (1-\alpha_{n_i})(\beta(\cos d(x_{n_i},p) - \cos d(Sx_{n_i},p)) + (1-\beta)(\cos d(x_{n_i},p) - \cos d(Tx_{n_i},p)))\\
&\le \varlimsup_{i\to\infty} (1-\alpha_{n_i})(\beta(\cos d(x_{n_i},p) - \cos d(Sx_{n_i},p)) + (1-\beta)(\cos d(x_{n_i},p) - \cos d(Tx_{n_i},p)))\\
&\le 0.
\end{aligned}$$

Hence we get

$$\lim_{i\to\infty} (1-\alpha_{n_i})(\beta(\cos d(x_{n_i},p) - \cos d(Sx_{n_i},p)) + (1-\beta)(\cos d(x_{n_i},p) - \cos d(Tx_{n_i},p))) = 0.$$

From Lemma 3.3, we have

$$\lim_{i\to\infty}(\cos d(x_{n_i},p) - \cos d(Sx_{n_i},p)) = 0 \text{ and } \lim_{i\to\infty}(\cos d(x_{n_i},p) - \cos d(Tx_{n_i},p)) = 0.$$

It follows that

$$\lim_{i\to\infty} \frac{\cos d(x_{n_i},p)}{\cos d(Sx_{n_i},p)} = \lim_{i\to\infty} \frac{\cos d(x_{n_i},p)}{\cos d(Tx_{n_i},p)} = 1.$$

Since S and T are strongly quasinonexpansive, we get

$$\lim_{i\to\infty} d(x_{n_i}, Sx_{n_i}) = \lim_{i\to\infty} d(x_{n_i}, Tx_{n_i}) = 0. \tag{3.1}$$

On the other hand, by Corollary 3.6, we have

$$\begin{aligned}
\cos d(y_{n_i},p) &\ge \beta_{n_i}\cos d(s_{n_i},p) + (1-\beta_{n_i})\cos d(t_{n_i},p)\\
&\ge \beta_{n_i}(\gamma_{n_i}\cos d(u,p) + (1-\gamma_{n_i})\cos d(Sx_{n_i},p))\\
&\quad + (1-\beta_{n_i})(\gamma_{n_i}\cos d(u,p) + (1-\gamma_{n_i})\cos d(Tx_{n_i},p))\\
&\ge \gamma_{n_i}\cos d(u,p) + (1-\gamma_{n_i})\cos d(x_{n_i},p).
\end{aligned}$$

To show $\lim_{i\to\infty} d(x_{n_i}, Ry_{n_i}) = 0$, we may assume $d(x_{n_i}, Ry_{n_i}) > 0$. Then, by Theorem 3.5 and Corollary 3.6 again, we get

$$\begin{aligned}
&\cos d(x_{n_i+1},p)\sin d(x_{n_i}, Ry_{n_i})\\
&\ge \cos d(x_{n_i},p)\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \cos d(Ry_{n_i},p)\sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})\\
&\ge \cos d(x_{n_i},p)\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \cos d(y_{n_i},p)\sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})\\
&\ge \cos d(x_{n_i},p)\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i})\\
&\quad + (\gamma_{n_i}\cos d(u,p) + (1-\gamma_{n_i})\cos d(x_{n_i},p))\sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})\\
&= \cos d(x_{n_i},p)(\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i}))\\
&\quad + \gamma_{n_i}\sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})(\cos d(u,p) - \cos d(x_{n_i},p)).
\end{aligned}$$

Thus, we obtain

$$1 \ge \frac{\cos d(x_{n_i},p)}{\cos d(x_{n_i+1},p)} \cdot \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})}$$

$$
\begin{aligned}
&+ \gamma_{n_i} \cdot \frac{\sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \cdot \frac{1}{\cos d(x_{n_i+1}, p)} \cdot (\cos d(u,p) - \cos d(x_{n_i}, p)) \\
\geq\ & \frac{\cos d(x_{n_i}, p)}{\cos d(x_{n_i+1}, p)} \cdot \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&- \gamma_{n_i} \cdot \frac{\sin(1-r)d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \cdot \frac{1}{\cos M} \cdot |\cos d(u,p) - \cos d(x_{n_i}, p)| \\
\geq\ & \frac{\cos d(x_{n_i}, p)}{\cos d(x_{n_i+1}, p)} \cdot \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&- \gamma_{n_i} \cdot (1-r) \cdot \frac{d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \cdot \frac{\sin(1-r)d(x_{n_i}, Ry_{n_i})}{(1-r)d(x_{n_i}, Ry_{n_i})} \cdot \frac{1}{\cos M} \cdot 2
\end{aligned}
$$

Since $\gamma_n \to 0$ and $\underline{\lim}_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(x_{n_i+1}, p)) \geq 0$, or equivalently,

$$\underline{\lim}_{i\to\infty} \frac{\cos d(x_{n_i}, p)}{\cos d(x_{n_i+1}, p)} \geq 1,$$

by Lemma 3.4, we get

$$
\begin{aligned}
1 &\geq \overline{\lim}_{i\to\infty} \left(\frac{\cos d(x_{n_i}, p)}{\cos d(x_{n_i+1}, p)} \cdot \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \right) \\
&\geq \underline{\lim}_{i\to\infty} \frac{\cos d(x_{n_i}, p)}{\cos d(x_{n_i+1}, p)} \cdot \overline{\lim}_{i\to\infty} \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\geq \overline{\lim}_{i\to\infty} \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\geq \underline{\lim}_{i\to\infty} \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i})d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\geq 1.
\end{aligned}
$$

Since $\{\alpha_{n_i}\} \subset [r, 1-r]$ and by Lemma 3.2, we have

$$\lim_{i\to\infty} d(x_{n_i}, Ry_{n_i}) = 0.$$

Let $U = \beta S \oplus (1-\beta)T$. Then U satisfies three conditions in Lemma 3.11 and from Lemma 3.8, we get

$$
\begin{aligned}
&d(y_{n_i}, Ux_{n_i}) \\
&= d(\beta_{n_i} s_{n_i} \oplus (1-\beta_{n_i})t_{n_i}, \beta S x_{n_i} \oplus (1-\beta)T x_{n_i}) \\
&\leq d(\beta_{n_i} s_{n_i} \oplus (1-\beta_{n_i})t_{n_i}, \beta_{n_i} S x_{n_i} \oplus (1-\beta_{n_i})t_{n_i}) \\
&\quad + d(\beta_{n_i} S x_{n_i} \oplus (1-\beta_{n_i})t_{n_i}, \beta_{n_i} S x_{n_i} \oplus (1-\beta_{n_i})T x_{n_i}) \\
&\quad\quad + d(\beta_{n_i} S x_{n_i} \oplus (1-\beta_{n_i})T x_{n_i}, \beta S x_{n_i} \oplus (1-\beta)T x_{n_i}) \\
&\leq \frac{\sin \beta_{n_i} M}{\sin M} d(s_{n_i}, S x_{n_i}) + \frac{\sin(1-\beta_{n_i})M}{\sin M} d(t_{n_i}, T x_{n_i}) + |\beta_{n_i} - \beta| d(S x_{n_i}, T x_{n_i}) \\
&\leq \gamma_{n_i} \left(\frac{\sin \beta_{n_i} M}{\sin M} d(u, S x_{n_i}) + \frac{\sin(1-\beta_{n_i})M}{\sin M} d(u, T x_{n_i}) \right) + |\beta_{n_i} - \beta| d(S x_{n_i}, T x_{n_i}).
\end{aligned}
$$

Hence we have $\lim_{i\to\infty} d(y_{n_i}, U y_{n_i}) = 0$. Further, since

$$d(x_{n_i}, RUx_{n_i}) \leq d(x_{n_i}, Ry_{n_i}) + d(Ry_{n_i}, RUx_{n_i}) \leq d(x_{n_i}, Ry_{n_i}) + d(y_{n_i}, Ux_{n_i}),$$

we also get $\lim_{i\to\infty} d(x_{n_i}, RUx_{n_i}) = 0$. Moreover, since

$$d(x_{n_i}, p) \le d(x_{n_i}, RUx_{n_i}) + d(RUx_{n_i}, p) \le d(x_{n_i}, RUx_{n_i}) + d(Ux_{n_i}, p)$$

for $p \in F$, we obtain

$$\begin{aligned}
0 &\le |\cos d(x_{n_i}, p) - \cos d(Ux_{n_i}, p)| \\
&= 2\sin \frac{d(x_{n_i}, p) + d(Ux_{n_i}, p)}{2} \left|\sin \frac{d(x_{n_i}, p) - d(Ux_{n_i}, p)}{2}\right| \\
&\le 2 \sin M \sin \frac{d(x_{n_i}, RUx_{n_i})}{2} \\
&\to 0
\end{aligned}$$

as $i \to \infty$. It follows $\lim_{i\to\infty} \cos d(x_{n_i}, p)/\cos d(Ux_{n_i}, p) = 1$. Since U is strongly quasinonexpansive, we get $\lim_{i\to\infty} d(x_{n_i}, Ux_{n_i}) = 0$. Thus, we have

$$d(x_{n_i}, Rx_{n_i}) \le d(x_{n_i}, RUx_{n_i}) + d(Ux_{n_i}, x_{n_i}) \to 0 \tag{3.2}$$

as $i \to \infty$.

Furthermore, there exists a subsequence $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that

$$\varlimsup_{i\to\infty} \frac{\beta_{n_i}\sigma_{n_i}b_{n_i} + (1-\beta_{n_i})\tau_{n_i}c_{n_i}}{\beta_{n_i}\sigma_{n_i} + (1-\beta_{n_i})\tau_{n_i}} = \lim_{j\to\infty} \frac{\beta_{n_{i_j}}\sigma_{n_{i_j}}b_{n_{i_j}} + (1-\beta_{n_{i_j}})\tau_{n_{i_j}}c_{n_{i_j}}}{\beta_{n_{i_j}}\sigma_{n_{i_j}} + (1-\beta_{n_{i_j}})\tau_{n_{i_j}}}.$$

From Theorem 3.9(ii), there exists a subsequence $\{z_k\} \subset \{x_{n_{i_j}}\}$ such that it Δ-converges to $x_0 \in X$. Then from the definition of Δ-demiclosedness with (3.1) and (3.2), we have $x_0 \in F$. Further, we can take a subsequence $\{z_{k_l}\} \subset \{z_k\}$ satisfying

$$\delta = \lim_{l\to\infty} d(u, z_{k_l}) = \varliminf_{k\to\infty} d(u, z_k) \ge d(u, x_0) \ge d(u, p) \tag{3.3}$$

by Lemma 3.10. Moreover, we get

$$\begin{aligned}
\frac{\sigma_{k_l}}{\gamma_{k_l}} &= \frac{\sin d(u, Sz_{k_l}) - \sin(1-\gamma_{k_l}) d(u, Sz_{k_l})}{\gamma_{k_l} \sin d(u, Sz_{k_l})} \\
&= \frac{1-\cos \gamma_{k_l} d(u, Sz_{k_l})}{\gamma_{k_l}} + \frac{d(u, Sz_{k_l})}{\tan d(u, Sz_{k_l})} \cdot \frac{\sin \gamma_{k_l} d(u, Sz_{k_l})}{\gamma_{k_l} d(u, Sz_{k_l})} \\
&\to 0 + \frac{\delta}{\tan\delta} \cdot 1 = \frac{\delta}{\tan\delta}
\end{aligned}$$

as $l \to \infty$. Similarly, we have $\tau_{k_l}/\gamma_{k_l} \to \delta/\tan\delta$. Then we obtain $\sigma_{k_l}/\tau_{k_l} \to 1$. Thus, using (3.3), we get

$$\begin{aligned}
&\varlimsup_{i\to\infty} \frac{\beta_{\varphi(i)}\sigma_{\varphi(i)}b_{\varphi(i)} + (1-\beta_{\varphi(i)})\tau_{\varphi(i)}c_{\varphi(i)}}{\beta_{\varphi(i)}\sigma_{\varphi(i)} + (1-\beta_{\varphi(i)})\tau_{\varphi(i)}} \\
&= \varlimsup_{i\to\infty} \frac{\beta_{n_i}\sigma_{n_i}b_{n_i} + (1-\beta_{n_i})\tau_{n_i}c_{n_i}}{\beta_{n_i}\sigma_{n_i} + (1-\beta_{n_i})\tau_{n_i}} \\
&= \lim_{l\to\infty} \frac{\beta_{k_l}\sigma_{k_l}b_{k_l} + (1-\beta_{k_l})\tau_{k_l}c_{k_l}}{\beta_{k_l}\sigma_{k_l} + (1-\beta_{k_l})\tau_{k_l}}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{l\to\infty} \frac{\beta_{k_l}\dfrac{\sigma_{k_l}}{\tau_{k_l}} b_{k_l} + (1-\beta_{k_l})c_{k_l}}{\beta_{k_l}\dfrac{\sigma_{k_l}}{\tau_{k_l}} + (1-\beta_{k_l})} \\
&= \beta\left(1 - \frac{\cos d(u,p)}{\cos\delta}\right) + (1-\beta)\left(1 - \frac{\cos d(u,p)}{\cos\delta}\right) \\
&\le 1 - \frac{\cos d(u,p)}{\cos d(u,x_0)} \\
&\le 0
\end{aligned}
$$

for any φ satisfying $\varliminf_{i\to\infty}\left(a_{\varphi(i)+1} - a_{\varphi(i)}\right) \ge 0$. Therefore, from Lemma 3.1, we obtain that $a_n \to 0$ *i.e.*, $\{x_n\}$ converges to $p = P_F u \in F$. $\square$

Chapter 4

Convergence rate of Mann iteration

In this chapter, we consider a convergence rate of $d(x_n, Tx_n) \to 0$, where $\{x_n\}$ is generated by the Mann iteration in the setting of CAT(κ) spaces. The main result in this chapter is based on Matsushita's result in the setting of Hilbert spaces:

Theorem 4.1 (Matsushita [18]). *Let H be a Hilbert space and C a nonempty closed convex subset of H. Let T be a nonexpansive mapping from C into itself with $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset]0,1[$ be a real sequence satisfying $\sum_{k=1}^{\infty} \alpha_k(1-\alpha_k) = \infty$. Define $\{x_n\} \subset C$ by $x_1 \in C$ and*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$$

for all $n \in \mathbb{N}$. Then $\|x_n - Tx_n\| = o\left(\dfrac{1}{\sqrt{\sigma_n}}\right)$, where $\sigma_n = \sum_{k=1}^{n} \alpha_k(1-\alpha_k)$.

Since a Hilbert space is a CAT(1) space, we generalize Theorem 4.1 and get the following:

Theorem 4.2 (Kimura and Nakagawa [12]). *Let X be an admissible CAT(1) space. Let T be a nonexpansive mapping from X into itself with $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset]0,1[$ be a real sequence satisfying $\sum_{k=1}^{\infty} \alpha_k(1-\alpha_k) = \infty$. Define $\{x_n\} \subset X$ by $x_1 \in X$ and*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n Tx_n$$

for all $n \in \mathbb{N}$. Then $d(x_n, Tx_n) = o\left(\dfrac{1}{\sqrt{\sigma_n}}\right)$, where $\sigma_n = \sum_{k=1}^{n} \alpha_k(1-\alpha_k)$.

This lemma is the most important result to get the main result.

Lemma 4.3 (Dong [7]). *Let $\{s_n\}, \{t_n\}$ be positive sequences satisfying the follows:*

$$\text{(i)} \sum_{k=1}^{\infty} s_k = \infty, \quad \text{(ii) } \{t_n\} \text{ is decreasing}, \quad \text{(iii)} \sum_{k=1}^{\infty} s_k t_k < \infty.$$

Then $t_n = o\left(1/\sum_{k=1}^{n} s_k\right)$, i.e., $\lim_{n\to\infty} t_n \cdot \sum_{k=1}^{n} s_k = 0$.

Next, we introduce a function c_κ to summarize the properties for each $\kappa \in \mathbb{R}$:

Definition 4.4 (Kajimura and Kimura [10]). For $\kappa \in \mathbb{R}$, we define a function $c_\kappa : [0, D_\kappa/2[\to \mathbb{R}$ as follows;

$$c_\kappa(t) = \begin{cases} \dfrac{1}{\kappa}\left(1 - \cos(\sqrt{\kappa}t)\right) & (\kappa > 0), \\ \dfrac{1}{2}t^2 & (\kappa = 0), \\ \dfrac{1}{-\kappa}\left(\cosh(\sqrt{-\kappa}t) - 1\right) & (\kappa < 0). \end{cases}$$

By the Definition 4.4, we get the following:

Theorem 4.5 (Kajimura and Kimura [10]). *Let* $c_\kappa : [0, D_\kappa/2[\to \mathbb{R}$ *for* $\kappa \in \mathbb{R}$ *as Definition 4.4. Then the following hold;*

$$c'_\kappa(t) = \begin{cases} \dfrac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}t) & (\kappa > 0), \\ t & (\kappa = 0), \\ \dfrac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}t) & (\kappa < 0), \end{cases}$$

and

$$c''_\kappa(t) = \begin{cases} \cos(\sqrt{\kappa}t) & (\kappa > 0), \\ 1 & (\kappa = 0), \\ \cosh(\sqrt{-\kappa}t) & (\kappa < 0). \end{cases}$$

Moreover, we know the following properties;

(i) c_κ *is convex and increasing with* $c_\kappa(0) = 0$,
(ii) c'_κ *is increasing with* $c'_\kappa(0) = 0$,
(iii) $c''_\kappa(0) = 1$,
(iv) *for any* $s, t \in [0, D_\kappa/2[$, $c'_\kappa(s) + c'_\kappa(t) = 2c'_\kappa((s+t)/2)c''_\kappa((s-t)/2)$, *in particular, if* $s = 0$, $c'_\kappa(t) = 2c'_\kappa(t/2)c''_\kappa(t/2)$.

Furthermore, we also obtain this lemma.

Lemma 4.6. *For* $\kappa \in \mathbb{R}$, *let* $t \in [0, D_\kappa/2[$. *Then the following hold;*

(i) $1 - \kappa c_\kappa(t) > 0$,
(ii) $c'_\kappa(t) \geq 2t/\pi$.

Proof. First we show (i). By definition of c_κ, we get $1 - \kappa c_\kappa(t) = c''_\kappa(t) > 0$ for any $\kappa \in \mathbb{R}$ and $t \in [0, D_\kappa/2[$. Hence we obtain (i).

Next we show (ii). For the case of $\kappa > 0$, we have c'_κ is concave for $t \in [0, D_\kappa/2]$. Then we get

$$\begin{aligned} c'_\kappa\left(\alpha \cdot 0 + (1-\alpha)\frac{D_\kappa}{2}\right) &\geq \alpha c'_\kappa(0) + (1-\alpha)c'_\kappa\left(\frac{D_\kappa}{2}\right) \\ &= \alpha \cdot 0 + (1-\alpha)\frac{1}{\sqrt{\kappa}} \\ &= \frac{2}{\pi}\left(\alpha \cdot 0 + (1-\alpha)\frac{D_\kappa}{2}\right) \end{aligned}$$

for any $\alpha \in]0, 1[$. Hence, for any $t \in [0, D_\kappa/2]$, we obtain $c'_\kappa(t) \geq 2t/\pi$.

For the case of $\kappa < 0$, since $\sinh t \geq t$ for $t \geq 0$, we get

$$c'_\kappa(t) = \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}t) \geq \frac{1}{\sqrt{-\kappa}}(\sqrt{-\kappa}t) = t \geq \frac{2}{\pi}t.$$

For $\kappa = 0$, it is obvious. Thus, we obtain $c'_\kappa(t) \geq 2t/\pi$ for any $\kappa \in \mathbb{R}$ and $t \in [0, D_\kappa/2[$. □

The following theorem is a generalization of Theorem 3.5.

Theorem 4.7 (Kajimura and Kimura [10])**.** *For $\kappa \in \mathbb{R}$, let $\triangle(x,y,z)$ be a geodesic triangle on a* $\mathrm{CAT}(\kappa)$ *space with $d(x,y)+d(y,z)+d(z,x) < 2D_\kappa$ and put $u = \alpha x \oplus (1-\alpha)y$ for some $\alpha \in [0,1]$. Then*

$$\begin{aligned} &c_\kappa(d(u,z))c'_\kappa(d(x,y)) \\ &\qquad \le c_\kappa(d(x,z))c'_\kappa(\alpha d(x,y)) + c_\kappa(d(y,z))c'_\kappa((1-\alpha)d(x,y)) \\ &\qquad\qquad - 4c'_\kappa\left(\frac{d(x,y)}{2}\right) c'_\kappa\left(\frac{\alpha d(x,y)}{2}\right) c'_\kappa\left(\frac{(1-\alpha)d(x,y)}{2}\right). \end{aligned}$$

Corollary 4.8. *For $\kappa \in \mathbb{R}$, let $\triangle(x,y,z)$ be a geodesic triangle on a* $\mathrm{CAT}(\kappa)$ *space X with $d(x,y)+d(y,z)+d(z,x) < 2D_\kappa$ and put $u = \alpha x \oplus (1-\alpha)y$ for some $\alpha \in [0,1]$. Then*

$$c_\kappa(d(u,z)) \le \alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z)).$$

Proof. For $\kappa > 0$, the inequality in Theorem 4.7 is equivalent to

$$\begin{aligned} &\cos(\sqrt{\kappa}d(u,z))\sin(\sqrt{\kappa}d(x,y)) \\ &\ge \cos(\sqrt{\kappa}d(x,z))\sin(\alpha\sqrt{\kappa}d(x,y)) + \cos(\sqrt{\kappa}d(y,z))\sin((1-\alpha)\sqrt{\kappa}d(x,y)). \end{aligned}$$

Further, since $\sin t$ is concave for $t \in \,]0,\pi/2[$, we have $\sin(\alpha t) \ge \alpha \sin t$ for $\alpha \in \,]0,1[$. Then we obtain

$$\begin{aligned} &\cos(\sqrt{\kappa}d(u,z))\sin(\sqrt{\kappa}d(x,y)) \\ &\ge \alpha\cos(\sqrt{\kappa}d(x,z))\sin(\sqrt{\kappa}d(x,y)) + (1-\alpha)\cos(\sqrt{\kappa}d(y,z))\sin(\sqrt{\kappa}d(x,y)). \end{aligned}$$

Thus, dividing by $\sin(\sqrt{\kappa}d(x,y)) > 0$, we get

$$\cos(\sqrt{\kappa}d(u,z)) \ge \alpha\cos(\sqrt{\kappa}d(x,z)) + (1-\alpha)\cos(\sqrt{\kappa}d(y,z))$$

and hence we have

$$1-\cos(\sqrt{\kappa}d(u,z)) \le \alpha(1-\cos(\sqrt{\kappa}d(x,z))) + (1-\alpha)(1-\cos(\sqrt{\kappa}d(y,z))).$$

We can also prove the case where $\kappa \le 0$ by using Theorem 4.7. In fact, for $\kappa \le 0$, c'_κ is convex. Then we have $c'_\kappa(\alpha t) \le \alpha c'_\kappa(t)$ for $t \in [0,\infty[$ and $\alpha \in [0,1]$. Since $c'_\kappa(t) > 0$ for $t \in \,]0,\infty[$, we obtain

$$\begin{aligned} &c_\kappa(d(u,z)) \\ &\le \frac{1}{c'_\kappa(d(x,y))}\left(c_\kappa(d(x,z))c'_\kappa(\alpha d(x,y)) + c_\kappa(d(y,z))c'_\kappa((1-\alpha)d(x,y))\right. \\ &\quad \left. -4c'_\kappa\left(\frac{d(x,y)}{2}\right) c'_\kappa\left(\frac{\alpha d(x,y)}{2}\right) c'_\kappa\left(\frac{(1-\alpha)d(x,y)}{2}\right)\right) \\ &\le \alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z)). \end{aligned}$$

□

Finally, we show the main result.

Theorem 4.9. *Let X be an admissible* $\mathrm{CAT}(\kappa)$ *space. Let T be a nonexpansive mapping from X into itself with $F(T) \ne \emptyset$. Let $\{\alpha_n\} \subset \,]0,1[$ be a real sequence satisfying $\sum_{k=1}^\infty \alpha_k(1-\alpha_k) = \infty$. Define $\{x_n\} \subset X$ by $x_1 \in X$ and*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T x_n$$

for all $n \in \mathbb{N}$. Then $d(x_n, Tx_n) = o\left(\frac{1}{\sqrt{\sigma_n}}\right)$, where $\sigma_n = \sum_{k=1}^n \alpha_k(1-\alpha_k)$.

Proof. Let $z \in F$. By assumption and Lemma 4.3, we may show the following:

(C1) $\{d(x_n, Tx_n)^2\}$ is decreasing,
(C2) $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)d(x_n, Tx_n)^2 < \infty$.

First we show (C1). Since $x_{n+1} \in [x_n, Tx_n]$ and nonexpansiveness of T, we get

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) &\leq d(x_{n+1}, Tx_n) + d(Tx_n, Tx_{n+1}) \\ &\leq d(x_{n+1}, Tx_n) + d(x_n, x_{n+1}) \\ &= d(x_n, Tx_n). \end{aligned}$$

Hence $\{d(x_n, Tx_n)\}$ is nonincreasing and it follows $\{d(x_n, Tx_n)^2\}$ is also nonincreasing.

Next we show (C2). By Corollary 4.8, we have

$$\begin{aligned} c_\kappa(d(x_{n+1}, z)) &\leq (1-\alpha_n)c_\kappa(d(x_n, z)) + \alpha_n c_\kappa(d(Tx_n, z)) \\ &\leq (1-\alpha_n)c_\kappa(d(x_n, z)) + \alpha_n c_\kappa(d(x_n, z)) \\ &= c_\kappa(d(x_n, z)). \end{aligned}$$

Hence $\{c_\kappa(d(x_n, z))\}$ is nonincreasing.

Moreover, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} &c_\kappa(d(x_{n+1}, z))c''_\kappa\left(\frac{d(x_n, Tx_n)}{2}\right) \\ &\qquad \leq c_\kappa(d(x_n, z))c''_\kappa\left(\frac{(1-2\alpha_n)d(x_n, Tx_n)}{2}\right) \\ &\qquad\quad - 2c'_\kappa\left(\frac{(1-\alpha_n)d(x_n, Tx_n)}{2}\right)c'_\kappa\left(\sin\frac{\alpha_n d(x_n, Tx_n)}{2}\right). \end{aligned} \tag{4.1}$$

In fact, if $d(x_n, Tx_n) = 0$, it is obvious. For $d(x_n, Tx_n) > 0$, by Theorem 4.5 (iv) and Theorem 4.7, we get

$$\begin{aligned} &2c_\kappa(d(x_{n+1}, z))c'_\kappa\left(\frac{d(x_n, Tx_n)}{2}\right)c''_\kappa\left(\frac{d(x_n, Tx_n)}{2}\right) \\ &= c_\kappa(d(x_{n+1}, z))c'_\kappa(d(x_n, Tx_n)) \\ &\leq c_\kappa(d(x_n, z))c'_\kappa\left(\frac{(1-\alpha_n)d(x_n, Tx_n)}{2}\right) + c_\kappa(d(Tx_n, z))c'_\kappa\left(\frac{\alpha_n d(x_n, Tx_n)}{2}\right) \\ &\quad - 4c'_\kappa\left(\frac{d(x_n, Tx_n)}{2}\right)c'_\kappa\left(\frac{(1-\alpha_n)d(x_n, Tx_n)}{2}\right)c'_\kappa\left(\frac{\alpha_n d(x_n, Tx_n)}{2}\right) \\ &\leq c_\kappa(d(x_n, z))c'_\kappa\left(\frac{(1-\alpha_n)d(x_n, Tx_n)}{2}\right) + c_\kappa(d(x_n, z))c'_\kappa\left(\frac{\alpha_n d(x_n, Tx_n)}{2}\right) \\ &\quad - 4c'_\kappa\left(\frac{d(x_n, Tx_n)}{2}\right)c'_\kappa\left(\frac{(1-\alpha_n)d(x_n, Tx_n)}{2}\right)c'_\kappa\left(\frac{\alpha_n d(x_n, Tx_n)}{2}\right) \\ &= c_\kappa(d(x_n, z))\left(c'_\kappa\left(\frac{(1-\alpha_n)d(x_n, Tx_n)}{2}\right) + c'_\kappa\left(\frac{\alpha_n d(x_n, Tx_n)}{2}\right)\right) \\ &\quad - 4c'_\kappa\left(\frac{d(x_n, Tx_n)}{2}\right)c'_\kappa\left(\frac{(1-\alpha_n)d(x_n, Tx_n)}{2}\right)c'_\kappa\left(\frac{\alpha_n d(x_n, Tx_n)}{2}\right) \\ &\leq c_\kappa(d(x_n, z))\left(c'_\kappa\left((1-\alpha_n)d(x_n, Tx_n)\right) + c'_\kappa\left(\alpha_n d(x_n, Tx_n)\right)\right) \end{aligned}$$

$$
\begin{aligned}
&- 4c_\kappa'\left(\frac{d(x_n,Tx_n)}{2}\right)c_\kappa'\left(\frac{(1-\alpha_n)d(x_n,Tx_n)}{2}\right)c_\kappa'\left(\frac{\alpha_n d(x_n,Tx_n)}{2}\right)\\
&= 2c_\kappa(d(x_n,z))c_\kappa'\left(\frac{d(x_n,Tx_n)}{2}\right)c_\kappa''\left(\frac{(1-2\alpha_n)d(x_n,Tx_n)}{2}\right)\\
&\quad - 4c_\kappa'\left(\frac{d(x_n,Tx_n)}{2}\right)c_\kappa'\left(\frac{(1-\alpha_n)d(x_n,Tx_n)}{2}\right)c_\kappa'\left(\frac{\alpha_n d(x_n,Tx_n)}{2}\right).
\end{aligned}
$$

Dividing by $2c_\kappa'\left(\frac{d(x_n,Tx_n)}{2}\right) > 0$, we have (4.1).

Further, from Lemma 4.6 and (4.1), we have

$$
\begin{aligned}
&(c_\kappa(d(x_n,z)) - c_\kappa(d(x_{n+1},z)))c_\kappa''\left(\frac{d(x_n,Tx_n)}{2}\right)\\
&= (c_\kappa(d(x_n,z))c_\kappa''\left(\frac{d(x_n,Tx_n)}{2}\right) - c_\kappa(d(x_{n+1},z))c_\kappa''\left(\frac{d(x_n,Tx_n)}{2}\right)\\
&\geq c_\kappa(d(x_n,z))\left(c_\kappa''\left(\frac{d(x_n,Tx_n)}{2}\right) - c_\kappa''\left(\frac{(1-2\alpha_n)d(x_n,Tx_n)}{2}\right)\right)\\
&\quad + 2c_\kappa'\left(\frac{(1-\alpha_n)d(x_n,Tx_n)}{2}\right)c_\kappa'\left(\frac{\alpha_n d(x_n,Tx_n)}{2}\right)\\
&= -2\kappa c_\kappa(d(x_n,z))c_\kappa'\left(\frac{(1-\alpha_n)d(x_n,Tx_n)}{2}\right)c_\kappa'\left(\frac{\alpha_n d(x_n,Tx_n)}{2}\right)\\
&\quad + 2c_\kappa'\left(\frac{(1-\alpha_n)d(x_n,Tx_n)}{2}\right)c_\kappa'\left(\frac{\alpha_n d(x_n,Tx_n)}{2}\right)\\
&= 2(1-\kappa c_\kappa(d(x_n,z)))c_\kappa'\left(\frac{(1-\alpha_n)d(x_n,Tx_n)}{2}\right)c_\kappa'\left(\frac{\alpha_n d(x_n,Tx_n)}{2}\right)\\
&\geq 2(1-\kappa c_\kappa(d(x_n,z)))\cdot\frac{2}{\pi}\frac{(1-\alpha_n)d(x_n,Tx_n)}{2}\cdot\frac{2}{\pi}\frac{\alpha_n d(x_n,Tx_n)}{2}\\
&= \frac{2(1-\kappa c_\kappa(d(x_n,z)))}{\pi^2}\alpha_n(1-\alpha_n)d(x_n,Tx_n)^2.
\end{aligned}
$$

On the other hand, there exists $M, m > 0$ such that

$$
c_\kappa''\left(\frac{d(x_n,Tx_n)}{2}\right) \leq M \quad \text{and} \quad 1-\kappa c_\kappa(d(x_n,z)) \geq m
$$

for any $\kappa \in \mathbb{R}$. Then, we get

$$
\begin{aligned}
\alpha_n(1-\alpha_n)d(x_n,Tx_n)^2 &\leq \frac{\pi^2(c_\kappa(d(x_n,z)) - c_\kappa(d(x_{n+1},z)))}{2(1-\kappa c_\kappa(d(x_n,z)))}c_\kappa''\left(\frac{d(x_n,Tx_n)}{2}\right)\\
&\leq \frac{\pi^2 M}{2m}(c_\kappa(d(x_n,z)) - c_\kappa(d(x_{n+1},z))).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\sum_{k=1}^{n}\alpha_k(1-\alpha_k)d(x_k,Tx_k)^2 &\leq \frac{\pi^2 M}{2m}(c_\kappa(d(x_1,z)) - c_\kappa(d(x_{n+1},z)))\\
&\leq \frac{\pi^2 M}{2m}c_\kappa(d(x_1,z)).
\end{aligned}
$$

Thus, we obtain

$$\sum_{k=1}^{\infty} \alpha_k (1-\alpha_k) d(x_k, Tx_k)^2 \leq \frac{\pi^2 M}{2m} c_\kappa(d(x_1, z)) < \infty$$

as $n \to \infty$.

Therefore, by Lemma 4.3, we get

$$d(x_n, Tx_n)^2 = o\left(\frac{1}{\sigma_n}\right), \text{ or equivalently, } \lim_{n\to\infty} \sigma_n d(x_n, Tx_n)^2 = 0,$$

and it follows

$$\lim_{n\to\infty} \sqrt{\sigma_n} d(x_n, Tx_n) = 0, \text{ or equivalently, } d(x_n, Tx_n) = o\left(\frac{1}{\sqrt{\sigma_n}}\right). \qquad \square$$

On the other hand, consider an iterative scheme defined on a CAT(-1) space X. Since X is also a CAT(1) space, the result in [12] can be applicable. However, if the diameter of X is greater than or equal to $\pi/2$, the space does not satisfy the assumption in [12]. For such a case, we can apply the main result of this chapter.

Chapter 5

Convergence rate of Halpern iteration

In this chapter, we consider a convergence rate of $d(x_n, Tx_n) \to 0$, where $\{x_n\}$ is generated by the Halpern iteration in the setting of CAT(1) spaces. First, we define the following three functions:

Definition 5.1 (Leuştean [16]). Let $\{\lambda_n\} \subset \mathbb{R}$ be a sequence such that $\lambda_n \to \lambda$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Then we define the following:

- $\alpha :]0, \infty[\to \mathbb{N}$ is called a rate of convergence of $\{\lambda_n\}$ if

$$\forall \varepsilon > 0 \text{ and } \forall n \geq \alpha(\varepsilon), |\lambda_n - \lambda| < \varepsilon.$$

- $\beta :]0, \infty[\to \mathbb{N}$ is called a Cauchy modulus of $\{S_n\}$ if

$$\forall \varepsilon > 0 \text{ and } \forall n \in \mathbb{N}, S_{\beta(\varepsilon)+n} - S_{\beta(\varepsilon)} < \varepsilon,$$

  where $S_n = \sum_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$.

- $\theta : \mathbb{N} \to \mathbb{N}$ is called a rate of divergence of $\sum_{n=1}^{\infty} \lambda_n$ if

$$\forall n \in \mathbb{N}, \sum_{i=1}^{\theta(n)} \lambda_i \geq n.$$

The main result in this chapter is based on Leuştean's result:

Theorem 5.2 (Leuştean [16]). *Let X be a normed space and let $T : X \to X$ a nonexpansive mapping. Let $\{\lambda_n\} \subset [0, 1]$ satisfying $\lambda_n \to 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Define a bounded sequence $\{x_n\} \subset X$ as $x_1, u \in X$ and*

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)Tx_n$$

for any $n \in \mathbb{N}$. Let $M \in]0, \infty[$ with $M \geq \|u - Tu\|, \|u - x_n\|$ for any $n \in \mathbb{N}$. Furhter, let $\alpha :]0, \infty[\to \mathbb{N}$ be a rate of convergence of $\{\lambda_n\}$, $\beta :]0, \infty[\to \mathbb{N}$ a Cauchy modulus of $S_n = \sum_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$, and $\theta : \mathbb{N} \to \mathbb{N}$ a rate of divergence of $\sum_{n=1}^{\infty} \lambda_n$. Then $\|x_n - Tx_n\| \to 0$. Moreover,

$$\forall \varepsilon \in]0, 2[, \forall n \geq \max \left\{ \theta \left(\beta \left(\frac{\varepsilon}{8M} \right) + 1 + \left\lceil \ln \left(\frac{8M}{\varepsilon} \right) \right\rceil \right), \alpha \left(\frac{\varepsilon}{4M} \right) \right\}, \|x_n - Tx_n\| < \varepsilon.$$

Next, we describe important lemmas to get the main result.

Lemma 5.3 (Leuştean [16]). *Let* $\{\lambda_n\} \subset [0,1]$ *and* $\{a_n\}, \{b_n\} \subset [0,\infty[$. *Suppose*

$$a_{n+1} \leq (1-\lambda_{n+1})a_n + b_n$$

for any $n \in \mathbb{N}$. *Then*

$$a_{n+m} \leq \left(\prod_{i=n}^{n+m-1}(1-\lambda_{i+1})\right)a_n + \sum_{i=n}^{n+m-1} b_i$$

for any $m, n \in \mathbb{N}$. *Moreover, if* $\sum_{n=1}^{\infty} b_n < \infty$, $\{a_n\}$ *is bounded.*

Lemma 5.4 (Leuştean [16]). *Let* $\{\lambda_n\} \subset [0,1]$ *and* $\{a_n\}, \{b_n\} \subset [0,\infty[$ *satisfying*

$$\sum_{n=1}^{\infty}\lambda_n = \infty, \sum_{n=1}^{\infty} b_n < \infty, \text{ and } a_{n+1} \leq (1-\lambda_{n+1})a_n + b_n$$

for any $n \in \mathbb{N}$. *Further, let* $\theta :]0,\infty[\to \mathbb{N}$ *be a rate of divergence of* $\sum_{n=1}^{\infty}\lambda_n$ *and* $\beta :]0,\infty[\to \mathbb{N}$ *a Cauchy modulus of* $S_n = \sum_{i=1}^{n} b_i$. *Then* $\lim_{n\to\infty} a_n = 0$. *Moreover,*

$$\forall \varepsilon \in]0,2[\,, \forall n \geq \theta\left(\beta\left(\frac{\varepsilon}{2}\right) + 1 + \left\lceil \ln\left(\frac{2D}{\varepsilon}\right)\right\rceil\right), a_n < \varepsilon,$$

where D *is an upper bound of* $\{a_n\}$.

For the sake of completeness, we give the proof.

Proof. By Lemma 5.3, $\{a_n\}$ is bounded so there exists $D \in]1,\infty[$ such that $a_n \leq D$ for any $n \in \mathbb{N}$. Let $\varepsilon \in]0,2[$ and put

$$N = \beta\left(\frac{\varepsilon}{2}\right) + 1.$$

Hence, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} a_{N+m} &\leq \left(\prod_{i=N}^{N+m-1}(1-\lambda_{i+1})\right)a_N + \sum_{i=N}^{N+m-1} b_i \\ &\leq \exp\left(-\sum_{i=N}^{N+m-1}\lambda_{i+1}\right)a_N + \left(S_{\beta\left(\frac{\varepsilon}{2}\right)+m} - S_{\beta\left(\frac{\varepsilon}{2}\right)}\right) \\ &< D\exp\left(-\sum_{i=N}^{N+m-1}\lambda_{i+1}\right) + \frac{\varepsilon}{2} \end{aligned}$$

since $\exp(-x) \geq 1 - x$ for any $x \in [0,\infty[$ and β is a Cauchy modulus of $\{S_n\}$.

Then we show $D\exp\left(-\sum_{i=N}^{N+m-1}\lambda_{i+1}\right) \leq \varepsilon/2$ for any $m \in \mathbb{N}$. Let us note that

$$\begin{aligned} D\exp\left(-\sum_{i=N}^{N+m-1}\lambda_{i+1}\right) \leq \frac{\varepsilon}{2} &\iff \exp\left(-\sum_{i=N}^{N+m-1}\lambda_{i+1}\right) \leq \frac{\varepsilon}{2D} \\ &\iff \sum_{i=N}^{N+m-1}\lambda_{i+1} \geq \ln\left(\frac{2D}{\varepsilon}\right) \\ &\iff \sum_{i=N+1}^{N+m}\lambda_i \geq \ln\left(\frac{2D}{\varepsilon}\right) \end{aligned}$$

$$\iff \sum_{i=1}^{N+m} \lambda_i \geq \sum_{i=1}^{N} \lambda_i + \ln\left(\frac{2D}{\varepsilon}\right).$$

Put

$$L = \theta\left(N + \left\lceil \ln\left(\frac{2D}{\varepsilon}\right)\right\rceil\right) - N.$$

Since θ is a rate of divergence of $\{\lambda_n\}$ and $\lambda_n \leq 1$, we have

$$n \leq \sum_{i=1}^{\theta(n)} \lambda_i \leq \theta(n)$$

for any $n \in \mathbb{N}$. Using the fact that $2D/\varepsilon > D > 1$, we also get $\ln(2D/\varepsilon) > 0$. Hence $L \in \mathbb{N}$. Further, for any $m \geq L$, we obtain

$$\sum_{i=1}^{N+m} \lambda_i \geq \sum_{i=1}^{N+L} \lambda_i \geq N + \left\lceil \ln\left(\frac{2D}{\varepsilon}\right)\right\rceil \geq \sum_{i=1}^{N} \lambda_i + \ln\left(\frac{2D}{\varepsilon}\right).$$

Thus, we get $D\exp\left(-\sum_{i=N}^{N+m-1} \lambda_{i+1}\right) \leq \varepsilon/2$ for any $m \in \mathbb{N}$ and it follows that $a_{N+m} < \varepsilon$ for any $m \in \mathbb{N}$. Therefore, we obtain $a_{N+L+n} < \varepsilon$ for any $n \in \mathbb{N}$ and get the desired result. □

Further, we show the following properties:

Lemma 5.5. *Let X be an admissible* CAT(1) *space and $T : X \to X$ a nonexpansive mapping. Let $\{\lambda_n\} \subset [0,1]$ and $\{x_n\} \subset X$ defined by $x_1, u \in X$,*

$$x_{n+1} = \lambda_n u \oplus (1-\lambda_n)Tx_n$$

for any $n \in \mathbb{N}$. If $\{x_n\}$ is bounded, then $\{Tx_n\}$ is also bounded. Moreover, the following hold:

(i) $d(x_{n+2}, x_{n+1}) \leq \dfrac{\sin(1-\lambda_{n+1})M}{\sin M} d(x_{n+1}, x_n) + 2M|\lambda_{n+1} - \lambda_n|$,

(ii) $d(x_n, Tx_n) \leq d(x_{n+1}, x_n) + 2M\lambda_n$,

where $M \in \,]0, \pi/2[$ satisfies $M \geq d(u, Tu), d(u, x_n)$ for any $n \in \mathbb{N}$.

Proof. Using the triangle inequality and nonexpansiveness of T, we have

$$d(u, Tx_n) \leq d(u, Tu) + d(Tu, Tx_n) \leq d(u, Tu) + d(u, x_n).$$

Since $\{x_n\}$ is bounded, $\{Tx_n\}$ is bounded. Moreover, by Lemma 3.8, we get

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &\leq d(\lambda_{n+1}u \oplus (1-\lambda_{n+1})Tx_{n+1}, \lambda_{n+1}u \oplus (1-\lambda_{n+1})Tx_n) \\ &\quad + d(\lambda_{n+1}u \oplus (1-\lambda_{n+1})Tx_n, \lambda_n u \oplus (1-\lambda_n)Tx_n) \\ &\leq \frac{\sin(1-\lambda_{n+1})M}{\sin M} d(Tx_{n+1}, Tx_n) + |\lambda_{n+1} - \lambda_n| d(u, Tx_n) \\ &\leq \frac{\sin(1-\lambda_{n+1})M}{\sin M} d(x_{n+1}, x_n) + 2M|\lambda_{n+1} - \lambda_n| \end{aligned}$$

for any $n \in \mathbb{N}$. Further, since $x_{n+1} \in [u, Tx_n]$, we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d(x_n, x_{n+1}) + \lambda_n d(u, Tx_n) \\ &\leq d(x_n, x_{n+1}) + 2M\lambda_n \end{aligned}$$

for any $n \in \mathbb{N}$. □

Finally, we prove the main result.

Theorem 5.6. *Let X be an admissible* CAT(1) *space and let $T : X \to X$ a nonexpansive mapping. Let $\{\lambda_n\} \subset [0,1]$ satisfying $\lambda_n \to 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Define a bounded sequence $\{x_n\} \subset X$ as $x_1, u \in X$ and*

$$x_{n+1} = \lambda_n u \oplus (1 - \lambda_n) T x_n$$

for any $n \in \mathbb{N}$. Let $M \in]0, \pi/2[$ with $M \geq d(u, Tu), d(u, x_n)$ for any $n \in \mathbb{N}$. Further, let $\alpha :]0, \infty[\to \mathbb{N}$ be a rate of convergence of $\{\lambda_n\}$, $\beta :]0, \infty[\to \mathbb{N}$ a Cauchy modulus of $S_n = \sum_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$, and $\theta : \mathbb{N} \to \mathbb{N}$ a rate of divergence of $\sum_{n=1}^{\infty} \mu_n$, where $\mu_n = 1 - \sin(1 - \lambda_n) M / \sin M$. Then $d(x_n, Tx_n) \to 0$. Moreover,

$$\forall \varepsilon \in]0, 2[, \forall n \geq \max \left\{ \theta \left(\beta \left(\frac{\varepsilon}{8M} \right) + 1 + \left\lceil \ln \left(\frac{8M}{\varepsilon} \right) \right\rceil \right), \alpha \left(\frac{\varepsilon}{4M} \right) \right\}, d(x_n, Tx_n) < \varepsilon.$$

Proof. First, we show

$$\forall \varepsilon \in]0, 2[, \forall n \geq \theta \left(\beta \left(\frac{\varepsilon}{8M} \right) + 1 + \left\lceil \ln \left(\frac{8M}{\varepsilon} \right) \right\rceil \right), d(x_{n+1}, x_n) < \frac{\varepsilon}{2}.$$

By Lemma 5.5(i), we have

$$d(x_{n+2}, x_{n+1}) \leq \frac{\sin(1 - \lambda_{n+1}) M}{\sin M} d(x_{n+1}, x_n) + 2M |\lambda_{n+1} - \lambda_n|$$

for any $n \in \mathbb{N}$. Put

$$a_n = d(x_{n+1}, x_n), b_n = 2M |\lambda_{n+1} - \lambda_n|, \text{ and } \mu_n = 1 - \frac{\sin(1 - \lambda_n) M}{\sin M}.$$

Then we obtain $a_{n+1} \leq (1 - \mu_{n+1}) a_n + b_n$ and

$$\begin{aligned}
\mu_n &= 1 - \frac{\sin(1 - \lambda_n) M}{\sin M} \\
&\geq \frac{\sin M - \sin(1 - \lambda_n) M}{\sin M} \\
&= \frac{2}{\sin M} \sin \frac{\lambda_n M}{2} \cos \left(1 - \frac{\lambda_n}{2} \right) M \\
&\geq \lambda_n \cos M
\end{aligned}$$

for any $n \in \mathbb{N}$. Further, according to Lemma 5.4, we have

$$\begin{aligned}
a_{N+m} &\leq \left(\prod_{i=N}^{N+m-1} (1 - \mu_{i+1}) \right) a_N + \sum_{i=N}^{N+m-1} b_i \\
&\leq \exp \left(- \sum_{i=N}^{N+m-1} \mu_{i+1} \right) a_N + 2M \left(S_{\beta(\frac{\varepsilon}{8M}) + m} - S_{\beta(\frac{\varepsilon}{8M})} \right) \\
&< 2M \exp \left(- \sum_{i=N}^{N+m-1} \mu_{i+1} \right) + 2M \cdot \frac{\varepsilon}{8M} \\
&= 2M \exp \left(- \sum_{i=N}^{N+m-1} \mu_{i+1} \right) + \frac{\varepsilon}{4},
\end{aligned}$$

where $N = \beta\left(\frac{\varepsilon}{8M}\right) + 1$. Thus, we get

$$\forall \varepsilon \in \,]0,2[\,, \forall n \geq \theta\left(\beta\left(\frac{\varepsilon}{8M}\right) + 1 + \left\lceil \ln\left(\frac{8M}{\varepsilon}\right)\right\rceil\right), a_n = d(x_{n+1}, x_n) < \frac{\varepsilon}{2}.$$

By Lemma 5.5(ii), we have

$$d(x_n, Tx_n) \leq d(x_{n+1}, x_n) + 2M\lambda_n$$

for any $n \in \mathbb{N}$. Since α is a rate of convergent of $\{\lambda_n\}$, we obtain

$$\forall \varepsilon > 0, \forall n \geq \alpha\left(\frac{\varepsilon}{4M}\right), 2M\lambda_n < 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}.$$

Therefore, we get the desired result. □

Acknowledgements

The author had spent a really great time as a Kimura's research student for 9 years. It was so long, but so short. The author was proud of he could meet Professor Yasunori Kimura, research members, and foreign friends. The author would like to thank them for giving him many ideas and his whereabouts. The author is looking forward to seeing them again.

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